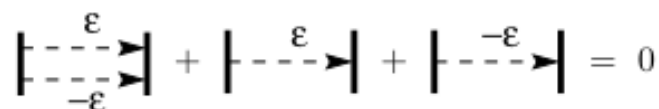
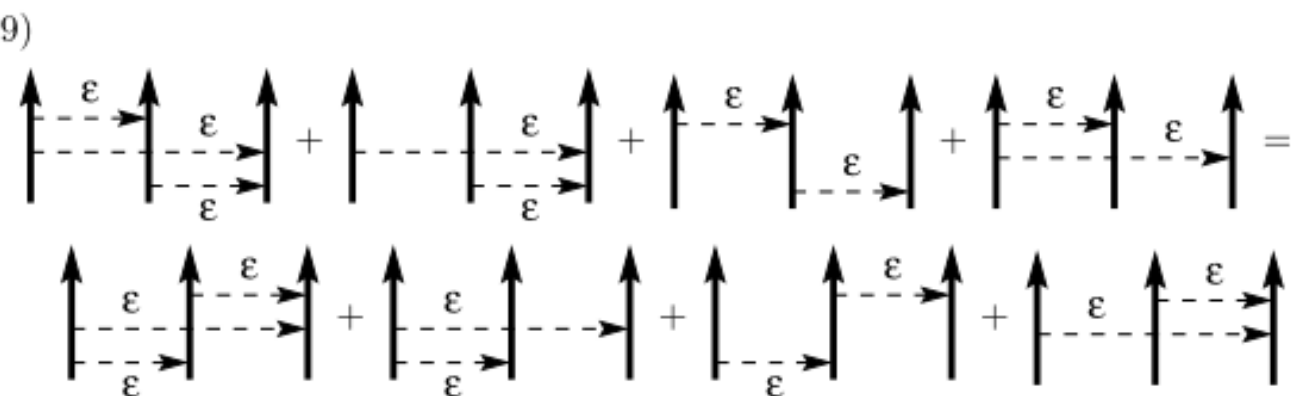


2.5. **The Polyak Algebra.** The *Polyak algebra* is the quotient of \mathcal{A} by the following relations:

(7)  = 0

(8)  = 0

(9)  =

R8T

```

SetAttributes[Diag, Orderless];
Place[Ar, {i_, j_}] := {Diag[Ar[i, j]], Diag[Ar[j, i]]};
Place[Ar, objs_, {i_, rest_}] := Flatten[Table[
  Outer[Join,
    Place[Ar, {i, {rest}[[k]]},
    Place[objs, Delete[{rest}, k]]
  ],
  {k, Length[{rest]}]
]];
Diagrams[k_.*Ar] := Place[Table[Ar, {k}], Range[2 k]]

```

```

Place[{R8T}, {i_, j_, k_}] :=
  Permutations[{i, j, k}] /. {i1_, j1_, k1_} => Diag[R8T[i1, j1, k1]];
Place[{r : (TC | R8T), objs__}, {i_, rest__}] := Flatten[Table[
  Outer[Join,
    Place[{r}, {i, {rest}[[j]], {rest}[[k]]},
    Place[{objs}, Delete[{rest}, {j}, {k}]]
  ],
  {k, 2, Length[{rest]}}, {j, 1, k-1}
]];
Diagrams[R8T] := Place[{R8T}, {1, 2, 3}];
Diagrams[R8T+k_.*Ar] /; k > 0 := Flatten[
  Place[#, Range[2k+3]] & /@ Permutations[Table[Ar, {k}]~Append~R8T]
];
Place[{TC}, {i_, j_, k_}] := Diag /@ {TC[i, j, k], TC[j, k, i], TC[k, i, j]};
Diagrams[TC] := Place[{TC}, {1, 2, 3}];
Diagrams[TC+k_.*Ar] /; k > 0 := Flatten[
  Place[#, Range[2k+3]] & /@ Permutations[Table[Ar, {k}]~Append~TC]
];
Diagrams[TC+k_.*Ar] /; k < 0 := {};

NormalizeDiag[diag_Diag] := Module[
  {indices = Union@@ (List @@ diag /. Ar -> List)},
  diag /. Thread[indices -> Range[Length[indices]]]
];
R[Diag[lft___, R8T[i_, j_, k_], rgt___]] := (
  (* RHS of (9) *)
  NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], Ar[j+0.5, k+0.5], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[j+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, k], Ar[j, k+0.5], rgt]]
  (* LHS of (9) *)
  - NormalizeDiag[Diag[lft, Ar[j, k], Ar[i, k+0.5], Ar[i+0.5, j+0.5], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k], Ar[i+0.5, j], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, j+0.5], Ar[j, k], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k+0.5], Ar[j, k], rgt]]
);
RMinus[n_, Diag[lft___, R8T[i_, j_, k_], rgt___]] := Module[
  {Expand2, Expand3, n1},
  n1 = n - Length[{lft, rgt}];
  Expand2[Diag[Ar[i1_, j1_], Ar[i2_, j2_]]] := Sum[
    (-1)^(k1+k2) * NormalizeDiag[Flatten[Diag[
      lft,

```

```

Diag@@Table[Ar[i1+0.01 q, j1+0.01 q], {q, k1}],
Diag@@Table[Ar[i2+0.01 q, j2+0.01 q], {q, k2}],
rgt
]],
{k1, n1-1}, {k2, n1-k1}
];
Expand3[Diag[Ar[i1_, j1_], Ar[i2_, j2_], Ar[i3_, j3_]] := Sum[
(-1)^(k1+k2+k3) * NormalizeDiag[Flatten[Diag[
lft,
Diag@@Table[Ar[i1+0.01 q, j1+0.01 q], {q, k1}],
Diag@@Table[Ar[i2+0.01 q, j2+0.01 q], {q, k2}],
Diag@@Table[Ar[i3+0.01 q, j3+0.01 q], {q, k3}],
rgt
]],
{k1, n1-2}, {k2, n1-k1-1}, {k3, n1-k1-k2}
];
Expand[Plus[
(
Diag[Ar[i, j], Ar[i+0.5, k]] +
Diag[Ar[i, j], Ar[j+0.5, k]] + Diag[Ar[i, k], Ar[j, k+0.5]]
- Diag[Ar[i, k], Ar[i+0.5, j]] - Diag[Ar[i, j+0.5], Ar[j, k]] -
Diag[Ar[i, k+0.5], Ar[j, k]]
) /. d_Diag => Expand2[d],
(Diag[Ar[i, j], Ar[i+0.5, k], Ar[j+0.5, k+0.5]] -
Diag[Ar[j, k], Ar[i, k+0.5], Ar[i+0.5, j+0.5]]) /. d_Diag => Expand3[d]
]]
];
R[Diag[lft___, TC[i_, j_, k_], rgt___]] := (
+NormalizeDiag[Diag[lft, Ar[k, i], Ar[k+0.5, j], rgt]]
-NormalizeDiag[Diag[lft, Ar[k+0.5, i], Ar[k, j], rgt]]
);
ContainsShortArrow[diag_Diag] := (1 == Min[List@@diag /. Ar[i_, j_] => Abs[i-j]]);

```

```

DimP[m_] /; m < 2 := Length[Diagrams[m Ar]];
DimP[m_, reotypes_List] /; m ≥ 2 := Module[
  {diags, rels, diagtoindex, mat, rel, i},
  Print[Date[], ": Starting work..."];
  diags = Join@@Table[Diagrams[k Ar], {k, m}];
  reotypes /. "R1" => (
    diags = Select[diags, !ContainsShortArrow[#] &]
  );
  Print[Date[], ": Computed ", Length[diags], " diagrams..."];
  rels = Join@@(reotypes /. {
    "R1" → {},
    "R8T" => (
      (R /@ Join@@Table[Diagrams[R8T+k Ar], {k, 0, m-2}]) /.
      diag_Diag /; Length[diag] > m => 0
    ),
    "R8Tminus" => (
      Rminus[m, #] & /@ Join@@Table[Diagrams[R8T+k Ar], {k, 0, m-2}]
    ),
    "TC" => (
      R /@ Join@@Table[Diagrams[TC+k Ar], {k, 0, m-2}]
    )
  });
  reotypes /. "R1" => (
    rels = rels /. {d_Diag /; ContainsShortArrow[d] => 0}
  );
  Print[Date[], ": Computed ", Length[rels], " relations..."];
  diagtoindex = Dispatch[Thread[Rule[diags, Range[Length[diags]]]];
  mat = SparseArray[
    Join@@Table[
      rel = rels[[i]];
      {i, # /. diagtoindex} → Coefficient[rel, #] & /@
      Cases[{rel}, diag_Diag, Infinity],
      {i, Length[rels]}
    ],
    {Length[rels], Length[diags]}
  ];
  Print[Date[], ": Computed mat..."];
  Length[diags] - MatrixRank[mat]
];
DimP[#, {"R8T"}] & /@ {2, 3, 4}

```

```

{2009, 8, 21, 14, 38, 1.349972}: Starting work...
{2009, 8, 21, 14, 38, 1.406922}: Computed 14 diagrams...
{2009, 8, 21, 14, 38, 1.469485}: Computed 6 relations...
{2009, 8, 21, 14, 38, 1.513586}: Computed mat...
{2009, 8, 21, 14, 38, 1.527676}: Starting work...
{2009, 8, 21, 14, 38, 1.528692}: Computed 134 diagrams...
{2009, 8, 21, 14, 38, 1.592648}: Computed 126 relations...
{2009, 8, 21, 14, 38, 1.604047}: Computed mat...
{2009, 8, 21, 14, 38, 1.606982}: Starting work...
{2009, 8, 21, 14, 38, 1.616321}: Computed 1814 diagrams...
{2009, 8, 21, 14, 38, 2.893941}: Computed 2646 relations...
{2009, 8, 21, 14, 38, 3.208169}: Computed mat...

```

```
{9, 36, 175}
```

```
DimP[#, {"R8T", "R8TMinus"}] & /@ {2, 3, 4}
```

```

{2009, 8, 21, 14, 38, 6.030791}: Starting work...
{2009, 8, 21, 14, 38, 6.031145}: Computed 14 diagrams...
{2009, 8, 21, 14, 38, 6.038043}: Computed 12 relations...
{2009, 8, 21, 14, 38, 6.039420}: Computed mat...
{2009, 8, 21, 14, 38, 6.039681}: Starting work...
{2009, 8, 21, 14, 38, 6.040472}: Computed 134 diagrams...
{2009, 8, 21, 14, 38, 6.213070}: Computed 252 relations...
{2009, 8, 21, 14, 38, 6.241989}: Computed mat...
{2009, 8, 21, 14, 38, 6.249765}: Starting work...
{2009, 8, 21, 14, 38, 6.258729}: Computed 1814 diagrams...
{2009, 8, 21, 14, 38, 9.603216}: Computed 5292 relations...
{2009, 8, 21, 14, 38, 10.102551}: Computed mat...

```

```
{9, 36, 175}
```

```
DimP[#, {"R8T", "R1"}] & /@ {2, 3, 4, 5}
```

```

{2009, 8, 21, 14, 38, 16.356902}: Starting work...
{2009, 8, 21, 14, 38, 16.398407}: Computed 4 diagrams...
{2009, 8, 21, 14, 38, 16.401230}: Computed 6 relations...
{2009, 8, 21, 14, 38, 16.401540}: Computed mat...
{2009, 8, 21, 14, 38, 16.401748}: Starting work...
{2009, 8, 21, 14, 38, 16.405914}: Computed 44 diagrams...
{2009, 8, 21, 14, 38, 16.483293}: Computed 126 relations...
{2009, 8, 21, 14, 38, 16.489173}: Computed mat...
{2009, 8, 21, 14, 38, 16.490030}: Starting work...
{2009, 8, 21, 14, 38, 16.547482}: Computed 620 diagrams...
{2009, 8, 21, 14, 38, 18.104403}: Computed 2646 relations...
{2009, 8, 21, 14, 38, 18.220519}: Computed mat...
{2009, 8, 21, 14, 38, 18.529263}: Starting work...
{2009, 8, 21, 14, 38, 19.387697}: Computed 11148 diagrams...
{2009, 8, 21, 14, 38, 58.367905}: Computed 63126 relations...
{2009, 8, 21, 14, 39, 1.073624}: Computed mat...
  {2, 9, 51, 297}

```

DimP[#, {"R8T", "TC"}] & /@ {2, 3, 4}

```

{2009, 8, 21, 15, 49, 5.965776}: Starting work...
{2009, 8, 21, 15, 49, 5.976610}: Computed 14 diagrams...
{2009, 8, 21, 15, 49, 5.982890}: Computed 9 relations...
{2009, 8, 21, 15, 49, 5.988301}: Computed mat...
{2009, 8, 21, 15, 49, 5.988576}: Starting work...
{2009, 8, 21, 15, 49, 5.989339}: Computed 134 diagrams...
{2009, 8, 21, 15, 49, 6.130737}: Computed 189 relations...
{2009, 8, 21, 15, 49, 6.231678}: Computed mat...
{2009, 8, 21, 15, 49, 6.286942}: Starting work...
{2009, 8, 21, 15, 49, 6.317844}: Computed 1814 diagrams...
{2009, 8, 21, 15, 49, 8.432075}: Computed 3969 relations...
{2009, 8, 21, 15, 49, 9.078183}: Computed mat...
  {6, 13, 25}

```

DimP[#, {"R8T", "TC", "R1"}] & /@ {2, 3, 4}

```
{2009, 8, 21, 15, 49, 10.896223}: Starting work...
{2009, 8, 21, 15, 49, 10.896897}: Computed 4 diagrams...
{2009, 8, 21, 15, 49, 10.900072}: Computed 9 relations...
{2009, 8, 21, 15, 49, 10.900495}: Computed mat...
{2009, 8, 21, 15, 49, 10.900709}: Starting work...
{2009, 8, 21, 15, 49, 10.904016}: Computed 44 diagrams...
{2009, 8, 21, 15, 49, 10.977745}: Computed 189 relations...
{2009, 8, 21, 15, 49, 11.006791}: Computed mat...
{2009, 8, 21, 15, 49, 11.035948}: Starting work...
{2009, 8, 21, 15, 49, 11.083248}: Computed 620 diagrams...
{2009, 8, 21, 15, 49, 12.947283}: Computed 3969 relations...
{2009, 8, 21, 15, 49, 13.113072}: Computed mat...
  {1, 2, 4}
```