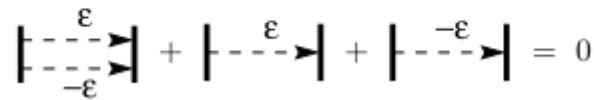
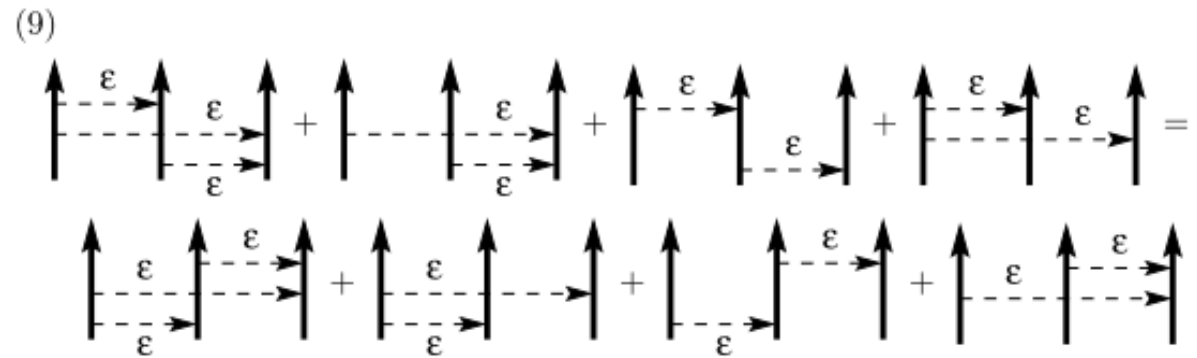


Pensieve Header: The Polyak Algebra.

2.5. **The Polyak Algebra.** The *Polyak algebra* is the quotient of \mathcal{A} by the following relations:

(7)  = 0

(8)  = 0

(9) 

R3b

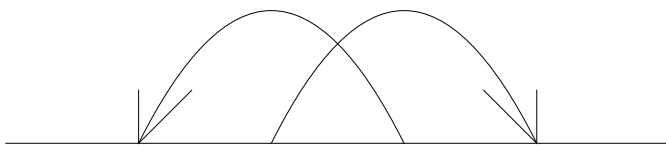
Program and Testing

Drawing Arrow Diagrams

```

SetAttributes[Diag, Orderless];
Draw[expr_] := expr /. diag_Diag => Draw[diag];
Draw[diag_Diag] := Module[
  {n = Length[diag]},
  Graphics[
    {
      Line[{{0, 0}, {2 n + 1, 0}}],
      (List@@diag) /. {
        Ar[i_, j_] => {
          BezierCurve[
            {
              {i, 0}, {(i + j) / 2, Abs[i - j]}, {j, 0}
            },
            Line[
              {{j - 0.2 + 0.2 Sign[i - j], 0.4}, {j, 0}, {j + 0.2 + 0.2 Sign[i - j], 0.4}}]
          }
        }
      ]
    }
  ];
Draw[Diag[Ar[3, 1], Ar[2, 4]]]

```



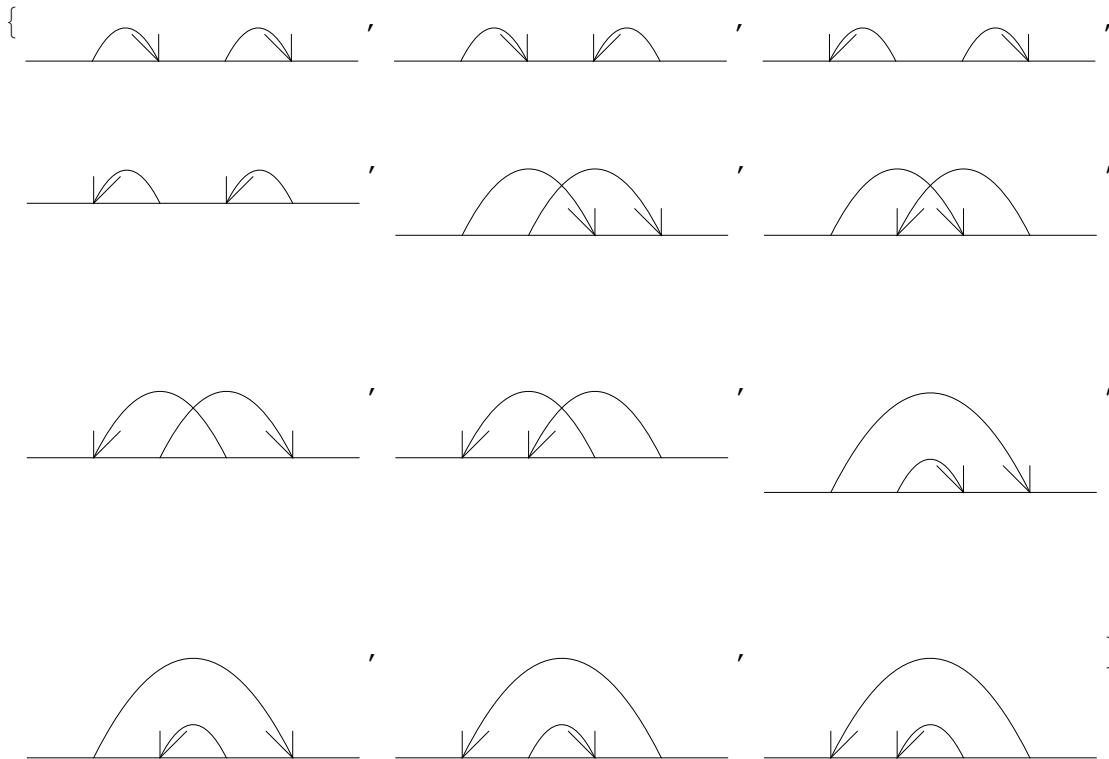
Generating Arrow Diagrams

```
Place[{Ar}, {i_, j_}] := {Diag[Ar[i, j]], Diag[Ar[j, i]]};
Place[{Ar, objs_}, {i_, rest_}] := Flatten[Table[
  Outer[Join,
    Place[{Ar}, {i, {rest}[[k]]}],
    Place[{objs}, Delete[{rest}, k]]
  ],
  {k, Length[{rest]}]
]];
Diagrams[k_.*Ar] := Place[Table[Ar, {k}], Range[2 k]];
```

Diagrams[2 Ar]

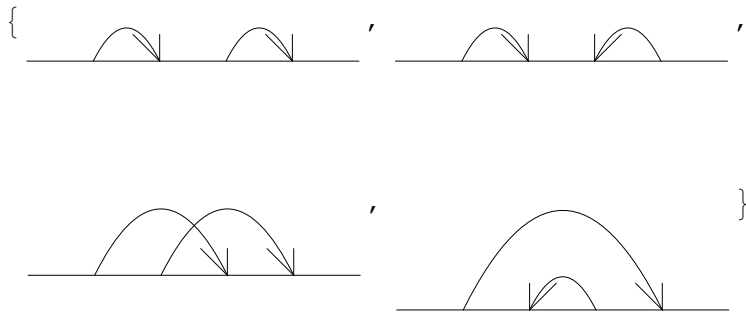
```
{Diag[Ar[1, 2], Ar[3, 4]], Diag[Ar[1, 2], Ar[4, 3]], Diag[Ar[2, 1], Ar[3, 4]],
  Diag[Ar[2, 1], Ar[4, 3]], Diag[Ar[1, 3], Ar[2, 4]], Diag[Ar[1, 3], Ar[4, 2]],
  Diag[Ar[2, 4], Ar[3, 1]], Diag[Ar[3, 1], Ar[4, 2]], Diag[Ar[1, 4], Ar[2, 3]],
  Diag[Ar[1, 4], Ar[3, 2]], Diag[Ar[2, 3], Ar[4, 1]], Diag[Ar[3, 2], Ar[4, 1]]}
```

Draw[Diagrams[2 Ar]]



The "Round" Case

```
DiagramRotateLeft[d_Diag] := Module[
  {labels = Union @@ (List @@ Apply[List, d, {1}])},
  d /. Thread[Rule[labels, RotateLeft[labels]]]
];
DiagramRotateToMinimal[expr_] := expr /. d_Diag => Module[
  {bd = d, rd = DiagramRotateLeft[d]},
  While[rd != d,
    bd = First[Sort[{bd, rd}]];
    rd = DiagramRotateLeft[rd]];
  bd
];
Union[DiagramRotateToMinimal[Diagrams[2 Ar]]]
{Diag[Ar[1, 2], Ar[3, 4]], Diag[Ar[1, 2], Ar[4, 3]],
  Diag[Ar[1, 3], Ar[2, 4]], Diag[Ar[1, 4], Ar[3, 2]]}
Draw[Union[DiagramRotateToMinimal[Diagrams[2 Ar]]]]
```



Generating Relators

```
Place[{r : (TC | R3b), objs__}, {i_, rest__}] := Flatten[Table[
  Outer[Join,
    Place[{r}, {i, {rest}[[j]], {rest}[[k]}],
    Place[{objs}, Delete[{rest}, {{j}, {k}}]]
  ],
  {k, 2, Length[{rest]}}, {j, 1, k - 1}
]];
Place[{R3b}, {i_, j_, k_}] :=
  Permutations[{i, j, k}] /. {i1_, j1_, k1_} => Diag[R3b[i1, j1, k1]];
Diagrams[R3b] := Place[{R3b}, {1, 2, 3}];
Diagrams[R3b + k_.* Ar] /; k > 0 := Flatten[
  Place[#, Range[2 k + 3]] & /@ Permutations[Table[Ar, {k}] ~ Append ~ R3b]
];
Place[{TC}, {i_, j_, k_}] := Diag /@ {TC[i, j, k], TC[j, k, i], TC[k, i, j]};
```

```

Diagrams[TC] := Place[{TC}, {1, 2, 3}];
Diagrams[TC + k_. * Ar] /; k > 0 := Flatten[
  Place[#, Range[2 k + 3]] & /@ Permutations[Table[Ar, {k}] ~Append~ TC]
];
Diagrams[TC + k_. * Ar] /; k < 0 := {};
Place[{R2c}, {i_, j_}] := {Diag[R2c[i, j]], Diag[R2c[j, i]]};
Place[{r: (R2c), objs__}, {i_, rest__}] := Flatten[Table[
  Outer[Join,
    Place[{r}, {i, {rest}[[j]]}],
    Place[{objs}, Delete[{rest}, j]]
  ],
  {j, 1, Length[{rest]}]
]];
Diagrams[R2c] := Place[{R2c}, {1, 2}];
Diagrams[R2c + k_. * Ar] /; k > 0 := Flatten[
  Place[#, Range[2 k + 2]] & /@ Permutations[Table[Ar, {k}] ~Append~ R2c]
];
Diagrams[R2c + k_. * Ar] /; k < 0 := {};
Place[{R1d}, {i_}] := {Diag[R1d[i]]};
Place[{R1a}, {i_}] := {Diag[R1a[i]]};
Place[{r: (R1d | R1a), objs__}, {i_, rest__}] := Flatten[
  Outer[Join,
    Place[{r}, {i}],
    Place[{objs}, {rest}]
  ]
];
Diagrams[R1d] := Place[{R1d}, {1}];
Diagrams[R1a] := Place[{R1a}, {1}];
Diagrams[R1d + k_. * Ar] /; k > 0 := Flatten[
  Place[#, Range[2 k + 1]] & /@ Permutations[Table[Ar, {k}] ~Append~ R1d]
];
Diagrams[R1a + k_. * Ar] /; k > 0 := Flatten[
  Place[#, Range[2 k + 1]] & /@ Permutations[Table[Ar, {k}] ~Append~ R1a]
];
Diagrams[R1d + k_. * Ar] /; k < 0 := {};
Diagrams[R1a + k_. * Ar] /; k < 0 := {};

Take[Diagrams[R3b + 1 Ar], 5]
{Diag[Ar[1, 2], R3b[3, 4, 5]],
 Diag[Ar[1, 2], R3b[3, 5, 4]], Diag[Ar[1, 2], R3b[4, 3, 5]],
 Diag[Ar[1, 2], R3b[4, 5, 3]], Diag[Ar[1, 2], R3b[5, 3, 4]]}

```

```
{Diagrams[R1d + Ar], Diagrams[R1a + Ar]}
{{Diag[Ar[1, 2], R1d[3]], Diag[Ar[2, 1], R1d[3]], Diag[Ar[1, 3], R1d[2]],
  Diag[Ar[3, 1], R1d[2]], Diag[Ar[2, 3], R1d[1]], Diag[Ar[3, 2], R1d[1]]},
 {Diag[Ar[1, 2], R1a[3]], Diag[Ar[2, 1], R1a[3]], Diag[Ar[1, 3], R1a[2]],
  Diag[Ar[3, 1], R1a[2]], Diag[Ar[2, 3], R1a[1]], Diag[Ar[3, 2], R1a[1]]}}
```

Generating Relations

```
NormalizeDiag[diag_Diag] := Module[
  {indices = Union@@ (List @@ diag /. Ar → List)},
  diag /. Thread[indices → Range[Length[indices]]]
];
R[Diag[lft___, R3b[i_, j_, k_], rgt___]] := (
  (* RHS of (9) *)
  NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], Ar[j+0.5, k+0.5], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[j+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, k], Ar[j, k+0.5], rgt]]
  (* LHS of (9) *)
  - NormalizeDiag[Diag[lft, Ar[j, k], Ar[i, k+0.5], Ar[i+0.5, j+0.5], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k], Ar[i+0.5, j], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, j+0.5], Ar[j, k], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k+0.5], Ar[j, k], rgt]]
);
RWrong1[Diag[lft___, R3b[i_, j_, k_], rgt___]] := (
  (* RHS of (9) *)
  NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], Ar[j+0.5, k+0.5], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[j+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, k], Ar[j, k+0.5], rgt]]
  (* LHS of (9) *)
  + NormalizeDiag[Diag[lft, Ar[j, k], Ar[i, k+0.5], Ar[i+0.5, j+0.5], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k], Ar[i+0.5, j], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, j+0.5], Ar[j, k], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k+0.5], Ar[j, k], rgt]]
);
RWrong2[Diag[lft___, R3b[i_, j_, k_], rgt___]] := (
  (* RHS of (9) *)
  NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], Ar[j+0.5, k+0.5], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[i+0.5, k], rgt]]
  + NormalizeDiag[Diag[lft, Ar[i, j], Ar[j+0.5, k], rgt]]
  - NormalizeDiag[Diag[lft, Ar[i, k], Ar[j, k+0.5], rgt]]
  (* LHS of (9) *)

```

```

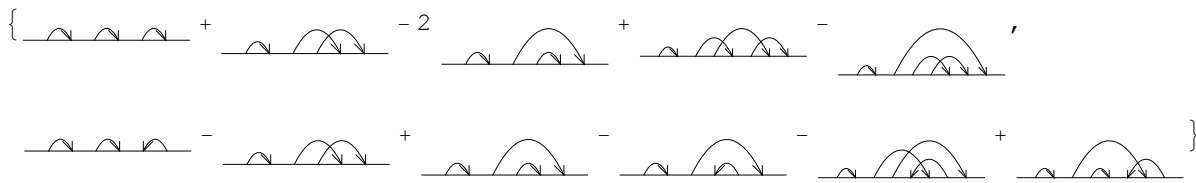
- NormalizeDiag[Diag[lft, Ar[j, k], Ar[i, k + 0.5], Ar[i + 0.5, j + 0.5], rgt]]
- NormalizeDiag[Diag[lft, Ar[i, k], Ar[i + 0.5, j], rgt]]
- NormalizeDiag[Diag[lft, Ar[i, j + 0.5], Ar[j, k], rgt]]
- NormalizeDiag[Diag[lft, Ar[i, k + 0.5], Ar[j, k], rgt]]
);
RMinus[n_, Diag[lft___, R3b[i_, j_, k_], rgt___]] := Module[
  {Expand2, Expand3, n1},
  n1 = n - Length[{lft, rgt}];
  Expand2[Diag[Ar[i1_, j1_], Ar[i2_, j2_]]] := Sum[
    (-1)^(k1 + k2) * NormalizeDiag[Flatten[Diag[
      lft,
      Diag@@Table[Ar[i1 + 0.01 q, j1 + 0.01 q], {q, k1}],
      Diag@@Table[Ar[i2 + 0.01 q, j2 + 0.01 q], {q, k2}],
      rgt
    ]]],
    {k1, n1 - 1}, {k2, n1 - k1}
  ];
  Expand3[Diag[Ar[i1_, j1_], Ar[i2_, j2_], Ar[i3_, j3_]]] := Sum[
    (-1)^(k1 + k2 + k3) * NormalizeDiag[Flatten[Diag[
      lft,
      Diag@@Table[Ar[i1 + 0.01 q, j1 + 0.01 q], {q, k1}],
      Diag@@Table[Ar[i2 + 0.01 q, j2 + 0.01 q], {q, k2}],
      Diag@@Table[Ar[i3 + 0.01 q, j3 + 0.01 q], {q, k3}],
      rgt
    ]]],
    {k1, n1 - 2}, {k2, n1 - k1 - 1}, {k3, n1 - k1 - k2}
  ];
  Expand[Plus[
    (
      Diag[Ar[i, j], Ar[i + 0.5, k]] +
      Diag[Ar[i, j], Ar[j + 0.5, k]] + Diag[Ar[i, k], Ar[j, k + 0.5]]
      - Diag[Ar[i, k], Ar[i + 0.5, j]] - Diag[Ar[i, j + 0.5], Ar[j, k]] -
      Diag[Ar[i, k + 0.5], Ar[j, k]]
    ) /. d_Diag => Expand2[d],
    (Diag[Ar[i, j], Ar[i + 0.5, k], Ar[j + 0.5, k + 0.5]] -
      Diag[Ar[j, k], Ar[i, k + 0.5], Ar[i + 0.5, j + 0.5]]) /. d_Diag => Expand3[d]
  ]
];
R[n_, Diag[lft___, R2c[i_, j_], rgt___]] := Module[
  {n1},
  n1 = n - Length[{lft, rgt}];
  Sum[
    Expand[(-1)^k * s * NormalizeDiag[Flatten[Diag[

```

```

    lft,
    Diag@@Table[Ar[i+0.01 q, j+s*0.01 q], {q, k}],
    rgt
  ]]],
  {k, 2, n1}, {s, -1, 1, 2}
]
];
R[n_, Diag[lft___, Rld[i_], rgt___]] := Module[
  {n1},
  n1 = n - Length[{lft, rgt}];
  Sum[
    Expand[(-1)^k * NormalizeDiag[Flatten[Diag[
      lft,
      Diag@@Table[Ar[i+0.01 q, i+0.5+0.01 q], {q, k}],
      rgt
    ]]],
    {k, 2, n1}
  ]
];
R[n_, Diag[lft___, Rla[i_], rgt___]] := Module[
  {n1},
  n1 = n - Length[{lft, rgt}];
  Sum[
    Expand[(-1)^k * NormalizeDiag[Flatten[Diag[
      lft,
      Diag@@Table[Ar[i+0.5+0.01 q, i+0.01 q], {q, k}],
      rgt
    ]]],
    {k, 2, n1}
  ]
];
R[Diag[lft___, TC[i_, j_, k_], rgt___]] := (
  +NormalizeDiag[Diag[lft, Ar[k, i], Ar[k+0.5, j], rgt]]
  - NormalizeDiag[Diag[lft, Ar[k+0.5, i], Ar[k, j], rgt]]
);
ContainsShortArrow[diag_Diag] := (1 == Min[List@@diag /. Ar[i_, j_] => Abs[i - j]]);
DescendingQ[diag_Diag] := And @@ (OrderedQ /@ diag);
Draw[R /@ Take[Diagrams[R3b + 1 Ar], 2]]

```

Producing "Quotient Spaces"

```
Options[PSpace] = {Debug → False, Gr → False, ReduceEquivalences → True};
PSpace[m_] /; m < 2 := QuotientSpace[Diagrams[m Ar], {}];
PSpace[m_, reltypes_List, opts___] /; m ≥ 2 := Module[
{
  debug = Debug /. {opts} /. Options[PSpace],
  gr = Gr /. {opts} /. Options[PSpace],
  reduceequivalences = ReduceEquivalences /. {opts} /. Options[PSpace],
  diags, rels, i, filter, equalities, equivclasses, pos, redrule
},
If[debug, Print[Date[], ": Starting work..."]];
diags = Join@@Table[Diagrams[k Ar], {k, If[gr, m, 1], m}];
filter = Identity;
reltypes /. {
  "R1" => (
    diags = Select[diags, !ContainsShortArrow[#] &]
  ),
  "Descending" => (
    filter = Select[#, DescendingQ] &;
    diags = Select[diags, DescendingQ]
  )
};
If[debug, Print[Date[], ": Computed ", Length[diags], " diagrams..."]];
rels = Join@@(reltypes /. {
  "R1" → {},
  "Descending" → {},
  "Round" => (diags /.
    d_Diag => (NormalizeDiag[d /. {1 → 2 Length[d], i_Integer => i - 1}] - d)
  ),
  "R3b" => (
    (R /@ Join@@Table[Diagrams[R3b + k Ar] // filter,
      {k, If[gr, m - 2, 0], m - 2}]) /. diag_Diag /; Length[diag] > m => 0
  ),
  "RWrong1" => (
    (RWrong1 /@ Join@@Table[Diagrams[R3b + k Ar],
      {k, If[gr, m - 2, 0], m - 2}]) /. diag_Diag /; Length[diag] > m => 0
  )
};
```

```

    ),
    "RWrong2" => (
      (RWrong2 /@ Join@@Table[Diagrams[R3b + k Ar],
        {k, If[gr, m - 2, 0], m - 2}] /. diag_Diag /; Length[diag] > m => 0
      ),
    ),
    "R3bMinus" => (
      RMinus[m, #] & /@
      Join@@Table[Diagrams[R3b + k Ar] // filter, {k, If[gr, m - 2, 0], m - 2}]
    ),
    "R2c" => (
      R[m, #] & /@
      Join@@Table[Diagrams[R2c + k Ar] // filter, {k, If[gr, m - 2, 0], m - 2}]
    ),
    "R1d" => (
      R[m, #] & /@
      Join@@Table[Diagrams[R1d + k Ar] // filter, {k, If[gr, m - 2, 0], m - 2}]
    ),
    "R1a" => (
      R[m, #] & /@
      Join@@Table[Diagrams[R1a + k Ar] // filter, {k, If[gr, m - 2, 0], m - 2}]
    ),
    "TC" => (
      R /@ Join@@Table[Diagrams[TC + k Ar], {k, If[gr, m - 2, 0], m - 2}]
    )
  });
reltypes /. {
  "R1" => (
    rels = DeleteCases[rels /. {d_Diag /; ContainsShortArrow[d] => 0}, 0]
  )
};
If[debug, Print[Date[], ": Computed ", Length[rels], " relations..."]];
If[reduceequivalences,
  equalities = Cases[rels, d1_Diag - d2_Diag => {d1, d2}];
  equivclasses = Fold[
    (
      pos = First /@ Position[#1, #2];
      Append[Delete[#1, List /@ pos], Union@@ (#1[[pos]])]
    ) &,
    equalities,
    Union@@equalities
  ];
redrule = Dispatch[Flatten[Table[
  class = equivclasses[[i]];

```

```
(# → First[class]) & /@ Rest[class],
{i, Length[equivclasses]}
]]];
diags = Union[diags /. redrule];
rels = Union[rels /. redrule];
];
QuotientSpace[diags, rels]
]
```

```
Draw[PSpace[2, {"R3b", "R2c"}]]
```



Computing Dimensions

```

Options[Dim] = {Debug → False, UseLinBox → False};
Dim[QuotientSpace[diags_List, rels_List], opts___] := Module[
{
  debug = Debug /. {opts} /. Options[Dim],
  uselinbox = UseLinBox /. {opts} /. Options[Dim],
  diagtoindex, mat, rel, matfile, rank
},
diagtoindex = Dispatch[Thread[Rule[diags, Range[Length[diags]]]];
If[uselinbox,
  matfile = OpenWrite["matfile.matrix"];
  WriteString[matfile, Length[rels], " ", Length[diags], " S\n"];
  Do[
    rel = rels[[i]];
    WriteString[matfile, StringJoin @@ Flatten[{
      ToString[Count[{rel}, _Diag, Infinity]], " ",
      {
        ToString[(# /. diagtoindex) - 1], " ",
        ToString[Coefficient[rel, #]], " "
      } & /@ Cases[{rel}, diag_Diag, Infinity],
      "\n"
    }]],
    {i, Length[rels]}
  ];
  matfile = Close[matfile];
  rank = RunThrough["./ComputeRank " <> matfile, Random[]][[2]];
  Length[diags] - rank,
  (* else *) mat = SparseArray[
    Join @@ Table[
      rel = rels[[i]];
      {i, # /. diagtoindex} → Coefficient[rel, #] & /@
        Cases[{rel}, diag_Diag, Infinity],
      {i, Length[rels]}
    ],
    {Length[rels], Length[diags]}
  ];
  If[debug, Print[Date[], ": Computed mat..."]];
  Length[diags] - MatrixRank[mat]
]
];
DimP[m_, reltypes_List, opts___] := Dim[PSpace[m, reltypes, opts], opts]

```

Analysis

```

Analyze[m_Integer, reltypes_List, opts___Rule] := Module[
  {qs},
  Table[
    {
      qs = PSpace[n, reltypes, Gr → True, opts];
      (Length /@ qs) → Dim[qs, opts],
      qs = PSpace[n, reltypes, Gr → False, opts];
      (Length /@ qs) → Dim[qs, opts]
    },
    {n, 2, m}
  ]
];
Analyze[reltypes_List, opts___Rule] := Analyze[4, reltypes, opts];

```

Runs

Runs as in the paper

```

Analyze["R3b", "R2c", "R1", "Round"] // MatrixForm
(
  QuotientSpace[1, 3] → 0      QuotientSpace[1, 3] → 0
  QuotientSpace[12, 29] → 1   QuotientSpace[12, 31] → 1
  QuotientSpace[107, 297] → 4  QuotientSpace[118, 336] → 5
)

Analyze["R3b", "R2c", "R1"] // MatrixForm
(
  QuotientSpace[4, 4] → 2      QuotientSpace[4, 4] → 2
  QuotientSpace[32, 37] → 7    QuotientSpace[34, 39] → 9
  QuotientSpace[460, 749] → 42 QuotientSpace[500, 829] → 51
)

Analyze["R3b", "R2c", "R1", "Descending"] // MatrixForm
(
  QuotientSpace[1, 1] → 0      QuotientSpace[1, 1] → 0
  QuotientSpace[3, 3] → 1      QuotientSpace[3, 4] → 1
  QuotientSpace[22, 23] → 6    QuotientSpace[27, 33] → 7
)

```

```

Analyze["R3b", "R2c", "Round"] // MatrixForm
(
  QuotientSpace[3, 3] → 1      QuotientSpace[4, 3] → 2
  QuotientSpace[16, 21] → 2    QuotientSpace[21, 24] → 4
  QuotientSpace[160, 329] → 7  QuotientSpace[187, 362] → 11
)

Analyze["R3b", "R2c"] // MatrixForm
(
  QuotientSpace[10, 7] → 5      QuotientSpace[12, 7] → 7
  QuotientSpace[96, 121] → 15   QuotientSpace[110, 129] → 22
  QuotientSpace[1332, 2401] → 67 QuotientSpace[1466, 2553] → 89
)

Analyze["R3b", "R2c", "Descending"] // MatrixForm
(
  QuotientSpace[2, 2] → 1      QuotientSpace[3, 2] → 2
  QuotientSpace[9, 11] → 2     QuotientSpace[13, 13] → 4
  QuotientSpace[63, 96] → 8    QuotientSpace[82, 114] → 12
)

```

Analyze["R3b", "R1", "R1d", "R1a", "Round"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[1, 3] \rightarrow 0 & \text{QuotientSpace}[1, 3] \rightarrow 0 \\ \text{QuotientSpace}[12, 29] \rightarrow 1 & \text{QuotientSpace}[12, 31] \rightarrow 1 \\ \text{QuotientSpace}[107, 297] \rightarrow 4 & \text{QuotientSpace}[118, 334] \rightarrow 5 \end{pmatrix}$$

Analyze["R3b", "R1d", "R1a", "R1"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[4, 4] \rightarrow 2 & \text{QuotientSpace}[4, 4] \rightarrow 2 \\ \text{QuotientSpace}[32, 37] \rightarrow 7 & \text{QuotientSpace}[34, 39] \rightarrow 9 \\ \text{QuotientSpace}[460, 749] \rightarrow 42 & \text{QuotientSpace}[500, 821] \rightarrow 51 \end{pmatrix}$$

Analyze["R3b", "R1d", "R1", "Descending"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[1, 1] \rightarrow 0 & \text{QuotientSpace}[1, 1] \rightarrow 0 \\ \text{QuotientSpace}[3, 3] \rightarrow 1 & \text{QuotientSpace}[3, 4] \rightarrow 1 \\ \text{QuotientSpace}[22, 23] \rightarrow 6 & \text{QuotientSpace}[27, 31] \rightarrow 7 \end{pmatrix}$$

Analyze["R3b", "Round"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[4, 3] \rightarrow 2 & \text{QuotientSpace}[5, 3] \rightarrow 3 \\ \text{QuotientSpace}[22, 21] \rightarrow 5 & \text{QuotientSpace}[27, 23] \rightarrow 8 \\ \text{QuotientSpace}[218, 353] \rightarrow 19 & \text{QuotientSpace}[245, 379] \rightarrow 27 \end{pmatrix}$$

Analyze["R3b"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[12, 6] \rightarrow 7 & \text{QuotientSpace}[14, 6] \rightarrow 9 \\ \text{QuotientSpace}[120, 120] \rightarrow 27 & \text{QuotientSpace}[134, 126] \rightarrow 36 \\ \text{QuotientSpace}[1680, 2520] \rightarrow 139 & \text{QuotientSpace}[1814, 2646] \rightarrow 175 \end{pmatrix}$$

Analyze["R3b", "Descending"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[3, 1] \rightarrow 2 & \text{QuotientSpace}[4, 1] \rightarrow 3 \\ \text{QuotientSpace}[15, 10] \rightarrow 6 & \text{QuotientSpace}[19, 11] \rightarrow 9 \\ \text{QuotientSpace}[105, 105] \rightarrow 24 & \text{QuotientSpace}[124, 116] \rightarrow 33 \end{pmatrix}$$

Analyze["R2c", "R1", "Round"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[1, 2] \rightarrow 0 & \text{QuotientSpace}[1, 2] \rightarrow 0 \\ \text{QuotientSpace}[12, 13] \rightarrow 4 & \text{QuotientSpace}[12, 13] \rightarrow 4 \\ \text{QuotientSpace}[107, 75] \rightarrow 44 & \text{QuotientSpace}[118, 82] \rightarrow 48 \end{pmatrix}$$

Analyze["R2c", "R1"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[4, 2] \rightarrow 2 & \text{QuotientSpace}[4, 2] \rightarrow 2 \\ \text{QuotientSpace}[32, 5] \rightarrow 28 & \text{QuotientSpace}[34, 5] \rightarrow 30 \\ \text{QuotientSpace}[460, 41] \rightarrow 420 & \text{QuotientSpace}[500, 51] \rightarrow 450 \end{pmatrix}$$

Analyze["R2c", "R1", "Descending"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[1, 1] \rightarrow 0 & \text{QuotientSpace}[1, 1] \rightarrow 0 \\ \text{QuotientSpace}[3, 2] \rightarrow 2 & \text{QuotientSpace}[3, 2] \rightarrow 2 \\ \text{QuotientSpace}[22, 5] \rightarrow 18 & \text{QuotientSpace}[27, 8] \rightarrow 20 \end{pmatrix}$$

Analyze["R2c", "Round"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[3, 1] \rightarrow 3 & \text{QuotientSpace}[4, 1] \rightarrow 4 \\ \text{QuotientSpace}[16, 1] \rightarrow 16 & \text{QuotientSpace}[21, 2] \rightarrow 20 \\ \text{QuotientSpace}[160, 1] \rightarrow 160 & \text{QuotientSpace}[187, 8] \rightarrow 180 \end{pmatrix}$$

Analyze["R2c"] // **MatrixForm**

$$\begin{pmatrix} \text{QuotientSpace}[10, 1] \rightarrow 10 & \text{QuotientSpace}[12, 1] \rightarrow 12 \\ \text{QuotientSpace}[96, 1] \rightarrow 96 & \text{QuotientSpace}[110, 3] \rightarrow 108 \\ \text{QuotientSpace}[1332, 1] \rightarrow 1332 & \text{QuotientSpace}[1466, 27] \rightarrow 1440 \end{pmatrix}$$

Analyze["R2c", "Descending"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[2, 1] \rightarrow 2 & \text{QuotientSpace}[3, 1] \rightarrow 3 \\ \text{QuotientSpace}[9, 1] \rightarrow 9 & \text{QuotientSpace}[13, 2] \rightarrow 12 \\ \text{QuotientSpace}[63, 1] \rightarrow 63 & \text{QuotientSpace}[82, 8] \rightarrow 75 \end{array} \right)$$

Other runs

Analyze["R3b", "TC", "R1"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[4, 5] \rightarrow 1 & \text{QuotientSpace}[4, 5] \rightarrow 1 \\ \text{QuotientSpace}[26, 51] \rightarrow 1 & \text{QuotientSpace}[30, 60] \rightarrow 2 \\ \text{QuotientSpace}[267, 777] \rightarrow 2 & \text{QuotientSpace}[297, 844] \rightarrow 4 \end{array} \right)$$

Analyze["R3b", "TC"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[9, 7] \rightarrow 4 & \text{QuotientSpace}[11, 7] \rightarrow 6 \\ \text{QuotientSpace}[64, 97] \rightarrow 7 & \text{QuotientSpace}[75, 103] \rightarrow 13 \\ \text{QuotientSpace}[625, 1501] \rightarrow 12 & \text{QuotientSpace}[700, 1603] \rightarrow 25 \end{array} \right)$$

Analyze["R3b", "TC", "R1", "Round"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[1, 3] \rightarrow 0 & \text{QuotientSpace}[1, 3] \rightarrow 0 \\ \text{QuotientSpace}[8, 29] \rightarrow 0 & \text{QuotientSpace}[9, 34] \rightarrow 0 \\ \text{QuotientSpace}[54, 253] \rightarrow 0 & \text{QuotientSpace}[63, 298] \rightarrow 0 \end{array} \right)$$

Analyze["R3b", "TC", "Round"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[3, 3] \rightarrow 1 & \text{QuotientSpace}[4, 3] \rightarrow 2 \\ \text{QuotientSpace}[11, 15] \rightarrow 1 & \text{QuotientSpace}[15, 17] \rightarrow 3 \\ \text{QuotientSpace}[70, 185] \rightarrow 1 & \text{QuotientSpace}[85, 205] \rightarrow 4 \end{array} \right)$$

Analyze["R3b", "R1", "Round"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[1, 3] \rightarrow 0 & \text{QuotientSpace}[1, 3] \rightarrow 0 \\ \text{QuotientSpace}[12, 29] \rightarrow 1 & \text{QuotientSpace}[13, 33] \rightarrow 1 \\ \text{QuotientSpace}[107, 297] \rightarrow 4 & \text{QuotientSpace}[122, 345] \rightarrow 5 \end{array} \right)$$

Analyze["R3b", "R1"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[4, 4] \rightarrow 2 & \text{QuotientSpace}[4, 4] \rightarrow 2 \\ \text{QuotientSpace}[32, 37] \rightarrow 7 & \text{QuotientSpace}[36, 43] \rightarrow 9 \\ \text{QuotientSpace}[460, 749] \rightarrow 42 & \text{QuotientSpace}[504, 827] \rightarrow 51 \end{array} \right)$$

Analyze["R3b", "R1", "Descending"] // **MatrixForm**

$$\left(\begin{array}{cc} \text{QuotientSpace}[1, 1] \rightarrow 0 & \text{QuotientSpace}[1, 1] \rightarrow 0 \\ \text{QuotientSpace}[3, 3] \rightarrow 1 & \text{QuotientSpace}[4, 4] \rightarrow 1 \\ \text{QuotientSpace}[22, 23] \rightarrow 6 & \text{QuotientSpace}[28, 30] \rightarrow 7 \end{array} \right)$$