Pensieve header: The 2-variable perturbed Alexander invariant: No g-quadratic solution.

```
In[•]:=  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
         Once[<< KnotTheory`; << ../APAI/Rot.m];
         TS = T S;
         (α_+)+ := α"++";  (* this is for cosmetic reasons only *)
         δ_{i_,j_} := If[i == j, 1, 0];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/APAI to compute rotation numbers.

Lower Rank 2 Gassner: (Modified from Rank2Gassner.nb)

```
In[•]:=  LR2G_{i_,j_}[ℰ_] := Expand[ℰ /. {e_i ⟼ T e_i + (1 - T) e_j, f_i ⟼ S f_i + (1 - S) f_j,
            g_i ⟼ T S g_i + (1 - T S) g_j
              + S (T - 1) e_j f_i + (S - 1) T e_i f_j + (S + T - 2 S T) e_j f_j}];
         LR2G̅_{i_,j_}[ℰ_] := Expand[ℰ /. {e_i ⟼ T^{-1} e_i + (1 - T^{-1}) e_j, f_i ⟼ S^{-1} f_i + (1 - S^{-1}) f_j,
            g_i ⟼ T^{-1} S^{-1} g_i + (1 - T^{-1} S^{-1}) g_j
              - S^{-1} (1 - T^{-1}) e_j f_i - T^{-1} (1 - S^{-1}) e_i f_j + (S^{-1} + T^{-1} - 2 S^{-1} T^{-1}) e_j f_j}];
         bas =
           {e_1, e_2, e_3, f_1, f_2, f_3, e_1 f_1, e_1 f_2, e_1 f_3, e_2 f_1, e_2 f_2, e_2 f_3, e_3 f_1, e_3 f_2, e_3 f_3, g_1, g_2, g_3};
         (bas // LR2G_{1,2} // LR2G̅_{1,2}) == bas
         (bas // LR2G_{3,2} // LR2G̅_{3,2}) == bas
         (lhs = bas // LR2G_{1,2} // LR2G_{1,3} // LR2G_{2,3}) == (bas // LR2G_{2,3} // LR2G_{1,3} // LR2G_{1,2})
```

Out[•]=
         True

Out[•]=
         True

Out[•]=
         True

```
In[*]:= R₁[s_, i_, j_] := TBD;  (* a sign(s)-dependent quadratic polynomial in g₁αβ, g₂αβ, g₃αβ,
           and yαβγ, where deg yαβγ=2 and α,β,γ∈{i,j}, with coefficients in ℤ[T^±1,S^±1]. *)
         CF[ℰ_] := Factor@Together[ℰ];
         λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
             {Cs, φ} = Rot[K];  n = Length[Cs];
             A = IdentityMatrix[2 n + 1];
             Cases[Cs, {s_, i_, j_} ⧴ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ) )];
                                                                     (  0    -1   )
             Δ = T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A];
             G = Inverse[A];
             gEval[ℰ_] := CF[ℰ /. α_⁺ ⧴ α + 1 /.
                 {g₁,α_,β_ ⧴ G[[α, β]], g₂,α_,β_ ⧴ (G[[α, β]] /. T → S), g₃,α_,β_ ⧴ (G[[α, β]] /. T → TS)}];
             Y[α_, β_, γ_] := Sum[{s, i, j} = c;
                 g₃,α,i (S^s (T^s - 1) g₁,j⁺,β g₂,i⁺,γ + (S^s - 1) T^s g₁,i⁺,β g₂,j⁺,γ + (S^s + T^s - 2 TS^s) g₁,j⁺,β g₂,j⁺,γ),
                 {c, Cs}];
             yEval[ℰ_] := CF[ℰ /. y_α_,β_,γ_ ⧴ Y[α, β, γ]];
             λ1 = ∑ₖ₌₁ⁿ R₁ @@ Cs[[k]] - ∑ₖ₌₁²ⁿ φ[[k]] (g₁,k,k + g₂,k,k + g₃,k,k);
             {Δ, Δ (Δ /. T → S) (Δ /. T → TS) λ1} // yEval // gEval
         ];
```

Step-by-step Run-Through

```
In[*]:= CF[ℰ_] := Factor@Together[ℰ];
```

```
In[*]:= K = {
           PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
           Knot[6, 2]
         }[[1]]
Out[*]=
       PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]
```

```
In[*]:= {Cs, φ} = Rot[K]
Out[*]=
       {{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

```
In[*]:= n = Length[Cs];
       A = IdentityMatrix[2 n + 1];
       Cases[Cs, {s_, i_, j_} ⧴ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ) )];
                                                                  (  0    -1   )
```

In[ ]:= **A // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:= **Δ = T$^{(-\text{Total}[\varphi]-\text{Total}[Cs[\![All,1]\!]])/2}$ Det[A]**

Out[ ]=

$$\frac{1 - T + T^2}{T}$$

In[ ]:= **G = Inverse[A];**
G // MatrixForm

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:= **gEval[$\mathcal{E}$_] := CF[$\mathcal{E}$ /. α_$^+$:→ α + 1 /.**
{g$_{1,\alpha\_,\beta\_}$:→ G[\![α, β]\!], g$_{2,\alpha\_,\beta\_}$:→ (G[\![α, β]\!] /. T → S), g$_{3,\alpha\_,\beta\_}$:→ (G[\![α, β]\!] /. T → TS)}];

In[ ]:= **Clear[Y]**
Y[α_, β_, γ_] := Y[α, β, γ] = Sum[{s, i, j} = c;
g$_{3,\alpha,i}$ (S^s (T^s – 1) g$_{1,j^+,\beta}$ g$_{2,i^+,\gamma}$ + (S^s – 1) T^s g$_{1,i^+,\beta}$ g$_{2,j^+,\gamma}$ + (S^s + T^s – 2 TS^s) g$_{1,j^+,\beta}$ g$_{2,j^+,\gamma}$),
{c, Cs}];

Column[MatrixForm /@ Table[Y[α, β, γ], {α, 3}, {β, 3}, {γ, 3}]]

Out[ ]=

$\left(\begin{array}{l} (S (-1 + T) g_{1,4^+,1} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,1} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,1} g_{2,4^+,1}) g_{3,1,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,2} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,2} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,2} g_{2,4^+,1}) g_{3,1,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,3} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,3} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,3} g_{2,4^+,1}) g_{3,1,1} + (S (-1 + T) g_1 \end{array}\right)$

$\left(\begin{array}{l} (S (-1 + T) g_{1,4^+,1} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,1} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,1} g_{2,4^+,1}) g_{3,2,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,2} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,2} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,2} g_{2,4^+,1}) g_{3,2,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,3} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,3} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,3} g_{2,4^+,1}) g_{3,2,1} + (S (-1 + T) g_1 \end{array}\right)$

$\left(\begin{array}{l} (S (-1 + T) g_{1,4^+,1} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,1} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,1} g_{2,4^+,1}) g_{3,3,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,2} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,2} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,2} g_{2,4^+,1}) g_{3,3,1} + (S (-1 + T) g_1 \\ (S (-1 + T) g_{1,4^+,3} g_{2,1^+,1} + (-1 + S) T g_{1,1^+,3} g_{2,4^+,1} + (S + T - 2 S T) g_{1,4^+,3} g_{2,4^+,1}) g_{3,3,1} + (S (-1 + T) g_1 \end{array}\right)$

In[◦]:= `Column[MatrixForm /@ Table[Y[α, β, γ] // gEval, {α, 3}, {β, 3}, {γ, 3}]]`

Out[◦]=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(-1+S)^2\,T}{1-S+S^2} \\ 0 & -\frac{S\,(-1+T)^2}{1-T+T^2} & -\frac{-S+S^2-T+2\,S\,T-2\,S^2\,T+T^2-2\,S\,T^2+2\,S^2\,T^2}{(1-S+S^2)\,(1-T+T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{S\,T\,(-S-T+2\,S\,T)}{(1-S+S^2)\,(1-T+T^2)\,(1-S\,T+S^2\,T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{S\,T\,(-S-T+2\,S\,T)}{(1-S+S^2)\,(1-T+T^2)\,(1-S\,T+S^2\,T^2)} \end{pmatrix}$$

In[◦]:= `yEval[ℰ_] := CF[ℰ /. y`$_{α\_,β\_,γ\_}$` :> Y[α, β, γ]]`

The g-Rules

In[◦]:= `gRules`$_{s\_,i\_,j\_}$` := {`

$g_{1,i,β\_} :\!\Rightarrow δ_{i,β} + T^s\,g_{1,i^+,β} + \left(1 - T^s\right)\,g_{1,j^+,β},\ g_{1,j,β\_} :\!\Rightarrow δ_{j,β} + g_{1,j^+,β},$

$g_{1,α\_,i} :\!\Rightarrow T^{-s}\,\left(g_{1,α,i^+} - δ_{α,i^+}\right),\ g_{1,α\_,j} :\!\Rightarrow g_{1,α,j^+} - \left(1 - T^s\right)\,g_{1,α,i} - δ_{α,j^+},$

$g_{2,i,β\_} :\!\Rightarrow δ_{i,β} + S^s\,g_{2,i^+,β} + \left(1 - S^s\right)\,g_{2,j^+,β},\ g_{2,j,β\_} :\!\Rightarrow δ_{j,β} + g_{2,j^+,β},$

$g_{2,α\_,i} :\!\Rightarrow S^{-s}\,\left(g_{2,α,i^+} - δ_{α,i^+}\right),\ g_{2,α\_,j} :\!\Rightarrow g_{2,α,j^+} - \left(1 - S^s\right)\,g_{2,α,i} - δ_{α,j^+},$

$g_{3,i,β\_} :\!\Rightarrow δ_{i,β} + TS^s\,g_{3,i^+,β} + \left(1 - TS^s\right)\,g_{3,j^+,β},\ g_{3,j,β\_} :\!\Rightarrow δ_{j,β} + g_{3,j^+,β},$

$g_{3,α\_,i} :\!\Rightarrow TS^{-s}\,\left(g_{3,α,i^+} - δ_{α,i^+}\right),\ g_{3,α\_,j} :\!\Rightarrow g_{3,α,j^+} - \left(1 - TS^s\right)\,g_{3,α,i} - δ_{α,j^+}$

`}`

In[◦]:= `(gs = Table[{g`$_{1,α,β}$`, g`$_{2,α,β}$`, g`$_{3,α,β}$`}, {α, 2 n + 1}, {β, 2 n + 1}])⟦1 ;; 2, 1 ;; 2, 1 ;; 2⟧`

Out[◦]=

$\{\{\{g_{1,1,1},\ g_{2,1,1}\},\ \{g_{1,1,2},\ g_{2,1,2}\}\},\ \{\{g_{1,2,1},\ g_{2,2,1}\},\ \{g_{1,2,2},\ g_{2,2,2}\}\}\}$

In[◦]:= `Cs`

Out[◦]=

$\{\{1, 1, 4\},\ \{1, 5, 2\},\ \{1, 3, 6\}\}$

In[◦]:= $\left(\text{gs /. gRules}_{\text{Sequence@@Cs⟦1⟧}}\right)$`⟦1 ;; 2, 1 ;; 2, 1 ;; 2⟧`

Out[◦]=

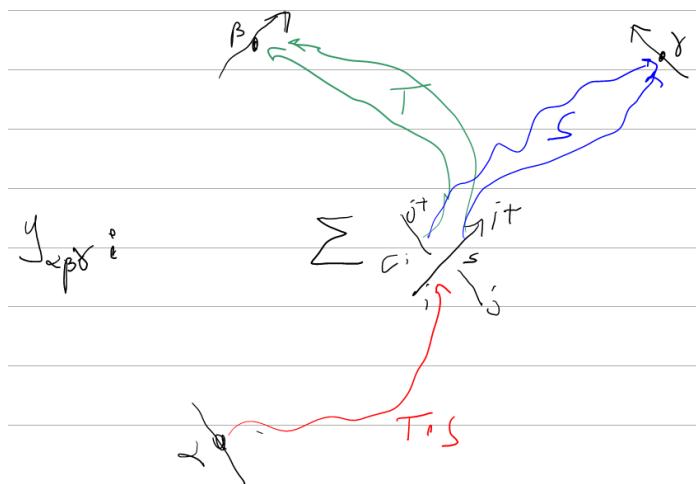$\left\{\left\{\left\{1 + T\,g_{1,1^+,1} + (1 - T)\,g_{1,4^+,1},\ 1 + S\,g_{2,1^+,1} + (1 - S)\,g_{2,4^+,1}\right\},\right.\right.$

$\left. \left\{T\,g_{1,1^+,2} + (1 - T)\,g_{1,4^+,2},\ S\,g_{2,1^+,2} + (1 - S)\,g_{2,4^+,2}\right\}\right\},$

$\left. \left\{\left\{\frac{-\text{If}[2 == 1^+, 1, 0] + g_{1,2,1^+}}{T},\ \frac{-\text{If}[2 == 1^+, 1, 0] + g_{2,2,1^+}}{S}\right\},\ \{g_{1,2,2},\ g_{2,2,2}\}\right\}\right\}$

$In[\circ]:=$ `gEval[gs - (gs /. Flatten@Table[gRules`$_{\text{Sequence@@c}}$`, {c, Cs}])]`

$Out[\circ]=$

$\{\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}\}$

The y-Rules

$In[\circ]:=$ `yRules`$_{s\_,i\_,j\_}$ `:= {`

$y_{i,\beta\_,\gamma\_} :\mapsto TS^s\, y_{i^+,\beta,\gamma} + (1 - TS^s)\, y_{j^+,\beta,\gamma} +$
$\quad (S^s\,(T^s - 1)\, g_{1,j^+,\beta}\, g_{2,i^+,\gamma} + (S^s - 1)\, T^s\, g_{1,i^+,\beta}\, g_{2,j^+,\gamma} + (S^s + T^s - 2\, TS^s)\, g_{1,j^+,\beta}\, g_{2,j^+,\gamma}),$

$y_{j,\beta\_,\gamma\_} :\mapsto y_{j^+,\beta,\gamma},$

$y_{\alpha\_,i,\gamma\_} :\mapsto T^{-s}\, y_{\alpha,i^+,\gamma} + (1 - S^s)\, g_{3,\alpha,i}\, g_{2,j^+,\gamma},$

$y_{\alpha\_,j,\gamma\_} :\mapsto y_{\alpha,j^+,\gamma} - (1 - T^s)\, y_{\alpha,i,\gamma} - g_{3,\alpha,i}\,(S^s\,(T^s - 1)\, g_{2,i^+,\gamma} + (S^s + T^s - 2\, TS^s)\, g_{2,j^+,\gamma}),$

$y_{\alpha\_,\beta\_,i} :\mapsto S^{-s}\, y_{\alpha,\beta,i^+} + (1 - T^s)\, g_{3,\alpha,i}\, g_{1,j^+,\beta},$

$y_{\alpha\_,\beta\_,j} :\mapsto y_{\alpha,\beta,j^+} - (1 - S^s)\, y_{\alpha,\beta,i} - g_{3,\alpha,i}\,(T^s\,(S^s - 1)\, g_{1,i^+,\beta} + (S^s + T^s - 2\, TS^s)\, g_{1,\beta,j^+})$

`}`

$In[\circ]:=$ `(ys = Table[y`$_{\alpha,\beta,\gamma}$`, {`α`, 2 n + 1}, {`β`, 2 n + 1}, {`γ`, 2 n + 1}]) `$[\![1\ ;;\ 2, 1\ ;;\ 2, 1\ ;;\ 2]\!]$

$Out[\circ]=$

$\{\{\{y_{1,1,1}, y_{1,1,2}\}, \{y_{1,2,1}, y_{1,2,2}\}\}, \{\{y_{2,1,1}, y_{2,1,2}\}, \{y_{2,2,1}, y_{2,2,2}\}\}\}$

$In[\circ]:=$ `Cs`

$Out[\circ]=$

$\{\{1, 1, 4\}, \{1, 5, 2\}, \{1, 3, 6\}\}$

$\textit{In[•]:=}$ $\left(\text{ys /. yRules}_{\text{Sequence@@Cs}[\![1]\!]}\right)[\![1 ;; 2, 1 ;; 2, 1 ;; 2]\!]$

$\textit{Out[•]=}$

$\Big\{\big\{\{S\ (-1 + T)\ g_{1,4^+,1}\ g_{2,1^+,1} + (-1 + S)\ T\ g_{1,1^+,1}\ g_{2,4^+,1} +$

$\quad (S + T - 2\ S\ T)\ g_{1,4^+,1}\ g_{2,4^+,1} + S\ T\ y_{1^+,1,1} + (1 - S\ T)\ y_{4^+,1,1},\ S\ (-1 + T)\ g_{1,4^+,1}\ g_{2,1^+,2} +$

$\quad (-1 + S)\ T\ g_{1,1^+,1}\ g_{2,4^+,2} + (S + T - 2\ S\ T)\ g_{1,4^+,1}\ g_{2,4^+,2} + S\ T\ y_{1^+,1,2} + (1 - S\ T)\ y_{4^+,1,2}\},$

$\quad \{S\ (-1 + T)\ g_{1,4^+,2}\ g_{2,1^+,1} + (-1 + S)\ T\ g_{1,1^+,2}\ g_{2,4^+,1} + (S + T - 2\ S\ T)\ g_{1,4^+,2}\ g_{2,4^+,1} +$

$\quad S\ T\ y_{1^+,2,1} + (1 - S\ T)\ y_{4^+,2,1},\ S\ (-1 + T)\ g_{1,4^+,2}\ g_{2,1^+,2} + (-1 + S)\ T\ g_{1,1^+,2}\ g_{2,4^+,2} +$

$\quad (S + T - 2\ S\ T)\ g_{1,4^+,2}\ g_{2,4^+,2} + S\ T\ y_{1^+,2,2} + (1 - S\ T)\ y_{4^+,2,2}\}\big\},$

$\quad \Big\{\Big\{(1 - S)\ g_{2,4^+,1}\ g_{3,2,1} + \dfrac{y_{2,1^+,1}}{T},\ (1 - S)\ g_{2,4^+,2}\ g_{3,2,1} + \dfrac{y_{2,1^+,2}}{T}\Big\},$

$\quad \Big\{(1 - T)\ g_{1,4^+,2}\ g_{3,2,1} + \dfrac{y_{2,2,1^+}}{S},\ y_{2,2,2}\Big\}\Big\}\Big\}$

$\textit{In[•]:=}$ $\text{yEval}\Big[\text{ys} - \left(\text{ys /. yRules}_{\text{Sequence@@Cs}[\![1]\!]}\right)\Big][\![1, 1, 1]\!]$

$\textit{Out[•]=}$

$S\ g_{1,4^+,1}\ g_{2,1^+,1} - S\ T\ g_{1,4^+,1}\ g_{2,1^+,1} + T\ g_{1,1^+,1}\ g_{2,4^+,1} - S\ T\ g_{1,1^+,1}\ g_{2,4^+,1} - S\ g_{1,4^+,1}\ g_{2,4^+,1} -$

$T\ g_{1,4^+,1}\ g_{2,4^+,1} + 2\ S\ T\ g_{1,4^+,1}\ g_{2,4^+,1} - S\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,1,1} + S\ T\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,1,1} -$

$T\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,1,1} + S\ T\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,1,1} + S\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1,1} + T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1,1} -$

$2\ S\ T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1,1} - S\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,1,3} + S\ T\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,1,3} - T\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,1,3} +$

$S\ T\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,1,3} + S\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1,3} + T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1,3} - 2\ S\ T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1,3} +$

$S\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1,5} + T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1,5} - 2\ S\ T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1,5} - T\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,1,5} +$

$S\ T\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,1,5} - S\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,1,5} + S\ T\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,1,5} + S^2\ T\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,1^+,1} -$

$S^2\ T^2\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,1^+,1} + S\ T^2\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,1^+,1} - S^2\ T^2\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,1^+,1} -$

$S^2\ T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1^+,1} - S\ T^2\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1^+,1} + 2\ S^2\ T^2\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,1^+,1} +$

$S^2\ T\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,1^+,3} - S^2\ T^2\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,1^+,3} + S\ T^2\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,1^+,3} -$

$S^2\ T^2\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,1^+,3} - S^2\ T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1^+,3} - S\ T^2\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1^+,3} +$

$2\ S^2\ T^2\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,1^+,3} - S^2\ T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1^+,5} - S\ T^2\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1^+,5} +$

$2\ S^2\ T^2\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,1^+,5} + S\ T^2\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,1^+,5} - S^2\ T^2\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,1^+,5} +$

$S^2\ T\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,1^+,5} - S^2\ T^2\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,1^+,5} + S\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,4^+,1} - S\ T\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,4^+,1} -$

$S^2\ T\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,4^+,1} + S^2\ T^2\ g_{1,4^+,1}\ g_{2,1^+,1}\ g_{3,4^+,1} + T\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} - S\ T\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} -$

$S\ T^2\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} + S^2\ T^2\ g_{1,1^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} - S\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} - T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} +$

$2\ S\ T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} + S^2\ T\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} + S\ T^2\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} -$

$2\ S^2\ T^2\ g_{1,4^+,1}\ g_{2,4^+,1}\ g_{3,4^+,1} + S\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,4^+,3} - S\ T\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,4^+,3} - S^2\ T\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,4^+,3} +$

$S^2\ T^2\ g_{1,6^+,1}\ g_{2,3^+,1}\ g_{3,4^+,3} + T\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} - S\ T\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} - S\ T^2\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} +$

$S^2\ T^2\ g_{1,3^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} - S\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} - T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} + 2\ S\ T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} +$

$S^2\ T\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} + S\ T^2\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} - 2\ S^2\ T^2\ g_{1,6^+,1}\ g_{2,6^+,1}\ g_{3,4^+,3} -$

$S\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} - T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} + 2\ S\ T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} + S^2\ T\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} +$

$S\ T^2\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} - 2\ S^2\ T^2\ g_{1,2^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} + T\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} -$

$S\ T\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} - S\ T^2\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} + S^2\ T^2\ g_{1,5^+,1}\ g_{2,2^+,1}\ g_{3,4^+,5} +$

$S\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,4^+,5} - S\ T\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,4^+,5} - S^2\ T\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,4^+,5} + S^2\ T^2\ g_{1,2^+,1}\ g_{2,5^+,1}\ g_{3,4^+,5}$

In[ ]:= **gEval@yEval$\left[\text{ys} - \left(\text{ys} /. \text{Flatten@Table}\left[\text{gRules}_{\text{Sequence@@c}}, \{c, Cs\}\right]\right)\right]$**

Out[ ]=

$\{\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\},$
$\{\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0\}\}\}$

Invariance of $y_{\alpha\beta\gamma}$ under remote R2s

In[ ]:= **Clear[i, j];**
Cs = $\{\{1, i, j\}, \{-1, i^+, j^+\}\}$
Z = Module$\left[\{s, i, j\}, \text{Sum}\left[\{s, i, j\} = c;\right.\right.$
    $g_{3,\alpha,i}\left(S^s\left(T^s - 1\right)g_{1,j^+,\beta}\,g_{2,i^+,\gamma} + \left(S^s - 1\right)T^s\,g_{1,i^+,\beta}\,g_{2,j^+,\gamma} + \left(S^s + T^s - 2\,TS^s\right)g_{1,j^+,\beta}\,g_{2,j^+,\gamma}\right),$
    $\{c, Cs\}\Big]\Big]$
Simplify$\left[\text{Z} /. \text{gRules}_{1,i,j}\bigcup\text{gRules}_{-1,i^+,j^+} /. \_\text{If} \to 0\right]$

Out[ ]=

$\{\{1, i, j\}, \{-1, i^+, j^+\}\}$

Out[ ]=

$\left(S\,(-1+T)\,g_{1,j^+,\beta}\,g_{2,i^+,\gamma} + (-1+S)\,T\,g_{1,i^+,\beta}\,g_{2,j^+,\gamma} + (S+T-2\,S\,T)\,g_{1,j^+,\beta}\,g_{2,j^+,\gamma}\right)g_{3,\alpha,i} +$
$\left(\frac{\left(-1+\frac{1}{T}\right)g_{1,j^{++},\beta}\,g_{2,i^{++},\gamma}}{S} + \frac{\left(-1+\frac{1}{S}\right)g_{1,i^{++},\beta}\,g_{2,j^{++},\gamma}}{T} + \left(\frac{1}{S}+\frac{1}{T}-\frac{2}{S\,T}\right)g_{1,j^{++},\beta}\,g_{2,j^{++},\gamma}\right)g_{3,\alpha,i^+}$

Out[ ]=

0

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

$In[\circ]:=$ `Clear[i, j, k];`
`Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}`
$Z = \text{Module}\big[\{s, i, j\}, \text{Sum}\big[\{s, i, j\} = c;$
$\quad g_{3,\alpha,i} \big(S^s (T^s - 1) g_{1,j^+,\beta} g_{2,i^+,\gamma} + (S^s - 1) T^s g_{1,i^+,\beta} g_{2,j^+,\gamma} + (S^s + T^s - 2 TS^s) g_{1,j^+,\beta} g_{2,j^+,\gamma}\big),$
$\quad \{c, Cs\}\big]\big]$
`lhs = Simplify`$\big[Z ~//.~ \text{gRules}_{1,i,j} \bigcup \text{gRules}_{1,i^+,k} \bigcup \text{gRules}_{1,j^+,k^+} ~/.~ \_\text{If} \to 0\big]$

$Out[\circ]=$
$\{\{1, i, j\}, \{1, i^+, k\}, \{1, j^+, k^+\}\}$

$Out[\circ]=$
$(S\,(-1+T)\,g_{1,j^+,\beta}\,g_{2,i^+,\gamma} + (-1+S)\,T\,g_{1,i^+,\beta}\,g_{2,j^+,\gamma} + (S+T-2\,S\,T)\,g_{1,j^+,\beta}\,g_{2,j^+,\gamma})\,g_{3,\alpha,i} +$
$(S\,(-1+T)\,g_{1,k^+,\beta}\,g_{2,i^{++},\gamma} + (-1+S)\,T\,g_{1,i^{++},\beta}\,g_{2,k^+,\gamma} + (S+T-2\,S\,T)\,g_{1,k^+,\beta}\,g_{2,k^+,\gamma})\,g_{3,\alpha,i^+} +$
$(S\,(-1+T)\,g_{1,k^{++},\beta}\,g_{2,j^{++},\gamma} + (-1+S)\,T\,g_{1,j^{++},\beta}\,g_{2,k^{++},\gamma} + (S+T-2\,S\,T)\,g_{1,k^{++},\beta}\,g_{2,k^{++},\gamma})\,g_{3,\alpha,j^+}$

$Out[\circ]=$
$\dfrac{1}{S^2\,T^2}\,\big(\,(\,(S\,(-1+T)\,(T\,g_{1,j^{++},\beta} - (-1+T)\,g_{1,k^{++},\beta})\,(S\,g_{2,i^{++},\gamma} - (-1+S)\,g_{2,k^{++},\gamma}) +$
$\qquad (-1+S)\,T\,(T\,g_{1,i^{++},\beta} - (-1+T)\,g_{1,k^{++},\beta})\,(S\,g_{2,j^{++},\gamma} - (-1+S)\,g_{2,k^{++},\gamma}) +$
$\qquad (S+T-2\,S\,T)\,(T\,g_{1,j^{++},\beta} - (-1+T)\,g_{1,k^{++},\beta})\,(S\,g_{2,j^{++},\gamma} - (-1+S)\,g_{2,k^{++},\gamma}))\,g_{3,\alpha,i^{++}} +$
$\quad S\,T\,(\,(-1+S)\,T\,g_{1,i^{++},\beta}\,g_{2,k^{++},\gamma} + g_{1,k^{++},\beta}\,(S\,(-1+T)\,g_{2,i^{++},\gamma} + (S+T-2\,S\,T)\,g_{2,k^{++},\gamma}))\,g_{3,\alpha,i^{++}} +$
$\quad S\,T\,(\,(-1+S)\,T\,g_{1,j^{++},\beta}\,g_{2,k^{++},\gamma} + g_{1,k^{++},\beta}\,(S\,(-1+T)\,g_{2,j^{++},\gamma} + (S+T-2\,S\,T)\,g_{2,k^{++},\gamma}))\,g_{3,\alpha,j^{++}})$

$In[\circ]:=$ `Clear[i, j, k];`
`Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}`
$Z = \text{Module}\big[\{s, i, j\}, \text{Sum}\big[\{s, i, j\} = c;$
$\quad g_{3,\alpha,i} \big(S^s (T^s - 1) g_{1,j^+,\beta} g_{2,i^+,\gamma} + (S^s - 1) T^s g_{1,i^+,\beta} g_{2,j^+,\gamma} + (S^s + T^s - 2 TS^s) g_{1,j^+,\beta} g_{2,j^+,\gamma}\big),$
$\quad \{c, Cs\}\big]\big]$
`rhs = Simplify`$\big[Z ~//.~ \text{gRules}_{1,j,k} \bigcup \text{gRules}_{1,i,k^+} \bigcup \text{gRules}_{1,i^+,j^+} ~/.~ \_\text{If} \to 0\big]$

$Out[\circ]=$
$\{\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}\}$

$Out[\circ]=$
$(\,(S+T-2\,S\,T)\,g_{1,k^{++},\beta}\,g_{2,k^{++},\gamma} + (-1+S)\,T\,g_{1,i^+,\beta}\,g_{2,k^{++},\gamma} + S\,(-1+T)\,g_{1,k^{++},\beta}\,g_{2,i^+,\gamma})\,g_{3,\alpha,i} +$
$(S\,(-1+T)\,g_{1,k^+,\beta}\,g_{2,j^+,\gamma} + (-1+S)\,T\,g_{1,j^+,\beta}\,g_{2,k^+,\gamma} + (S+T-2\,S\,T)\,g_{1,k^+,\beta}\,g_{2,k^+,\gamma})\,g_{3,\alpha,j} +$
$(S\,(-1+T)\,g_{1,j^{++},\beta}\,g_{2,i^{++},\gamma} + (-1+S)\,T\,g_{1,i^{++},\beta}\,g_{2,j^{++},\gamma} + (S+T-2\,S\,T)\,g_{1,j^{++},\beta}\,g_{2,j^{++},\gamma})\,g_{3,\alpha,i^+}$

$Out[\circ]=$
$\dfrac{1}{S^2\,T^2}\,(S\,T\,(\,(-1+S)\,T\,g_{1,i^{++},\beta}\,g_{2,j^{++},\gamma} + g_{1,j^{++},\beta}\,(S\,(-1+T)\,g_{2,i^{++},\gamma} + (S+T-2\,S\,T)\,g_{2,j^{++},\gamma}))\,g_{3,\alpha,i^{++}} +$
$\quad (S\,(-1+T)\,g_{1,k^{++},\beta}\,(S\,g_{2,i^{++},\gamma} - (-1+S)\,g_{2,j^{++},\gamma}) +$
$\qquad (-1+S)\,T\,(T\,g_{1,i^{++},\beta} - (-1+T)\,g_{1,j^{++},\beta})\,g_{2,k^{++},\gamma} + (S+T-2\,S\,T)\,g_{1,k^{++},\beta}\,g_{2,k^{++},\gamma})\,g_{3,\alpha,i^{++}} +$
$\quad (\,(-1+S)\,T\,g_{1,j^{++},\beta}\,g_{2,k^{++},\gamma} + g_{1,k^{++},\beta}\,(S\,(-1+T)\,g_{2,j^{++},\gamma} + (S+T-2\,S\,T)\,g_{2,k^{++},\gamma}))$
$\qquad (\,(-1+S\,T)\,g_{3,\alpha,i^{++}} + S\,T\,g_{3,\alpha,j^{++}}))$

In[ ]:= **Simplify[lhs == rhs]**

Out[ ]=

True

Setting up R_1

In[ ]:= **gs = Flatten@Table[{$g_{p,i,i}$, $g_{p,i,j}$, $g_{p,j,i}$, $g_{p,j,j}$}, {p, 3}]**
bas = {1} ⋃ gs ⋃ Flatten@Table[gs[[p]] gs[[q]], {p, Length[gs]}, {q, p, Length[gs]}] ⋃
    {$y_{i,i,i}$, $y_{i,i,j}$, $y_{i,j,i}$, $y_{i,j,j}$, $y_{j,i,i}$, $y_{j,i,j}$, $y_{j,j,i}$, $y_{j,j,j}$}
len = Length[bas];
vars = Table[c_p = 0; c_p =.; c_p, {p, 2 len}]
R_1[1, $i\_$, $j\_$] := Evaluate[Sum[c_p bas[[p]], {p, len}]];
R_1[-1, $i\_$, $j\_$] := Evaluate[Sum[c_{len+p} bas[[p]], {p, Length[bas]}]];

Out[ ]=

$\{g_{1,i,i}, g_{1,i,j}, g_{1,j,i}, g_{1,j,j}, g_{2,i,i}, g_{2,i,j}, g_{2,j,i}, g_{2,j,j}, g_{3,i,i}, g_{3,i,j}, g_{3,j,i}, g_{3,j,j}\}$

Out[ ]=

$\{1, g_{1,i,i}, g_{1,i,i}^2, g_{1,i,j}, g_{1,i,i} g_{1,i,j}, g_{1,i,j}^2, g_{1,j,i}, g_{1,i,i} g_{1,j,i}, g_{1,i,j} g_{1,j,i}, g_{1,j,i}^2, g_{1,j,j},$
$g_{1,i,i} g_{1,j,j}, g_{1,i,j} g_{1,j,j}, g_{1,j,i} g_{1,j,j}, g_{1,j,j}^2, g_{2,i,i}, g_{1,i,i} g_{2,i,i}, g_{1,i,j} g_{2,i,i}, g_{1,j,i} g_{2,i,i},$
$g_{1,j,j} g_{2,i,i}, g_{2,i,i}^2, g_{2,i,j}, g_{1,i,i} g_{2,i,j}, g_{1,i,j} g_{2,i,j}, g_{1,j,i} g_{2,i,j}, g_{1,j,j} g_{2,i,j}, g_{2,i,i} g_{2,i,j},$
$g_{2,i,j}^2, g_{2,j,i}, g_{1,i,i} g_{2,j,i}, g_{1,i,j} g_{2,j,i}, g_{1,j,i} g_{2,j,i}, g_{1,j,j} g_{2,j,i}, g_{2,i,i} g_{2,j,i}, g_{2,i,j} g_{2,j,i},$
$g_{2,j,i}^2, g_{2,j,j}, g_{1,i,i} g_{2,j,j}, g_{1,i,j} g_{2,j,j}, g_{1,j,i} g_{2,j,j}, g_{1,j,j} g_{2,j,j}, g_{2,i,i} g_{2,j,j}, g_{2,i,j} g_{2,j,j},$
$g_{2,j,i} g_{2,j,j}, g_{2,j,j}^2, g_{3,i,i}, g_{1,i,i} g_{3,i,i}, g_{1,i,j} g_{3,i,i}, g_{1,j,i} g_{3,i,i}, g_{1,j,j} g_{3,i,i}, g_{2,i,i} g_{3,i,i},$
$g_{2,i,j} g_{3,i,i}, g_{2,j,i} g_{3,i,i}, g_{2,j,j} g_{3,i,i}, g_{3,i,i}^2, g_{3,i,j}, g_{1,i,i} g_{3,i,j}, g_{1,i,j} g_{3,i,j}, g_{1,j,i} g_{3,i,j},$
$g_{1,j,j} g_{3,i,j}, g_{2,i,i} g_{3,i,j}, g_{2,i,j} g_{3,i,j}, g_{2,j,i} g_{3,i,j}, g_{2,j,j} g_{3,i,j}, g_{3,i,i} g_{3,i,j}, g_{3,i,j}^2,$
$g_{3,j,i}, g_{1,i,i} g_{3,j,i}, g_{1,i,j} g_{3,j,i}, g_{1,j,i} g_{3,j,i}, g_{1,j,j} g_{3,j,i}, g_{2,i,i} g_{3,j,i}, g_{2,i,j} g_{3,j,i},$
$g_{2,j,i} g_{3,j,i}, g_{2,j,j} g_{3,j,i}, g_{3,i,i} g_{3,j,i}, g_{3,i,j} g_{3,j,i}, g_{3,j,i}^2, g_{3,j,j}, g_{1,i,i} g_{3,j,j}, g_{1,i,j} g_{3,j,j},$
$g_{1,j,i} g_{3,j,j}, g_{1,j,j} g_{3,j,j}, g_{2,i,i} g_{3,j,j}, g_{2,i,j} g_{3,j,j}, g_{2,j,i} g_{3,j,j}, g_{2,j,j} g_{3,j,j}, g_{3,i,i} g_{3,j,j},$
$g_{3,i,j} g_{3,j,j}, g_{3,j,i} g_{3,j,j}, g_{3,j,j}^2, y_{i,i,i}, y_{i,i,j}, y_{i,j,i}, y_{i,j,j}, y_{j,i,i}, y_{j,i,j}, y_{j,j,i}, y_{j,j,j}\}$

Out[ ]=

$\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{21}, c_{22},$
$c_{23}, c_{24}, c_{25}, c_{26}, c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}, c_{37}, c_{38}, c_{39}, c_{40}, c_{41},$
$c_{42}, c_{43}, c_{44}, c_{45}, c_{46}, c_{47}, c_{48}, c_{49}, c_{50}, c_{51}, c_{52}, c_{53}, c_{54}, c_{55}, c_{56}, c_{57}, c_{58}, c_{59}, c_{60},$
$c_{61}, c_{62}, c_{63}, c_{64}, c_{65}, c_{66}, c_{67}, c_{68}, c_{69}, c_{70}, c_{71}, c_{72}, c_{73}, c_{74}, c_{75}, c_{76}, c_{77}, c_{78}, c_{79},$
$c_{80}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}, c_{86}, c_{87}, c_{88}, c_{89}, c_{90}, c_{91}, c_{92}, c_{93}, c_{94}, c_{95}, c_{96}, c_{97}, c_{98},$
$c_{99}, c_{100}, c_{101}, c_{102}, c_{103}, c_{104}, c_{105}, c_{106}, c_{107}, c_{108}, c_{109}, c_{110}, c_{111}, c_{112}, c_{113}, c_{114}, c_{115},$
$c_{116}, c_{117}, c_{118}, c_{119}, c_{120}, c_{121}, c_{122}, c_{123}, c_{124}, c_{125}, c_{126}, c_{127}, c_{128}, c_{129}, c_{130}, c_{131}, c_{132},$
$c_{133}, c_{134}, c_{135}, c_{136}, c_{137}, c_{138}, c_{139}, c_{140}, c_{141}, c_{142}, c_{143}, c_{144}, c_{145}, c_{146}, c_{147}, c_{148}, c_{149},$
$c_{150}, c_{151}, c_{152}, c_{153}, c_{154}, c_{155}, c_{156}, c_{157}, c_{158}, c_{159}, c_{160}, c_{161}, c_{162}, c_{163}, c_{164}, c_{165}, c_{166},$
$c_{167}, c_{168}, c_{169}, c_{170}, c_{171}, c_{172}, c_{173}, c_{174}, c_{175}, c_{176}, c_{177}, c_{178}, c_{179}, c_{180}, c_{181}, c_{182},$
$c_{183}, c_{184}, c_{185}, c_{186}, c_{187}, c_{188}, c_{189}, c_{190}, c_{191}, c_{192}, c_{193}, c_{194}, c_{195}, c_{196}, c_{197}, c_{198}\}$

$In[\circ]:=$ **$R_1[-1, i^+, 7]$**

$Out[\circ]=$

$c_{100} + c_{110}\, g_{1,7,7} + c_{114}\, g_{1,7,7}^2 + c_{106}\, g_{1,7,i^+} + c_{113}\, g_{1,7,7}\, g_{1,7,i^+} + c_{109}\, g_{1,7,i^+}^2 + c_{103}\, g_{1,i^+,7} + c_{112}\, g_{1,7,7}\, g_{1,i^+,7} +$

$c_{108}\, g_{1,7,i^+}\, g_{1,i^+,7} + c_{105}\, g_{1,i^+,7}^2 + c_{101}\, g_{1,i^+,i^+} + c_{111}\, g_{1,7,7}\, g_{1,i^+,i^+} + c_{107}\, g_{1,7,i^+}\, g_{1,i^+,i^+} +$

$c_{104}\, g_{1,i^+,7}\, g_{1,i^+,i^+} + c_{102}\, g_{1,i^+,i^+}^2 + c_{136}\, g_{2,7,7} + c_{140}\, g_{1,7,7}\, g_{2,7,7} + c_{139}\, g_{1,7,i^+}\, g_{2,7,7} + c_{138}\, g_{1,i^+,7}\, g_{2,7,7} +$

$c_{137}\, g_{1,i^+,i^+}\, g_{2,7,7} + c_{144}\, g_{2,7,7}^2 + c_{128}\, g_{2,7,i^+} + c_{132}\, g_{1,7,7}\, g_{2,7,i^+} + c_{131}\, g_{1,7,i^+}\, g_{2,7,i^+} + c_{130}\, g_{1,i^+,7}\, g_{2,7,i^+} +$

$c_{129}\, g_{1,i^+,i^+}\, g_{2,7,i^+} + c_{143}\, g_{2,7,7}\, g_{2,7,i^+} + c_{135}\, g_{2,7,i^+}^2 + c_{121}\, g_{2,i^+,7} + c_{125}\, g_{1,7,7}\, g_{2,i^+,7} +$

$c_{124}\, g_{1,7,i^+}\, g_{2,i^+,7} + c_{123}\, g_{1,i^+,7}\, g_{2,i^+,7} + c_{122}\, g_{1,i^+,i^+}\, g_{2,i^+,7} + c_{142}\, g_{2,7,7}\, g_{2,i^+,7} + c_{134}\, g_{2,7,i^+}\, g_{2,i^+,7} +$

$c_{127}\, g_{2,i^+,7}^2 + c_{115}\, g_{2,i^+,i^+} + c_{119}\, g_{1,7,7}\, g_{2,i^+,i^+} + c_{118}\, g_{1,7,i^+}\, g_{2,i^+,i^+} + c_{117}\, g_{1,i^+,7}\, g_{2,i^+,i^+} +$

$c_{116}\, g_{1,i^+,i^+}\, g_{2,i^+,i^+} + c_{141}\, g_{2,7,7}\, g_{2,i^+,i^+} + c_{133}\, g_{2,7,i^+}\, g_{2,i^+,i^+} + c_{126}\, g_{2,i^+,7}\, g_{2,i^+,i^+} + c_{120}\, g_{2,i^+,i^+}^2 +$

$c_{178}\, g_{3,7,7} + c_{182}\, g_{1,7,7}\, g_{3,7,7} + c_{181}\, g_{1,7,i^+}\, g_{3,7,7} + c_{180}\, g_{1,i^+,7}\, g_{3,7,7} + c_{179}\, g_{1,i^+,i^+}\, g_{3,7,7} +$

$c_{186}\, g_{2,7,7}\, g_{3,7,7} + c_{185}\, g_{2,7,i^+}\, g_{3,7,7} + c_{184}\, g_{2,i^+,7}\, g_{3,7,7} + c_{183}\, g_{2,i^+,i^+}\, g_{3,7,7} + c_{190}\, g_{3,7,7}^2 + c_{166}\, g_{3,7,i^+} +$

$c_{170}\, g_{1,7,7}\, g_{3,7,i^+} + c_{169}\, g_{1,7,i^+}\, g_{3,7,i^+} + c_{168}\, g_{1,i^+,7}\, g_{3,7,i^+} + c_{167}\, g_{1,i^+,i^+}\, g_{3,7,i^+} + c_{174}\, g_{2,7,7}\, g_{3,7,i^+} +$

$c_{173}\, g_{2,7,i^+}\, g_{3,7,i^+} + c_{172}\, g_{2,i^+,7}\, g_{3,7,i^+} + c_{171}\, g_{2,i^+,i^+}\, g_{3,7,i^+} + c_{189}\, g_{3,7,7}\, g_{3,7,i^+} + c_{177}\, g_{3,7,i^+}^2 +$

$c_{155}\, g_{3,i^+,7} + c_{159}\, g_{1,7,7}\, g_{3,i^+,7} + c_{158}\, g_{1,7,i^+}\, g_{3,i^+,7} + c_{157}\, g_{1,i^+,7}\, g_{3,i^+,7} + c_{156}\, g_{1,i^+,i^+}\, g_{3,i^+,7} +$

$c_{163}\, g_{2,7,7}\, g_{3,i^+,7} + c_{162}\, g_{2,7,i^+}\, g_{3,i^+,7} + c_{161}\, g_{2,i^+,7}\, g_{3,i^+,7} + c_{160}\, g_{2,i^+,i^+}\, g_{3,i^+,7} + c_{188}\, g_{3,7,7}\, g_{3,i^+,7} +$

$c_{176}\, g_{3,7,i^+}\, g_{3,i^+,7} + c_{165}\, g_{3,i^+,7}^2 + c_{145}\, g_{3,i^+,i^+} + c_{149}\, g_{1,7,7}\, g_{3,i^+,i^+} + c_{148}\, g_{1,7,i^+}\, g_{3,i^+,i^+} +$

$c_{147}\, g_{1,i^+,7}\, g_{3,i^+,i^+} + c_{146}\, g_{1,i^+,i^+}\, g_{3,i^+,i^+} + c_{153}\, g_{2,7,7}\, g_{3,i^+,i^+} + c_{152}\, g_{2,7,i^+}\, g_{3,i^+,i^+} + c_{151}\, g_{2,i^+,7}\, g_{3,i^+,i^+} +$

$c_{150}\, g_{2,i^+,i^+}\, g_{3,i^+,i^+} + c_{187}\, g_{3,7,7}\, g_{3,i^+,i^+} + c_{175}\, g_{3,7,i^+}\, g_{3,i^+,i^+} + c_{164}\, g_{3,i^+,7}\, g_{3,i^+,i^+} + c_{154}\, g_{3,i^+,i^+}^2 +$

$c_{198}\, y_{7,7,7} + c_{197}\, y_{7,7,i^+} + c_{196}\, y_{7,i^+,7} + c_{195}\, y_{7,i^+,i^+} + c_{194}\, y_{i^+,7,7} + c_{193}\, y_{i^+,7,i^+} + c_{192}\, y_{i^+,i^+,7} + c_{191}\, y_{i^+,i^+,i^+}$

Solving R2b

$In[\circ]:=$ **Clear[i, j];**
eqn = Expand$\big[$R$_1$[1, i, j] + R$_1$[-1, i$^+$, j$^+$] //.
gRules$_{1,i,j}$ ⋃ gRules$_{-1,i^+,j^+}$ ⋃ yRules$_{1,i,j}$ ⋃ yRules$_{-1,i^+,j^+}$ /. _If → 0$\big]$

$Out[\circ]=$

$c_1 + c_{100} + c_2\, g_{1,i^+,i^+} - c_{93}\, g_{1,i^+,i^+} + 2\, S\, c_{93}\, g_{1,i^+,i^+} + \dfrac{S\, c_{93}\, g_{1,i^+,i^+}}{T^2} + \dfrac{c_{93}\, g_{1,i^+,i^+}}{T} - \dfrac{3\, S\, c_{93}\, g_{1,i^+,i^+}}{T} +$

$\boxed{\cdots 2412 \cdots} + c_{194}\, y_{j^+,j^+,i^+} - S\, c_{194}\, y_{j^+,j^+,i^+} + \dfrac{c_{194}\, y_{j^+,j^+,i^+}}{T} - \dfrac{c_{194}\, y_{j^+,j^+,i^+}}{S\, T} + S\, c_{197}\, y_{j^+,j^+,i^+} +$

$c_{198}\, y_{j^+,j^+,i^+} - S\, c_{198}\, y_{j^+,j^+,i^+} + c_{99}\, y_{j^+,j^+,j^+} + c_{194}\, y_{j^+,j^+,j^+} - \dfrac{c_{194}\, y_{j^+,j^+,j^+}}{S\, T} + c_{198}\, y_{j^+,j^+,j^+}$

Full expression not available (original memory size: 1.2 MB) ⚙

$\text{\textit{In[○]:=}}$ **covars = Cases[eqn, g$_{\_\_\_}$, ∞] ⋃ Cases[eqn, c$_{\_}$. * (g1 : g$_{\_,\_,\_}$) (g2 : g$_{\_,\_,\_}$) ⧴ g1 g2, ∞] ⋃**
    Cases[eqn, g$^2_{\_\_}$, ∞] ⋃ Cases[eqn, y$_{\_\_}$, ∞]

$\text{\textit{Out[○]=}}$

$\Big\{ g_{1,i^{++},i^{++}}, g^2_{1,i^{++},i^{++}}, g_{1,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,i^{++},j^{++}}, g^2_{1,i^{++},j^{++}}, g_{1,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},i^{++}},$

$g_{1,i^{++},j^{++}} g_{1,j^{++},i^{++}}, g^2_{1,j^{++},i^{++}}, g_{1,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{1,j^{++},j^{++}},$

$g_{1,j^{++},i^{++}} g_{1,j^{++},j^{++}}, g^2_{1,j^{++},j^{++}}, g_{2,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},i^{++}},$

$g_{1,j^{++},j^{++}} g_{2,i^{++},i^{++}}, g^2_{2,i^{++},i^{++}}, g_{2,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},j^{++}},$

$g_{1,j^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g^2_{2,i^{++},j^{++}}, g_{2,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},i^{++}},$

$g_{1,j^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{2,j^{++},i^{++}}, g^2_{2,j^{++},i^{++}}, g_{2,j^{++},j^{++}},$

$g_{1,i^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},j^{++}},$

$g_{2,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g^2_{2,j^{++},j^{++}}, g_{3,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},i^{++}},$

$g_{1,j^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},i^{++}},$

$g_{2,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g^2_{3,i^{++},i^{++}}, g_{3,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{3,i^{++},j^{++}},$

$g_{1,j^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},j^{++}} g_{3,i^{++},j^{++}},$

$g_{3,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g^2_{3,i^{++},j^{++}}, g_{3,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{3,j^{++},i^{++}},$

$g_{1,j^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},j^{++}} g_{3,j^{++},i^{++}},$

$g_{3,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g^2_{3,j^{++},i^{++}}, g_{3,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},j^{++}},$

$g_{1,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},j^{++}},$

$g_{2,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g^2_{3,j^{++},j^{++}},$

$y_{i^{++},i^{++},i^{++}}, y_{i^{++},i^{++},j^{++}}, y_{i^{++},j^{++},i^{++}}, y_{i^{++},j^{++},j^{++}}, y_{j^{++},i^{++},i^{++}}, y_{j^{++},i^{++},j^{++}}, y_{j^{++},j^{++},i^{++}}, y_{j^{++},j^{++},j^{++}} \Big\}$

$\text{\textit{In[○]:=}}$ **Short[eqns =**
    {eqn == 0} ⋃ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y)$_{\_,\_,\_}$ → 0}, 5]

$\text{\textit{Out[○]//Short=}}$

$\Big\{ c_6 + \dfrac{c_{105}}{T^2} == 0, \ c_3 + c_{102} - c_{104} + \dfrac{c_{104}}{T} + c_{105} + \dfrac{c_{105}}{T^2} - \dfrac{2\,c_{105}}{T} == 0, \ \ll 95 \gg,$

$c_{67} + 2\,c_{99} - S\,c_{99} - T\,c_{99} - c_{145} + S\,T\,c_{145} + 2\,c_{155} - \dfrac{c_{155}}{S\,T} - S\,T\,c_{155} + S\,T\,c_{166} + c_{178} - S\,T\,c_{178} + c_{192} - \dfrac{c_{192}}{S} +$

$T\,c_{192} - S\,T\,c_{192} + c_{193} + S\,c_{193} - \dfrac{c_{193}}{T} - S\,T\,c_{193} + \dfrac{2\,c_{194}}{S} - 2\,S\,c_{194} + \dfrac{2\,c_{194}}{T} - \dfrac{2\,c_{194}}{S\,T} - 2\,T\,c_{194} +$

$2\,S\,T\,c_{194} + T\,c_{196} - S\,T\,c_{196} + S\,c_{197} - S\,T\,c_{197} + 2\,c_{198} - 2\,S\,c_{198} - 2\,T\,c_{198} + 2\,S\,T\,c_{198} == 0, \ c_1 + c_{100} == 0 \Big\}$

$In[\bullet]:=$ **Short[{sol} = Solve[eqns, vars], 5]**

⋯ **Solve**: Equations may not give solutions for all "solve" variables. ⓘ

$Out[\bullet]//Short=$

$$\left\{\left\{c_2 \to -c_{101} - \left(-1 + \frac{1}{T}\right) c_{103} - \frac{\left(-S - T + 3\,S\,T + T^2 - 2\,S\,T^2\right) c_{192}}{S\,T^2}\right.,\right.$$

$$c_3 \to -c_{102} - \left(-1 + \frac{1}{T}\right) c_{104} - \frac{\left(1 - 2\,T + T^2\right) c_{105}}{T^2},\ c_4 \to -\frac{c_{103}}{T} - \frac{\left(-S - T + 2\,S\,T\right) c_{192}}{S\,T^2},$$

$$c_5 \to -\frac{c_{104}}{T} + \frac{2\,(-1 + T)\,c_{105}}{T^2},\ c_6 \to -\frac{c_{105}}{T^2},\ \ll90\gg,$$

$$c_{97} \to -\frac{(-1 + S\,T)\,c_{192}}{S} - \frac{\left(-1 + T + S\,T - S\,T^2\right) c_{194}}{S\,T} - T\,c_{196} - (1 - T)\,c_{198},$$

$$c_{98} \to -\frac{(-1 + S\,T)\,c_{193}}{T} - \frac{\left(-1 + S + S\,T - S^2\,T\right) c_{194}}{S\,T} - S\,c_{197} - (1 - S)\,c_{198},$$

$$\left.\left.c_{99} \to -\frac{(-1 + S\,T)\,c_{194}}{S\,T} - c_{198},\ c_{100} \to -c_1\right\}\right\}$$

$In[\bullet]:=$ **Short[sol /. ($v\_ \to val\_$) :→ (v = CF[val])]**

$Out[\bullet]//Short=$

$$\left\{-\frac{S\,T^2\,c_{101} + \ll11\gg}{S\,T^2},\ -\frac{T^2\,c_{102} + \ll6\gg + T^2\,c_{105}}{T^2},\right.$$

$$\left.-\frac{\ll1\gg}{S\,\ll1\gg},\ \ll93\gg,\ -\frac{\ll1\gg}{\ll1\gg},\ -\frac{-c_{194} + \ll1\gg + S\,T\,c_{\ll3\gg}}{S\,T},\ -c_1\right\}$$

$In[\bullet]:=$ **Short[R_1[1, i, j], 5]**

$Out[\bullet]//Short=$

$\ll1\gg$

Solving R3

$In[\bullet]:=$ **Clear[i, j, k];**
lhs = Expand[R_1[1, i, j] + R_1[1, i$^+$, k] + R_1[1, j$^+$, k$^+$] //. gRules$_{1,i,j}$ ⋃
   gRules$_{1,i^+,k}$ ⋃ gRules$_{1,j^+,k^+}$ ⋃ yRules$_{1,i,j}$ ⋃ yRules$_{1,i^+,k}$ ⋃ yRules$_{1,j^+,k^+}$ /. _If → 0];
rhs = Expand[R_1[1, j, k] + R_1[1, i, k$^+$] + R_1[1, i$^+$, j$^+$] //. gRules$_{1,j,k}$ ⋃
   gRules$_{1,i,k^+}$ ⋃ gRules$_{1,i^+,j^+}$ ⋃ yRules$_{1,j,k}$ ⋃ yRules$_{1,i,k^+}$ ⋃ yRules$_{1,i^+,j^+}$ /. _If → 0];
eqn = lhs − rhs

$Out[\bullet]=$

$$-2\,c_{194} + \frac{c_{194}}{S} + \frac{c_{194}}{T} - c_{103}\,g_{1,i^{++},j^{++}} + \frac{c_{103}\,g_{1,i^{++},j^{++}}}{T} - \frac{c_{192}\,g_{1,i^{++},j^{++}}}{T^2} + \frac{2\,c_{192}\,g_{1,i^{++},j^{++}}}{T} -$$
$$\frac{c_{192}\,g_{1,i^{++},j^{++}}}{S\,T} - c_{104}\,g_{1,i^{++},i^{++}}\,g_{1,i^{++},j^{++}} + \boxed{\cdots 3208 \cdots} + \frac{c_{198}\,y_{k^{++},j^{++},j^{++}}}{S} + \frac{c_{198}\,y_{k^{++},j^{++},j^{++}}}{T} - \frac{2\,c_{198}\,y_{k^{++},j^{++},j^{++}}}{S\,T} -$$
$$c_{198}\,y_{k^{++},j^{++},k^{++}} + \frac{c_{198}\,y_{k^{++},j^{++},k^{++}}}{T} - \frac{c_{198}\,y_{k^{++},k^{++},i^{++}}}{S^2} + \frac{c_{198}\,y_{k^{++},k^{++},i^{++}}}{S} - c_{198}\,y_{k^{++},k^{++},j^{++}} + \frac{c_{198}\,y_{k^{++},k^{++},j^{++}}}{S}$$

Full expression not available (original memory size: 1.6 MB)   ⚙

$In[\circ]:=$ **vars = Union@Cases[eqn, c_ , ∞]**

$Out[\circ]=$

$\{c_{101}, c_{102}, c_{103}, c_{104}, c_{105}, c_{106}, c_{107}, c_{108}, c_{109}, c_{110}, c_{111}, c_{112}, c_{113}, c_{114}, c_{115}, c_{116}, c_{117},$
$c_{118}, c_{119}, c_{120}, c_{121}, c_{122}, c_{123}, c_{124}, c_{125}, c_{126}, c_{127}, c_{128}, c_{129}, c_{130}, c_{131}, c_{132}, c_{133}, c_{134},$
$c_{135}, c_{136}, c_{137}, c_{138}, c_{139}, c_{140}, c_{141}, c_{142}, c_{143}, c_{144}, c_{145}, c_{146}, c_{147}, c_{148}, c_{149}, c_{150},$
$c_{151}, c_{152}, c_{153}, c_{154}, c_{155}, c_{156}, c_{157}, c_{158}, c_{159}, c_{160}, c_{161}, c_{162}, c_{163}, c_{164}, c_{165}, c_{166},$
$c_{167}, c_{168}, c_{169}, c_{170}, c_{171}, c_{172}, c_{173}, c_{174}, c_{175}, c_{176}, c_{177}, c_{178}, c_{179}, c_{180}, c_{181}, c_{182},$
$c_{183}, c_{184}, c_{185}, c_{186}, c_{187}, c_{188}, c_{189}, c_{190}, c_{191}, c_{192}, c_{193}, c_{194}, c_{195}, c_{196}, c_{197}, c_{198}\}$

$In[\circ]:=$ **Short[**
    covars = Cases[eqn, c_ . * (g1 : g_,_,_) (g2 : g_,_,_) :> g1 g2, ∞] ⋃
      Cases[eqn, g_ $_{\_\_}^2$, ∞] ⋃ Cases[eqn, y__ , ∞],
    5]

$Out[\circ]//Short=$

$\{g_{1,i^{+},i^{+}} g_{1,i^{+},j^{+}}, g_{1,i^{+},j^{+}}^2, g_{1,i^{+},i^{+}} g_{1,i^{+},k^{+}}, g_{1,i^{+},j^{+}} g_{1,i^{+},k^{+}},$
$g_{1,i^{+},k^{+}}^2, g_{1,i^{+},i^{+}} g_{1,j^{+},i^{+}}, g_{1,i^{+},k^{+}} g_{1,j^{+},i^{+}}, \ll298\gg, y_{k^{+},i^{+},j^{+}},$
$y_{k^{+},i^{+},k^{+}}, y_{k^{+},j^{+},i^{+}}, y_{k^{+},j^{+},j^{+}}, y_{k^{+},j^{+},k^{+}}, y_{k^{+},k^{+},i^{+}}, y_{k^{+},k^{+},j^{+}}\}$

$In[\circ]:=$ **Short[eqns =**
    {eqn == 0} ⋃ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. $\{(g \mid y)_{\_,\_,\_} \to 0\}$, 5]

$Out[\circ]//Short=$

$\left\{2 c_{102} - \dfrac{2 c_{102}}{T} == 0, -2 c_{102} + \dfrac{2 c_{102}}{T} == 0, c_{104} - \dfrac{c_{104}}{T} == 0,\right.$

$-c_{104} + \dfrac{c_{104}}{T} == 0, \ll293\gg, -c_{182} + \dfrac{c_{182}}{S\,T} + 2 c_{196} - 6 T c_{196} + \dfrac{2 T c_{196}}{S} + 4 T^2 c_{196} -$

$\dfrac{2 T^2 c_{196}}{S} - 5 c_{198} + \dfrac{c_{198}}{S} + \dfrac{c_{198}}{T} + \dfrac{c_{198}}{S\,T} + 5 T c_{198} - \dfrac{2 T c_{198}}{S} - 2 T^2 c_{198} + \dfrac{T^2 c_{198}}{S} == 0,$

$-c_{191} + \dfrac{c_{191}}{S} + \dfrac{c_{192}}{S^2} - \dfrac{c_{192}}{S} - 2 c_{196} + \dfrac{c_{196}}{S} + \dfrac{c_{196}}{T} - T c_{196} + 2 T^2 c_{196} - \dfrac{T^2 c_{196}}{S} - c_{198} +$

$\left. 3 T c_{198} - \dfrac{T c_{198}}{S} - 2 T^2 c_{198} + \dfrac{T^2 c_{198}}{S} == 0, -2 c_{194} + \dfrac{c_{194}}{S} + \dfrac{c_{194}}{T} == 0\right\}$

In[ ]:= **Short[{sol} = Solve[eqns, vars], 5]**

⬤ Solve: Equations may not give solutions for all "solve" variables. ⓘ

Out[ ]//Short=

$$\left\{\left\{c_{102} \to 0,\ c_{104} \to 0,\ c_{105} \to 0,\ c_{108} \to -\frac{T\,c_{107}}{1+T} - \frac{T\,c_{113}}{1+T},\right.\right.$$

$$c_{109} \to -\frac{(-1+T)\,c_{107}}{1+T} - \frac{(1-T)\,c_{113}}{T\,(1+T)},\ c_{111} \to -\frac{T\,c_{107}}{1+T} - \frac{T\,c_{113}}{1+T},\ c_{112} \to 0,\ c_{114} \to 0,\ c_{116} \to 0,$$

$$c_{117} \to 0,\ c_{119} \to \frac{T\,c_{118}}{-1+T} - \frac{S\,c_{129}}{-1+S},\ c_{120} \to 0,\ c_{122} \to 0,\ c_{123} \to 0,\ c_{124} \to 0,\ \ll 45 \gg,$$

$$c_{184} \to 0,\ c_{185} \to -\frac{(-1+S)\,T\,c_{171}}{-1+S\,T},\ c_{186} \to 0,\ c_{187} \to -\frac{S\,T\,c_{175}}{1+S\,T} - \frac{S\,T\,c_{189}}{1+S\,T},\ c_{188} \to 0,$$

$$\left.\left.c_{190} \to 0,\ c_{191} \to 0,\ c_{192} \to 0,\ c_{193} \to 0,\ c_{194} \to 0,\ c_{195} \to 0,\ c_{196} \to 0,\ c_{197} \to 0,\ c_{198} \to 0\right\}\right\}$$

In[ ]:= **Short[sol /. (v_ → val_) :→ (v = CF[val])]**

Out[ ]//Short=

$$\left\{0,\ 0,\ 0,\ -\frac{T\,(c_{107}+c_{113})}{1+T},\ \ll 66 \gg,\ 0,\ 0,\ 0,\ 0\right\}$$

In[ ]:= **Short[R₁[1, i, j], 5]**

Out[ ]//Short=

$$c_1 - \frac{\left(S\,T^2\,c_{101} + S\,T\,c_{103} - S\,T^2\,c_{103}\right)\,g_{1,i,i}}{S\,T^2} - \frac{c_{103}\,g_{1,i,j}}{T} -$$

$$\frac{\left(-S\,T^2\,c_{101} + \ll 10 \gg\right)\,g_{1,j,i}}{S\,T^2} - \frac{\ll 1 \gg}{T^2} + \ll 73 \gg + \frac{S\,(-c_{152} + \ll 4 \gg)\,g_{\ll 1 \gg}\,g_{3,j,j}}{(-1+S)\,(-1+S\,T)} -$$

$$\frac{\left(-\frac{(-1+S)\,S^3\,T^2\,c_{171}}{-1+S\,T} + \frac{S^3\,T\,(\ll 1 \gg)}{(-1+S)\,\ll 1 \gg} - \frac{S^4\,T\,(-c_{152} + \ll 4 \gg)}{(-1+S)\,(-1+S\,T)}\right)\,\ll 1 \gg\,g_{\ll 1 \gg}}{S^2\,T} +$$

$$\frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,i}\,g_{3,j,j}}{1+S\,T} - \frac{\left(S^3\,T^3\,c_{189} + \frac{2\,S^3\,T^3\,(c_{175}+c_{189})}{1+S\,T} - \frac{2\,S^4\,T^4\,(c_{175}+c_{189})}{1+S\,T}\right)\,g_{3,j,i}\,g_{3,j,j}}{S^2\,T^2}$$

$In[\bullet]:=$ **Simplify[R_1[1, i, j] /. {$c_{155} \to 0$, $c_{189} \to 0$, $c_{175} \to 0$, $c_{178} \to 0$, $c_{166} \to 0$, $c_{145} \to 0$}]**

$Out[\bullet]=$

$$c_1 - \frac{(T\,c_{101} - (-1+T)\,c_{103})\,g_{1,i,i}}{T} - \frac{c_{103}\,g_{1,i,j}}{T} -$$

$$\frac{\left((-1+T)\,T\,c_{101} - (-1+T)^2\,c_{103} + T\,(T\,c_{106} - (-1+T)\,c_{110})\right)\,g_{1,j,i}}{T} -$$

$$\frac{T\,((-1+3\,T)\,c_{107} + 2\,(-1+T)\,c_{113})\,g_{1,i,i}\,g_{1,j,i}}{1+T} + \frac{T\,(c_{107}+c_{113})\,g_{1,i,j}\,g_{1,j,i}}{1+T} -$$

$$\frac{(-1+T)\,T\,((-1+2\,T)\,c_{107} + (-2+T)\,c_{113})\,g_{1,j,i}^2}{1+T} - \frac{((-1+T)\,c_{103} + T\,c_{110})\,g_{1,j,j}}{T} +$$

$$\frac{T\,(c_{107}+c_{113})\,g_{1,i,i}\,g_{1,j,j}}{1+T} + \frac{T\,(2\,(-1+T)\,c_{107} + (-3+T)\,c_{113})\,g_{1,j,i}\,g_{1,j,j}}{1+T} -$$

$$\frac{(S\,c_{115} - (-1+S)\,c_{121})\,g_{2,i,i}}{S} - \frac{S\,(-1+T)\,c_{129}\,g_{1,j,i}\,g_{2,i,i}}{-1+S} + \left(-\frac{T\,c_{118}}{-1+T} + \frac{S\,c_{129}}{-1+S}\right)\,g_{1,j,j}\,g_{2,i,i} -$$

$$\frac{c_{121}\,g_{2,i,j}}{S} - \frac{\left((-1+S)\,S\,c_{115} - (-1+S)^2\,c_{121} + S\,(S\,c_{128} - (-1+S)\,c_{136})\right)\,g_{2,j,i}}{S} -$$

$$\frac{(-1+S)\,T\,c_{118}\,g_{1,i,i}\,g_{2,j,i}}{-1+T} - \left((-1+S)\,T\,c_{118} + S\,(-1+T)\,c_{129}\right)\,g_{1,j,i}\,g_{2,j,i} + S\,c_{129}\,g_{1,j,j}\,g_{2,j,i} -$$

$$\frac{S\,((-1+3\,S)\,c_{133} + 2\,(-1+S)\,c_{143})\,g_{2,i,i}\,g_{2,j,i}}{1+S} + \frac{S\,(c_{133}+c_{143})\,g_{2,i,j}\,g_{2,j,i}}{1+S} -$$

$$\frac{(-1+S)\,S\,((-1+2\,S)\,c_{133} + (-2+S)\,c_{143})\,g_{2,j,i}^2}{1+S} - \frac{((-1+S)\,c_{121} + S\,c_{136})\,g_{2,j,j}}{S} +$$

$$\frac{((-1+S)\,T\,c_{118} - S\,(-1+T)\,c_{129})\,g_{1,i,i}\,g_{2,j,j}}{(-1+S)\,(-1+T)} + T\,c_{118}\,g_{1,j,i}\,g_{2,j,j} + \frac{S\,(c_{133}+c_{143})\,g_{2,i,i}\,g_{2,j,j}}{1+S} +$$

$$\frac{S\,(2\,(-1+S)\,c_{133} + (-3+S)\,c_{143})\,g_{2,j,i}\,g_{2,j,j}}{1+S} - \frac{S\,(-1+T)\,T\,c_{167}\,g_{1,j,i}\,g_{3,i,i}}{-1+S\,T} +$$

$$T\,\left(-\frac{c_{148}}{-1+T} + \frac{S\,c_{167}}{-1+S\,T}\right)\,g_{1,j,j}\,g_{3,i,i} - \frac{(-1+S)\,S\,T\,c_{171}\,g_{2,j,i}\,g_{3,i,i}}{-1+S\,T} +$$

$$S\,\left(-\frac{c_{152}}{-1+S} + \frac{T\,c_{171}}{-1+S\,T}\right)\,g_{2,j,j}\,g_{3,i,i} - \frac{T\,(-1+S\,T)\,c_{148}\,g_{1,i,i}\,g_{3,j,i}}{-1+T} -$$

$$T\,((-1+S\,T)\,c_{148} + S\,(-1+T)\,c_{167})\,g_{1,j,i}\,g_{3,j,i} + S\,T\,c_{167}\,g_{1,j,j}\,g_{3,j,i} -$$

$$\frac{S\,(-1+S\,T)\,c_{152}\,g_{2,i,i}\,g_{3,j,i}}{-1+S} - S\,((-1+S\,T)\,c_{152} + (-1+S)\,T\,c_{171})\,g_{2,j,i}\,g_{3,j,i} +$$

$$S\,T\,c_{171}\,g_{2,j,j}\,g_{3,j,i} + \frac{T\,((-1+S\,T)\,c_{148} - S\,(-1+T)\,c_{167})\,g_{1,i,i}\,g_{3,j,j}}{(-1+T)\,(-1+S\,T)} +$$

$$T\,c_{148}\,g_{1,j,i}\,g_{3,j,j} + \frac{S\,((-1+S\,T)\,c_{152} - (-1+S)\,T\,c_{171})\,g_{2,i,i}\,g_{3,j,j}}{(-1+S)\,(-1+S\,T)} + S\,c_{152}\,g_{2,j,i}\,g_{3,j,j}$$

The most general no-mixed-terms solution:

$In[\bullet]:=$ **Simplify[{R_1[-1, i, j], R_1[1, i, j]} /. {$c_{118|129|148|152|167|171} \to 0$}]**

$Out[\bullet]=$

$$\left\{-c_1 + c_{101}\,g_{1,i,i} + c_{103}\,g_{1,i,j} + c_{106}\,g_{1,j,i} + c_{107}\,g_{1,i,i}\,g_{1,j,i} - \frac{T\,(c_{107}+c_{113})\,g_{1,i,j}\,g_{1,j,i}}{1+T} - \right.$$

$$\frac{(-1+T)\,(T\,c_{107}-c_{113})\,g_{1,j,i}^2}{T\,(1+T)} + c_{110}\,g_{1,j,j} - \frac{T\,(c_{107}+c_{113})\,g_{1,i,i}\,g_{1,j,j}}{1+T} + c_{113}\,g_{1,j,i}\,g_{1,j,j} +$$

$$c_{115}\,g_{2,i,i} + c_{121}\,g_{2,i,j} + c_{128}\,g_{2,j,i} + c_{133}\,g_{2,i,i}\,g_{2,j,i} - \frac{S\,(c_{133}+c_{143})\,g_{2,i,j}\,g_{2,j,i}}{1+S} -$$

$$\frac{(-1+S)\,(S\,c_{133}-c_{143})\,g_{2,j,i}^2}{S\,(1+S)} + c_{136}\,g_{2,j,j} - \frac{S\,(c_{133}+c_{143})\,g_{2,i,i}\,g_{2,j,j}}{1+S} + c_{143}\,g_{2,j,i}\,g_{2,j,j} +$$

$$c_{145}\,g_{3,i,i} + c_{155}\,g_{3,i,j} + c_{166}\,g_{3,j,i} + c_{175}\,g_{3,i,i}\,g_{3,j,i} - \frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,j}\,g_{3,j,i}}{1+S\,T} -$$

$$\frac{(-1+S\,T)\,(S\,T\,c_{175}-c_{189})\,g_{3,j,i}^2}{S\,T\,(1+S\,T)} + c_{178}\,g_{3,j,j} - \frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,i}\,g_{3,j,j}}{1+S\,T} + c_{189}\,g_{3,j,i}\,g_{3,j,j}\,,$$

$$c_1 - \frac{(T\,c_{101} - (-1+T)\,c_{103})\,g_{1,i,i}}{T} - \frac{c_{103}\,g_{1,i,j}}{T} -$$

$$\frac{\left((-1+T)\,T\,c_{101} - (-1+T)^2\,c_{103} + T\,(T\,c_{106} - (-1+T)\,c_{110})\right)\,g_{1,j,i}}{T} -$$

$$\frac{T\,((-1+3\,T)\,c_{107} + 2\,(-1+T)\,c_{113})\,g_{1,i,i}\,g_{1,j,i}}{1+T} + \frac{T\,(c_{107}+c_{113})\,g_{1,i,j}\,g_{1,j,i}}{1+T} -$$

$$\frac{(-1+T)\,T\,((-1+2\,T)\,c_{107} + (-2+T)\,c_{113})\,g_{1,j,i}^2}{1+T} -$$

$$\frac{((-1+T)\,c_{103} + T\,c_{110})\,g_{1,j,j}}{T} + \frac{T\,(c_{107}+c_{113})\,g_{1,i,i}\,g_{1,j,j}}{1+T} +$$

$$\frac{T\,(2\,(-1+T)\,c_{107} + (-3+T)\,c_{113})\,g_{1,j,i}\,g_{1,j,j}}{1+T} - \frac{(S\,c_{115} - (-1+S)\,c_{121})\,g_{2,i,i}}{S} -$$

$$\frac{c_{121}\,g_{2,i,j}}{S} - \frac{\left((-1+S)\,S\,c_{115} - (-1+S)^2\,c_{121} + S\,(S\,c_{128} - (-1+S)\,c_{136})\right)\,g_{2,j,i}}{S} -$$

$$\frac{S\,((-1+3\,S)\,c_{133} + 2\,(-1+S)\,c_{143})\,g_{2,i,i}\,g_{2,j,i}}{1+S} + \frac{S\,(c_{133}+c_{143})\,g_{2,i,j}\,g_{2,j,i}}{1+S} -$$

$$\frac{(-1+S)\,S\,((-1+2\,S)\,c_{133} + (-2+S)\,c_{143})\,g_{2,j,i}^2}{1+S} - \frac{((-1+S)\,c_{121} + S\,c_{136})\,g_{2,j,j}}{S} +$$

$$\frac{S\,(c_{133}+c_{143})\,g_{2,i,i}\,g_{2,j,j}}{1+S} + \frac{S\,(2\,(-1+S)\,c_{133} + (-3+S)\,c_{143})\,g_{2,j,i}\,g_{2,j,j}}{1+S} -$$

$$\frac{(S\,T\,c_{145} + (1-S\,T)\,c_{155})\,g_{3,i,i}}{S\,T} - \frac{c_{155}\,g_{3,i,j}}{S\,T} -$$

$$\frac{\left(S\,T\,(-1+S\,T)\,c_{145} - (-1+S\,T)^2\,c_{155} + S\,T\,(S\,T\,c_{166} + (1-S\,T)\,c_{178})\right)\,g_{3,j,i}}{S\,T} -$$

$$\frac{S\,T\,((-1+3\,S\,T)\,c_{175} + 2\,(-1+S\,T)\,c_{189})\,g_{3,i,i}\,g_{3,j,i}}{1+S\,T} + \frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,j}\,g_{3,j,i}}{1+S\,T} -$$

$$\frac{S\,T\,(-1+S\,T)\,((-1+2\,S\,T)\,c_{175} + (-2+S\,T)\,c_{189})\,g_{3,j,i}^2}{1+S\,T} - \frac{((-1+S\,T)\,c_{155} + S\,T\,c_{178})\,g_{3,j,j}}{S\,T} +$$

$$\frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,i}\,g_{3,j,j}}{1+S\,T} + \frac{S\,T\,(2\,(-1+S\,T)\,c_{175} + (-3+S\,T)\,c_{189})\,g_{3,j,i}\,g_{3,j,j}}{1+S\,T}\Big\}$$

The most general no-mixed-terms quadratic-only solution:

$In[•]:=$ `Simplify[{R₁[-1, i, j], R₁[1, i, j]} /.`
    `{c₁|₁₀₁|₁₀₃|₁₀₆|₁₁₀|₁₁₅|₁₁₈|₁₂₁|₁₂₈|₁₂₉|₁₃₆|₁₄₅|₁₄₈|₁₅₂|₁₅₅|₁₆₆|₁₆₇|₁₇₁|₁₇₈ → 0}]`

$Out[•]=$

$$\left\{ c_{107}\, g_{1,i,i}\, g_{1,j,i} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,j}\, g_{1,j,i}}{1+T} - \right.$$

$$\frac{(-1+T)\,(T\, c_{107}-c_{113})\, g_{1,j,i}^2}{T\,(1+T)} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,i}\, g_{1,j,j}}{1+T} + c_{113}\, g_{1,j,i}\, g_{1,j,j} +$$

$$c_{133}\, g_{2,i,i}\, g_{2,j,i} - \frac{S\,(c_{133}+c_{143})\, g_{2,i,j}\, g_{2,j,i}}{1+S} - \frac{(-1+S)\,(S\, c_{133}-c_{143})\, g_{2,j,i}^2}{S\,(1+S)} -$$

$$\frac{S\,(c_{133}+c_{143})\, g_{2,i,i}\, g_{2,j,j}}{1+S} + c_{143}\, g_{2,j,i}\, g_{2,j,j} + c_{175}\, g_{3,i,i}\, g_{3,j,i} - \frac{S\,T\,(c_{175}+c_{189})\, g_{3,i,j}\, g_{3,j,i}}{1+S\,T} -$$

$$\frac{(-1+S\,T)\,(S\,T\, c_{175}-c_{189})\, g_{3,j,i}^2}{S\,T\,(1+S\,T)} - \frac{S\,T\,(c_{175}+c_{189})\, g_{3,i,i}\, g_{3,j,j}}{1+S\,T} + c_{189}\, g_{3,j,i}\, g_{3,j,j},$$

$$-\frac{T\,((-1+3\,T)\, c_{107}+2\,(-1+T)\, c_{113})\, g_{1,i,i}\, g_{1,j,i}}{1+T} + \frac{T\,(c_{107}+c_{113})\, g_{1,i,j}\, g_{1,j,i}}{1+T} -$$

$$\frac{(-1+T)\,T\,((-1+2\,T)\, c_{107}+(-2+T)\, c_{113})\, g_{1,j,i}^2}{1+T} + \frac{T\,(c_{107}+c_{113})\, g_{1,i,i}\, g_{1,j,j}}{1+T} +$$

$$\frac{T\,(2\,(-1+T)\, c_{107}+(-3+T)\, c_{113})\, g_{1,j,i}\, g_{1,j,j}}{1+T} - \frac{S\,((-1+3\,S)\, c_{133}+2\,(-1+S)\, c_{143})\, g_{2,i,i}\, g_{2,j,i}}{1+S} +$$

$$\frac{S\,(c_{133}+c_{143})\, g_{2,i,j}\, g_{2,j,i}}{1+S} - \frac{(-1+S)\,S\,((-1+2\,S)\, c_{133}+(-2+S)\, c_{143})\, g_{2,j,i}^2}{1+S} +$$

$$\frac{S\,(c_{133}+c_{143})\, g_{2,i,i}\, g_{2,j,j}}{1+S} + \frac{S\,(2\,(-1+S)\, c_{133}+(-3+S)\, c_{143})\, g_{2,j,i}\, g_{2,j,j}}{1+S} -$$

$$\frac{S\,T\,((-1+3\,S\,T)\, c_{175}+2\,(-1+S\,T)\, c_{189})\, g_{3,i,i}\, g_{3,j,i}}{1+S\,T} + \frac{S\,T\,(c_{175}+c_{189})\, g_{3,i,j}\, g_{3,j,i}}{1+S\,T} -$$

$$\frac{S\,T\,(-1+S\,T)\,((-1+2\,S\,T)\, c_{175}+(-2+S\,T)\, c_{189})\, g_{3,j,i}^2}{1+S\,T} +$$

$$\left. \frac{S\,T\,(c_{175}+c_{189})\, g_{3,i,i}\, g_{3,j,j}}{1+S\,T} + \frac{S\,T\,(2\,(-1+S\,T)\, c_{175}+(-3+S\,T)\, c_{189})\, g_{3,j,i}\, g_{3,j,j}}{1+S\,T} \right\}$$

The most general quadratic-only solution: (has mixed terms!)

$In[•]:=$ `Simplify[{R₁[-1, i, j], R₁[1, i, j]} /. {c₁|₁₀₁|₁₀₃|₁₀₆|₁₁₀|₁₁₅|₁₂₁|₁₂₈|₁₃₆|₁₄₅|₁₅₅|₁₆₆|₁₇₈ → 0}]`

$Out[•]=$

$$\left\{ c_{107}\, g_{1,i,i}\, g_{1,j,i} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,j}\, g_{1,j,i}}{1+T} - \right.$$

$$\frac{(-1+T)\,(T\, c_{107}-c_{113})\, g_{1,j,i}^2}{T\,(1+T)} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,i}\, g_{1,j,j}}{1+T} + c_{113}\, g_{1,j,i}\, g_{1,j,j} +$$

$$c_{118}\, g_{1,j,i}\, g_{2,i,i} + \frac{((-1+S)\,T\, c_{118}-S\,(-1+T)\, c_{129})\, g_{1,j,j}\, g_{2,i,i}}{(-1+S)\,(-1+T)} + c_{129}\, g_{1,i,i}\, g_{2,j,i} -$$

$$\frac{\left(\,(-1+S)\;T\;c_{118}+S\;(-1+T)\;c_{129}\right)\,g_{1,j,i}\,g_{2,j,i}}{S\,T}-\frac{(-1+S)\;T\;c_{118}\,g_{1,j,j}\,g_{2,j,i}}{S\,(-1+T)}+$$

$$c_{133}\,g_{2,i,i}\,g_{2,j,i}-\frac{S\,(c_{133}+c_{143})\,g_{2,i,j}\,g_{2,j,i}}{1+S}-\frac{(-1+S)\;(S\,c_{133}-c_{143})\,g_{2,j,i}^{2}}{S\,(1+S)}+$$

$$\left(-\frac{T\,c_{118}}{-1+T}+\frac{S\,c_{129}}{-1+S}\right)g_{1,i,i}\,g_{2,j,j}-\frac{S\,(-1+T)\,c_{129}\,g_{1,j,i}\,g_{2,j,j}}{(-1+S)\;T}-\frac{S\,(c_{133}+c_{143})\,g_{2,i,i}\,g_{2,j,j}}{1+S}+$$

$$c_{143}\,g_{2,j,i}\,g_{2,j,j}+c_{148}\,g_{1,j,i}\,g_{3,i,i}+\frac{T\,\left(\,(-1+S\,T)\;c_{148}-S\;(-1+T)\;c_{167}\right)\,g_{1,j,j}\,g_{3,i,i}}{(-1+T)\;(-1+S\,T)}+$$

$$c_{152}\,g_{2,j,i}\,g_{3,i,i}+\frac{S\,\left(\,(-1+S\,T)\;c_{152}-(-1+S)\;T\,c_{171}\right)\,g_{2,j,j}\,g_{3,i,i}}{(-1+S)\;(-1+S\,T)}+c_{167}\,g_{1,i,i}\,g_{3,j,i}-$$

$$\frac{\left(\,(-1+S\,T)\;c_{148}+S\;(-1+T)\;c_{167}\right)\,g_{1,j,i}\,g_{3,j,i}}{S\,T}-\frac{(-1+S\,T)\;c_{148}\,g_{1,j,j}\,g_{3,j,i}}{S\,(-1+T)}+$$

$$c_{171}\,g_{2,i,i}\,g_{3,j,i}-\frac{\left(\,(-1+S\,T)\;c_{152}+(-1+S)\;T\,c_{171}\right)\,g_{2,j,i}\,g_{3,j,i}}{S\,T}-$$

$$\frac{(-1+S\,T)\;c_{152}\,g_{2,j,j}\,g_{3,j,i}}{(-1+S)\;T}+c_{175}\,g_{3,i,i}\,g_{3,j,i}-\frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,j}\,g_{3,j,i}}{1+S\,T}-$$

$$\frac{(-1+S\,T)\;(S\,T\,c_{175}-c_{189})\,g_{3,j,i}^{2}}{S\,T\,(1+S\,T)}+T\left(-\frac{c_{148}}{-1+T}+\frac{S\,c_{167}}{-1+S\,T}\right)g_{1,i,i}\,g_{3,j,j}-$$

$$\frac{S\,(-1+T)\,c_{167}\,g_{1,j,i}\,g_{3,j,j}}{-1+S\,T}+S\left(-\frac{c_{152}}{-1+S}+\frac{T\,c_{171}}{-1+S\,T}\right)g_{2,i,i}\,g_{3,j,j}-$$

$$\frac{(-1+S)\;T\,c_{171}\,g_{2,j,i}\,g_{3,j,j}}{-1+S\,T}-\frac{S\,T\,(c_{175}+c_{189})\,g_{3,i,i}\,g_{3,j,j}}{1+S\,T}+c_{189}\,g_{3,j,i}\,g_{3,j,j},$$

$$-\frac{T\,\left(\,(-1+3\,T)\;c_{107}+2\;(-1+T)\;c_{113}\right)\,g_{1,i,i}\,g_{1,j,i}}{1+T}+\frac{T\,(c_{107}+c_{113})\,g_{1,i,j}\,g_{1,j,i}}{1+T}-$$

$$\frac{(-1+T)\;T\,\left(\,(-1+2\,T)\;c_{107}+(-2+T)\;c_{113}\right)\,g_{1,j,i}^{2}}{1+T}+\frac{T\,(c_{107}+c_{113})\,g_{1,i,i}\,g_{1,j,j}}{1+T}+$$

$$\frac{T\,\left(2\;(-1+T)\;c_{107}+(-3+T)\;c_{113}\right)\,g_{1,j,i}\,g_{1,j,j}}{1+T}-\frac{S\,(-1+T)\,c_{129}\,g_{1,j,i}\,g_{2,i,i}}{-1+S}+$$

$$\left(-\frac{T\,c_{118}}{-1+T}+\frac{S\,c_{129}}{-1+S}\right)g_{1,j,j}\,g_{2,i,i}-\frac{(-1+S)\;T\,c_{118}\,g_{1,i,i}\,g_{2,j,i}}{-1+T}-$$

$$\left(\,(-1+S)\;T\,c_{118}+S\;(-1+T)\;c_{129}\right)\,g_{1,j,i}\,g_{2,j,i}+S\,c_{129}\,g_{1,j,j}\,g_{2,j,i}-$$

$$\frac{S\,\left(\,(-1+3\,S)\;c_{133}+2\;(-1+S)\;c_{143}\right)\,g_{2,i,i}\,g_{2,j,i}}{1+S}+\frac{S\,(c_{133}+c_{143})\,g_{2,i,j}\,g_{2,j,i}}{1+S}-$$

$$\frac{(-1+S)\;S\,\left(\,(-1+2\,S)\;c_{133}+(-2+S)\;c_{143}\right)\,g_{2,j,i}^{2}}{1+S}+\frac{\left(\,(-1+S)\;T\,c_{118}-S\;(-1+T)\;c_{129}\right)\,g_{1,i,i}\,g_{2,j,j}}{(-1+S)\;(-1+T)}+$$

$$T\,c_{118}\,g_{1,j,i}\,g_{2,j,j}+\frac{S\,(c_{133}+c_{143})\,g_{2,i,i}\,g_{2,j,j}}{1+S}+\frac{S\,\left(2\;(-1+S)\;c_{133}+(-3+S)\;c_{143}\right)\,g_{2,j,i}\,g_{2,j,j}}{1+S}-$$

$$\frac{S\,(-1+T)\;T\,c_{167}\,g_{1,j,i}\,g_{3,i,i}}{-1+S\,T}+T\left(-\frac{c_{148}}{-1+T}+\frac{S\,c_{167}}{-1+S\,T}\right)g_{1,j,j}\,g_{3,i,i}-$$

$$\frac{(-1+S)\;S\,T\,c_{171}\,g_{2,j,i}\,g_{3,i,i}}{-1+S\,T}+S\left(-\frac{c_{152}}{-1+S}+\frac{T\,c_{171}}{-1+S\,T}\right)g_{2,j,j}\,g_{3,i,i}-\frac{T\,(-1+S\,T)\;c_{148}\,g_{1,i,i}\,g_{3,j,i}}{-1+T}-$$

$$T\left(\left(-1+S\,T\right)c_{148}+S\left(-1+T\right)c_{167}\right)g_{1,j,i}\,g_{3,j,i}+S\,T\,c_{167}\,g_{1,j,j}\,g_{3,j,i}-$$

$$\frac{S\left(-1+S\,T\right)c_{152}\,g_{2,i,i}\,g_{3,j,i}}{-1+S}-S\left(\left(-1+S\,T\right)c_{152}+\left(-1+S\right)T\,c_{171}\right)g_{2,j,i}\,g_{3,j,i}+$$

$$S\,T\,c_{171}\,g_{2,j,j}\,g_{3,j,i}-\frac{S\,T\left(\left(-1+3\,S\,T\right)c_{175}+2\left(-1+S\,T\right)c_{189}\right)g_{3,i,i}\,g_{3,j,i}}{1+S\,T}+$$

$$\frac{S\,T\left(c_{175}+c_{189}\right)g_{3,i,j}\,g_{3,j,i}}{1+S\,T}-\frac{S\,T\left(-1+S\,T\right)\left(\left(-1+2\,S\,T\right)c_{175}+\left(-2+S\,T\right)c_{189}\right)g_{3,j,i}^{2}}{1+S\,T}+$$

$$\frac{T\left(\left(-1+S\,T\right)c_{148}-S\left(-1+T\right)c_{167}\right)g_{1,i,i}\,g_{3,j,j}}{\left(-1+T\right)\left(-1+S\,T\right)}+T\,c_{148}\,g_{1,j,i}\,g_{3,j,j}+$$

$$\frac{S\left(\left(-1+S\,T\right)c_{152}-\left(-1+S\right)T\,c_{171}\right)g_{2,i,i}\,g_{3,j,j}}{\left(-1+S\right)\left(-1+S\,T\right)}+S\,c_{152}\,g_{2,j,i}\,g_{3,j,j}+$$

$$\frac{S\,T\left(c_{175}+c_{189}\right)g_{3,i,i}\,g_{3,j,j}}{1+S\,T}+\frac{S\,T\left(2\left(-1+S\,T\right)c_{175}+\left(-3+S\,T\right)c_{189}\right)g_{3,j,i}\,g_{3,j,j}}{1+S\,T}\Biggr\}$$

The most general no-$g_{3,\_,\_}$ solution: (has mixed terms)

$\textit{In[•]:=}$ **Simplify[{R$_1$[-1, i, j], R$_1$[1, i, j]} /. {c$_{145|148|152|155|166|167|171|175|178|189}$ → 0}]**

$\textit{Out[•]=}$

$$\Big\{ -c_1 + c_{101}\, g_{1,i,i} + c_{103}\, g_{1,i,j} + c_{106}\, g_{1,j,i} + c_{107}\, g_{1,i,i}\, g_{1,j,i} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,j}\, g_{1,j,i}}{1+T} -$$

$$\frac{(-1+T)\,(T\,c_{107}-c_{113})\, g_{1,j,i}^2}{T\,(1+T)} + c_{110}\, g_{1,j,j} - \frac{T\,(c_{107}+c_{113})\, g_{1,i,i}\, g_{1,j,j}}{1+T} + c_{113}\, g_{1,j,i}\, g_{1,j,j} +$$

$$c_{115}\, g_{2,i,i} + c_{118}\, g_{1,j,i}\, g_{2,i,i} + \frac{((-1+S)\,T\,c_{118} - S\,(-1+T)\,c_{129})\, g_{1,j,j}\, g_{2,i,i}}{(-1+S)\,(-1+T)} + c_{121}\, g_{2,i,j} + c_{128}\, g_{2,j,i} +$$

$$c_{129}\, g_{1,i,i}\, g_{2,j,i} - \frac{((-1+S)\,T\,c_{118} + S\,(-1+T)\,c_{129})\, g_{1,j,i}\, g_{2,j,i}}{S\,T} - \frac{(-1+S)\,T\,c_{118}\, g_{1,j,j}\, g_{2,j,i}}{S\,(-1+T)} +$$

$$c_{133}\, g_{2,i,i}\, g_{2,j,i} - \frac{S\,(c_{133}+c_{143})\, g_{2,i,j}\, g_{2,j,i}}{1+S} - \frac{(-1+S)\,(S\,c_{133}-c_{143})\, g_{2,j,i}^2}{S\,(1+S)} +$$

$$c_{136}\, g_{2,j,j} + \Big(-\frac{T\,c_{118}}{-1+T} + \frac{S\,c_{129}}{-1+S}\Big)\, g_{1,i,i}\, g_{2,j,j} - \frac{S\,(-1+T)\,c_{129}\, g_{1,j,i}\, g_{2,j,j}}{(-1+S)\,T} -$$

$$\frac{S\,(c_{133}+c_{143})\, g_{2,i,i}\, g_{2,j,j}}{1+S} + c_{143}\, g_{2,j,i}\, g_{2,j,j},\ \ c_1 - \frac{(T\,c_{101} - (-1+T)\,c_{103})\, g_{1,i,i}}{T} -$$

$$\frac{c_{103}\, g_{1,i,j}}{T} - \frac{\big((-1+T)\,T\,c_{101} - (-1+T)^2\,c_{103} + T\,(T\,c_{106} - (-1+T)\,c_{110})\big)\, g_{1,j,i}}{T} -$$

$$\frac{T\,((-1+3\,T)\,c_{107} + 2\,(-1+T)\,c_{113})\, g_{1,i,i}\, g_{1,j,i}}{1+T} + \frac{T\,(c_{107}+c_{113})\, g_{1,i,j}\, g_{1,j,i}}{1+T} -$$

$$\frac{(-1+T)\,T\,((-1+2\,T)\,c_{107} + (-2+T)\,c_{113})\, g_{1,j,i}^2}{1+T} - \frac{((-1+T)\,c_{103} + T\,c_{110})\, g_{1,j,j}}{T} +$$

$$\frac{T\,(c_{107}+c_{113})\, g_{1,i,i}\, g_{1,j,j}}{1+T} + \frac{T\,(2\,(-1+T)\,c_{107} + (-3+T)\,c_{113})\, g_{1,j,i}\, g_{1,j,j}}{1+T} -$$

$$\frac{(S\,c_{115} - (-1+S)\,c_{121})\, g_{2,i,i}}{S} - \frac{S\,(-1+T)\,c_{129}\, g_{1,j,i}\, g_{2,i,i}}{-1+S} + \Big(-\frac{T\,c_{118}}{-1+T} + \frac{S\,c_{129}}{-1+S}\Big)\, g_{1,j,j}\, g_{2,i,i} -$$

$$\frac{c_{121}\, g_{2,i,j}}{S} - \frac{\big((-1+S)\,S\,c_{115} - (-1+S)^2\,c_{121} + S\,(S\,c_{128} - (-1+S)\,c_{136})\big)\, g_{2,j,i}}{S} -$$

$$\frac{(-1+S)\,T\,c_{118}\, g_{1,i,i}\, g_{2,j,i}}{-1+T} - ((-1+S)\,T\,c_{118} + S\,(-1+T)\,c_{129})\, g_{1,j,i}\, g_{2,j,i} +$$

$$S\,c_{129}\, g_{1,j,j}\, g_{2,j,i} - \frac{S\,((-1+3\,S)\,c_{133} + 2\,(-1+S)\,c_{143})\, g_{2,i,i}\, g_{2,j,i}}{1+S} +$$

$$\frac{S\,(c_{133}+c_{143})\, g_{2,i,j}\, g_{2,j,i}}{1+S} - \frac{(-1+S)\,S\,((-1+2\,S)\,c_{133} + (-2+S)\,c_{143})\, g_{2,j,i}^2}{1+S} -$$

$$\frac{((-1+S)\,c_{121} + S\,c_{136})\, g_{2,j,j}}{S} + \frac{((-1+S)\,T\,c_{118} - S\,(-1+T)\,c_{129})\, g_{1,i,i}\, g_{2,j,j}}{(-1+S)\,(-1+T)} +$$

$$T\,c_{118}\, g_{1,j,i}\, g_{2,j,j} + \frac{S\,(c_{133}+c_{143})\, g_{2,i,i}\, g_{2,j,j}}{1+S} + \frac{S\,(2\,(-1+S)\,c_{133} + (-3+S)\,c_{143})\, g_{2,j,i}\, g_{2,j,j}}{1+S} \Big\}$$