Dimensions of Spaces of Finite Type invariants of Virtual Knots

August 27, 2009

D. BAR-NATAN, I. HALACHEVA, L. LEUNG, F. ROUKEMA

Abstract

The study of finite type invariants is central to the development of knot theory. Much of the theory still needs to be extended to the newer virtual context. In this article we calculate the dimensions of the spaces of virtual finite type knot invariants and associated graded algebras for several classes of virtual knots to orders four and five. The data obtained highlights a certain pattern on all the "reasonable" classes of knots that we considered, and in turn supports the conjecture that all weight systems integrate.

1 Finite Type Invariants of Virtual Knots

Kauffman's theory of virtual knots extends the standard theory, see [Kau]. A type n invariant in the virtual context vanishes on virtual knots containing greater than n semi-virtual crossings, where a semi-virtual over crossing, $\semivirtualover$, is given by $\overcrossing - \virtualcrossing$, and a semi-virtual under crossing, $\semivirtualunder$, by $-\virtualcrossing + \undercrossing$. Thus double points are the difference of semi-virtual crossings, and virtual type n invariants restrict to standard finite type invariants of type less than or equal to n.

The first result in this article is the computation of the dimensions of the spaces of virtual finite type invariants and long virtual finite type invariants. They are given to order five in table 1, and were calculated using a code written for the purposes of this paper, see Section 3. The original source code can be found at [Code].

Using the same code, we were able to compute the dimensions of the associated graded spaces as in the sense of Polyak, see [Pol], but also with framing independence. The results are tabulated in table 1.

k	round	long
2	0	2
3	1	9
4	5	51
5	22	?

Table 1: Dimensions of the spaces of virtual finite type invariants

k	round	long
2	0	2
3	1	7
4	4	42
5	17	?

Table 2: Dimensions of the associated graded spaces

k	2	3	4	5
round	0	1	5	?
long	2	9	51	297

Table 3: Dimensions of the spaces of virtual finite type invariants

k	2	3	4	5
round	0	1	4	?
long	2	7	42	246

Table 4: Dimensions of the associated graded spaces

Dimensions of the associated spaces of weight systems

Looking at the table, we see a striking contrast between how the size of the integers in the round column and the long column differs, but there is much more. The k^{th} dimension of the associated algebras plus the $k-1^{st}$ dimension of the space of finite type invariants equals the k^{th} dimension of the space of finite type invariants. This pattern empiricaly supports the following conjecture which is well known in lore, see Section **??** for details.

Conjecture 1.1 *Every weight system integrates to a virtual finite type invariant.*

Section 2 looks at further classes of virtual knots and how the same pattern continues, Section **??** looks at what this means and how the data suports conjecture 1.1, and Section 3 gives details about the code.

Acknowledgement. Bradford Hovinen, NSERC...

2 Variations of the finite type invariant spaces and their dimensions

Instead of the space of long virtual knot diagrams, $\mathcal{VKD}$, modulo the R1, R2 and R3 moves, we can consider other quotient spaces where we take the quotient of $\mathcal{VKD}$ by a subset of the set the Reidemester moves R1, $R2^b$, $R2^c$, $R3^b$ ans $R2^c$ where the superscripts b and c stand for "braid-like" and "cyclic", referring to whether the orientations of the strands all point in one irection or form a cycle. These moves are shown in Figures 2 and 2 below.

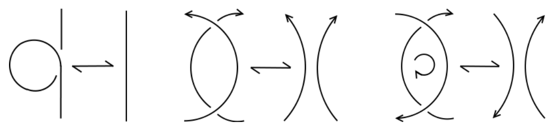


Figure 1: RI, $R2^b$ and $R2^c$

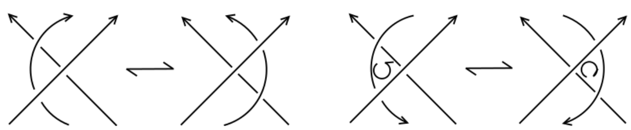


Figure 2: $R3^b$ and $R3^c$

The Reidemeister three move can also be split into $R3^-$ and $R3^+$, namely $R3$ with only positive crossings. Yet, given $R2^b$, $R3^+$ implies $R3^-$ so we need only consider $R3^+$. Further relationships between those six moves are as follows:

1. $R1$, $R2^b$, and $R3^c$ together imply $R2^c$.
2. $R3^b$ and $R2^c$ give us $R3^c$.

It is of interest to consider the spaces of finite type invariants on these different quotient spaces of virtual knot diagrams. We have computed the dimensions of some of these spaces in the table below. For the original code see [Code]. The notation for the table is as follows: If $\mathcal{W}_m$ refers to the quotient space of virtual knot diagrams $\mathcal{VKD}/\{R3^b, R2^b\}$ where we set diagrams with more than m real crossings to be 0, then $\mathcal{V}_m$ is the space of finite type invariants on $\mathcal{W}_m$. The superscripts in the table indicate further relations that have been included in the quotient: "1" refers to the R1 move and leads to framing independence, "2c" refers to the $R2^c$ move, "no3b" means not including the $R3^b$ relation in the quotient,"o" refers to the round knot case and "oc" refers to the "overcrossings commute" relation, illustrated in Figure 2, which deals with the w-knots case described in [BN2]. In each box of the table, the first number gives the non-graded and the second number gives the graded space dimension.

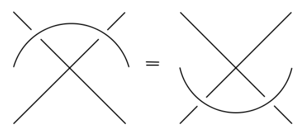


Figure 3: Overcrossings Commute

m	$\mathcal{V}_m$	$\mathcal{V}_m^{oc}$	$\mathcal{V}_m^{2c}$	$\mathcal{V}_m^{o,no3b}$	$\mathcal{V}_m^{2c,no3b}$
2	9, 7	6, 4	7, 5	5, 4	12, 10
3	36, 27	13, 7	22, 15	27, 22	108, 96
4	175, 139	25, 12	89, 67	245, 218	1440, 1332
5	?	?	?	?	?

Table 5: Computed Dimensions without Framing Independence

m	$\mathcal{V}_m^1$	$\mathcal{V}_m^{1,oc}$	$\mathcal{V}_m^{1,2c}$	$\mathcal{V}_m^{1,oc,o}$	$\mathcal{V}_m^{1,2c,o}$	$\mathcal{V}_m^{1,2c,no3b}$
2	2, 2	1, 1	2, 2	0, 0	0, 0	2, 2
3	9, 7	2, 1	9, 7	0, 0	1, 1	30, 28
4	51, 42	4, 2	51, 42	0, 0	5, 4	450, 420
5	?	?	297, 242	?	?	8258, 7808

Table 6: Computed Dimensions with Framing Independence

Comparison of the obtained results with those in [BN2] suggests that Conjecture 1. can be extended to all studied classes of knots:

Conjecture 2.1 *For all considered classes of virtual knot diagrams, every weight system integrates.*

Even more generally, it is reasonable to conjecture, based on the wide range of supporting results, that using any set of meaningful knot theoretic relations every weight system integrates. The additivity of dimensions described in the previous section between the spaces of finite type invariants and the corresponding graded spaces is also observable for the considered variations, which further supports the conjecture.

3 The Method

We start with the motivation for the method of computation. The origins lie with the Polyak algebra, and the account we present here comes mostly from [GPV]. To begin, we need the following definition:

Definition The map s on a virtual knot diagram is the sum of all subdiagrams, while the map s_n is the map that truncates this sum to contain only diagrams with less than or equal to n crossings.

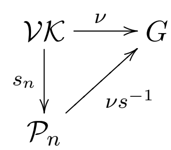
Here, a subdiagram is the obvious thing, simply a diagram obtained by virtualising some number of crossings. With this in mind, the maps s and s_n are written schematically by ⤧ → ⤧ + ×, thus s^{-1} exists and sends real crossings to semi-virtual crossings. For details see [GPV], [Rou].

Now, the map s_n is only well defined on diagrams, and we require a map on knots. To account for this, we need to keep track of the Reidemeister moves. In particular, we need to factor out our domain by the Reidemeister moves and our range by s of the Reidemeister relations. This gives a map from the space of virtual knots, $\mathcal{VK}$, to the so called *Polyak Algebra*, $\mathcal{P}$. Of course, now s becomes an isomorphism of the space generated by knots.

The point is that any virtual type n invariant, ν, factors through the map $s_n : \mathcal{VK} \to \mathcal{P}_n$ precisely because s^{-1} sends real crossings to semi-virtual crossings and ν is a of type n;

$$\nu(D) = \nu s^{-1} s(D) = \nu s^{-1} s_n(D) + \nu s^{-1}(\Sigma_{|C_i|>n} C_i) = \nu s^{-1} s_n(D)$$

We obtain a commutative diagram;



In other words, $s_n : \mathcal{VK} \to \mathcal{P}_n$ is a universal type n invariant.

Now $\mathcal{P}_n$ is finite dimensional, and so ν is determined by its values on a basis of $\mathcal{P}_n$, a finite set. Letting $\mathcal{B}$ be a basis for $\mathcal{P}_n$ we can see that $\Sigma_{D \in \mathcal{B}}(\nu s^{-1}(D)D)$ is a *Gauss Diagram Formula* for ν, see [GPV], [**?**], [Rou]. As a consequence, every virtual finite type invariant has a Gauss diagram formula.

Starting at the other end, all elements of $\mathcal{P}_n^*$ correspond to virtual finite type invariants under composition with s_n because real crossings may be expressed as a sum of semi-virtual crossings vitualized crossings, and s_n sends these crossings to real crossings before truncating. Further, elements of $\mathcal{P}_n$ implicitly subsume s_n of the Reidemeister moves making the composition of our functional with s_n an invariant.

Collecting the commentary together we get:

Proposition 3.1 *The space of virtual type n invariants and $\mathcal{P}_n$ are isomorphic.*

Thence, knowing the dimension of the space of virtual type n invariants boils down to knowing the dimension of $\mathcal{P}_n$, and this is what we calculate. To do this we use the original relations given by Polyak, **??**, which contain all variations of the Reidemeister moves, see Figures 4, 5 and 6.

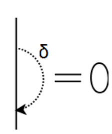


Figure 4: The image of R1 under s.

In Section 2 we considered different classes of virtual knots and the again computed the dimensions of the corresponding spaces of virtual type n invariants. This was done in the analagous way, with the relations on the space of arrow diagrams now being obtained by pushing forward combinations of the relations from Figures 2 and 2.

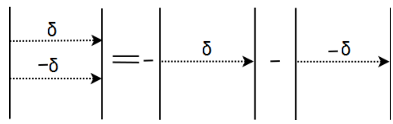


Figure 5: The image of R2 under s.

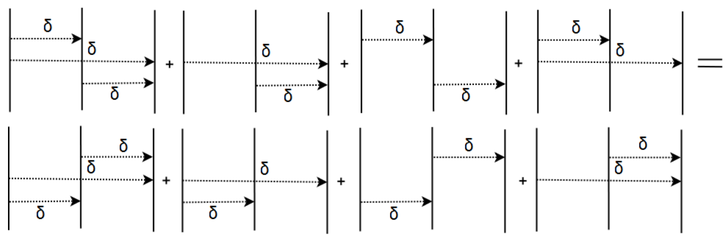


Figure 6: The image of R3 under s.

The actual computations in this paper come from a computer program, [?], so, what does the program do? Firstly the program finds all arrow diagrams of fixed order, followed by all relations which are generated from the initial relations we choose to impose, finally we are left with linear algebra and it remains only to find the rank of a sparse matrix. We collect the number of diagrams and number of relations for the spaces we considered in Sections **??** and **??** in table 7.

The two numbers for each space and dimension are collected in boxes with the top number representing the number of diagrams and the bottom number representing the number of relations.

A final comment is that the spaces of arrow algebras modulo the specified relations are all isomorphic to their dual spaces and the elements correspond to the Gauss diagram formulas for virtual finite type invariants by the above commentary. Thus, by computing the basis of the relevant arrow diagram space, our code computes the dual basis which, in turn, corresponds to the all Gauss diagram formulas for virtual finite type invariants. The code is able to draw the elements of the arrow spaces, thereby finding all Gauss diagram formulas to order four or five depending on the class of virtual knot considered.

m	$\mathcal{V}_m$	$\mathcal{V}_m^1$	$\mathcal{V}_m^{tc}$	$\mathcal{V}_m^{1,tc}$	$\mathcal{V}_m^{2c}$	$\mathcal{V}_m^{2c,gr}$	$\mathcal{V}_m^{1,2c}$	$\mathcal{V}_m^{1,2c,gr}$	$\mathcal{V}_m^{o}$	$\mathcal{V}_m^{1,tc,o}$	$\mathcal{V}_m^{1,2c,o}$
2	14	4	14	4	?	?	?	?	?	?	?
	6	6	9	9	?	?	?	?	?	?	?
3	134	44	134	44	?	?	?	?	?	?	?
	126	126	189	189	?	?	?	?	?	?	?
4	1814	620	1814	620	?	?	?	?	?	?	?
	2646	2646	3969	3969	?	?	?	?	?	?	?
5	?	11148	?	?	?	?	?	?	?	?	?
	?	63126	?	?	?	?	?	?	?	?	?

Table 7: Number of Diagrams and Relations

4 Polyak's Other Algebra and Integration of Weight Systems

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This section generalizes the notions of weight system and Polyak's algebra of arrow diagrams in [Pol] to our new setting where not all Reidemeister moves are allowed. We state a conjecture, based on low-dimension calculations, that in this new setting all weight systems integrate to corresponding finite type invariants.

Let $\mathcal{P}_n^R$ be any variation of the Polyak algebra (i.e. the image under s of virtual knot diagrams modulo R where R contains $R2^b$, $R3^b$ and possibly other relations). We consider the sequence of subsets

$$0 = \Pi_{n,n+1}^R \subseteq \Pi_{n,n}^R \subseteq \cdots \subseteq \Pi_{n,1}^R \subseteq \Pi_{n,0}^R = \mathcal{P}_n^R$$

where each $\Pi_{n,k}$ is comprised of diagrams of degree $\geq k$ and $\leq n$. Note for $m, n \geq k$, $\Pi_{n,k}^R/\Pi_{n,k+1}^R \cong \Pi_{m,k}^R/\Pi_{m,k+1}^R$. For each k, we define

$$\mathcal{G}_k^R := \Pi_{n,k}^R/\Pi_{n,k+1}^R,$$

where n is some number greater than or equal to k. Relations in $\mathcal{G}_k^R$ are now different from those in $\mathcal{G}_k^R$. For example, since diagrams with more than k arrows are quotiented out, the two terms of the higher degree in the 8T relation are dropped, and 8T becomes 6T. (See figure 7.)

We have a function

$$\iota_k : \mathcal{G}_k^R \longrightarrow \mathcal{P}_k^R$$

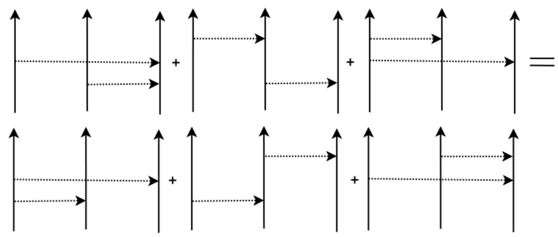

Figure 7: The 6T relation

which maps each diagram in $\mathcal{G}_k^R$ to itself, so every diagram in $\mathcal{P}_k^R$ with k arrows has a preimage uner this map. Since ι_k may or may not be injective,

$$\sum_{k=1}^{n} \dim(\mathcal{G}_k^R) = \dim(\bigoplus_{k=1}^{n} \mathcal{G}_k^R) \geq \dim(\mathcal{P}_n^R). \tag{1}$$

We can think of $\mathcal{G}_k^R$ as the "associated graded space" of degree k of $\mathcal{P}_n^R$. The map ι_k induces a map $\iota_k^* : (\mathcal{P}_k^R)^* \to (\mathcal{G}_k^R)^*$ A weight system is an element of $(\mathcal{G}_k^R)^*$. We say a weight system of degree k integrates to a type-k invariant if it is in the image of ι_k^*. Since $(\mathcal{P}_n^R)^*$ is the space of invariants of at most type n ([GPV]), if all weight systems integrate to finite type invariants, the adjoint map $\iota_n^* : (\mathcal{P}_n^R)^* \to (\mathcal{G}_n^R)^*$ is surjective. This is equivalent to ι_n being injective.

Therefore all weight systems of associated graded spaces integrate to finite type invariants if all ι_n are injective. We have the following proposition.

(a) $\dim(\mathcal{P}_n^R) \leq \sum_{k=1}^{n} \dim(\mathcal{G}_k^R)$.

(b) $\dim(\mathcal{P}_n^R) = \sum_{k=1}^{n} \dim(\mathcal{G}_k^R)$ if and only if $\iota_k : \mathcal{G}_k^R \to \mathcal{P}_k$ is injective for all $k \leq n$.

Proof First we prove (a). The proof is by induction. Since there are no relations equating diagrams of degree 0 and diagrams of degree 1, $\mathcal{G}_1 = \mathcal{P}_1$, and $\dim(\mathcal{G}_1) = \dim(\mathcal{P}_1)$. Suppose, as our inductive hypothesis,

$$\dim(\mathcal{P}_{m-1}) \leq \sum_{k=1}^{m-1} \dim(\mathcal{G}_k^R).$$

As a vector space, $\mathcal{P}_m$ is isomorphic to the direct sum of $\mathcal{P}_m/\iota_m(\mathcal{G}_m^R)$ and $\iota_m(\mathcal{G}_m^R)$. Since $\iota_m(\mathcal{G}_m^R)$ contains exactly the diagrams with m arrows, $\mathcal{P}_m/\iota_m(\mathcal{G}_m^R) =$

$\mathcal{P}_{m-1}$. Therefore

$$\dim(\mathcal{P}_m) = \dim(\mathcal{P}_m/\iota_m(\mathcal{G}_m^R))+\dim(\iota_m(\mathcal{G}_m^R)) \leq \dim(\mathcal{P}_{m-1})+\dim(\mathcal{G}_m^R) \leq \sum_{k=1}^{m} \dim(\mathcal{G}_k^R). \tag{2}$$

To prove (b), we observe that if ι_k is injective for all $k \leq n$, the two inequalities above become equalities. If $\iota_{k'}$ is not injective for some k', then the middle inequality becomes a strict inequality for $m = k'$, so for all $n \geq k$, $\dim(\mathcal{P}_n^R) < \sum_{k=1}^{n} \dim(\mathcal{G}_k^R)$

Since injectivity of ι_k is equivalent to all degree-k weight systems integrate, the proposition above, together with Tables **??** and **??**, tells us that all weight systems integrate up to degree 4. Hoping that the same trend continues into higher dimensions, we conjecture the following:

Weight systems of $\mathcal{G}_n^{\{R2^b,R3^b\}}$, $\mathcal{G}_n^{\{R2^b,R2^c,R3^b\}}$ and $\mathcal{G}_n^{\{R1,R2^b,R2^c,R3^b\}}$ integrate to corresponding finite type invariants.

In particular, if R consists of only $R2^b$ and $R3^b$, $\mathcal{G}_n^R$ is the algebra $\vec{\mathcal{A}}_n$ of arrow diagrams mod 6T ([Pol]). (Recall figure 7.) The spaces $\vec{\mathcal{A}}_n$ are of particular interest because of their relations to Lie bialgebras ([BN2],[Pol],[Leu]). Proposition 4 tells us that all weight systems of $\vec{\mathcal{A}}_n$ integrate to finite type invariants of virtual knot diagrams modulo $R2^b$ and $R3^b$ up to degree 4. Conjecture 4 suggests that the same is true for all degrees. Note, however, that since 27¿22 and 139¿89, the column $\mathcal{V}_m^{2c}$ and Proposition 4 together tell us that not all weight systems (functionals) of arrow diagrams modulo 6T integrate to finite type invariants which respect $R2^c$. This suggests that if our study of finite type invariants is entirely motivated by $\vec{\mathcal{A}}_n$, we should focus on virtual knot diagrams modulo only $R2^b$ and $R3^b$. A challenge is to come up with topological interpretations of such objects.

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