Dimensions of Spaces of Finite Type Invariants of Virtual Knots

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Abstract

The study of finite type invariants is central to the development of knot theory. Much of the theory still needs to be extended to the newer virtual context. In this article, we calculate the dimensions of the spaces of virtual finite type knot invariants and associated graded algebras for several classes of virtual knots to orders four and five. The data obtained highlights a certain pattern on all the "reasonable" classes of knots that we considered, and in turn supports the conjecture that all weight systems integrate.

1 Finite Type Invariants of Virtual Knots

Kauffman's theory of virtual knots extends the standard theory, see [5]. A type n invariant in the virtual context vanishes on virtual knots containing greater than n semi-virtual crossings, where a semi-virtual overcrossing, $\times$, is given by $\times - \times$, and a semi-virtual undercrossing, $\times$, by $-\times + \times$. Thus, double points are the difference of semi-virtual crossings and virtual type n invariants restrict to standard finite type invariants of type less than or equal to n.

The first result in this article is the computation of the dimensions of the spaces of virtual finite type invariants and long virtual finite type invariants. They are given to order five in table 1, and were calculated using a code written for the purposes of this paper, see Section 4. The original source code can be found at [10]. [BN3]

k	round	long
2	0	2
3	1	9
4	5	51
5	?	?

Transpose table

297

Table 1: Dimensions of the spaces of virtual finite type invariants

Using the same code, we were able to compute the dimensions of the associated graded algebras. The results are tabulated in table 2.

Spaces

Looking at the table, we see a striking contrast between how the size of the integers in the round column and the long column differs, but there is much more. The k^{th} dimension of the associated algebras plus the $(k-1)^{st}$ dimension of the space of finite type invariants equals

associated space of arrow diagrams in the sense of Polyak [Pol] (also moded out by FI, see Figure ...)

k	round	long
2	0	2
3	1	7
4	4	42
5	?	?

Table 2: Dimensions of the associated graded spaces

the k^{th} dimension of the space of finite type invariants. This pattern empiricaly supports the following conjecture which is well known in lore, see Section 3 for details.

Conjecture 1.1. *Every weight system integrates to a virtual finite type invariant.*

Section 2 looks at further classes of knots and how the same pattern continues, Section 3 looks at what this means and how the data suports Conjecture 1.1, and Section 4 gives details about the code.

2 Variations of the Finite Type Invariant Spaces and Their Dimensions

Instead of the space of long virtual knot diagrams, $\mathcal{VKD}$, modulo the R1, R2 and R3 moves, we can consider other quotient spaces where we take the quotient of $\mathcal{VKD}$ by a subset of the set the Reidemester moves. The map $s : \mathcal{VKD} \longrightarrow \mathcal{VKD}$ taking a diagram to the sum of all its subdiagrams, which is discussed in [9] and [3] and leads to the construction of a universal finite-type invariant, takes the Reidemeister moves to the following relations respectively, which also appear in [3]. The dotted arrows represent semivirtual crossings.

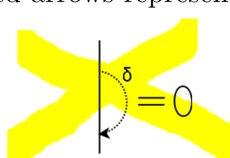


Figure 1: The image of R1 under s.

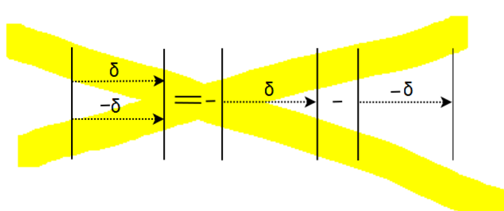
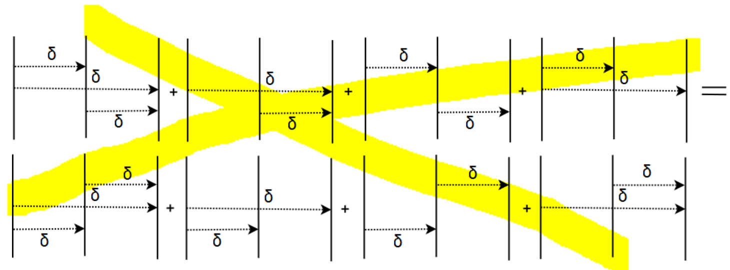


Figure 2: The image of R2 under s.

Depending on the orientation of the vertical arrows, each of these relations corresponds to two versions of a Reidemeister move. For the first relation, we have the correspondence with $R1^+$ or $R1^-$, where the sign refers to the sign of the crossing. The second relation corresponds to $R2^b$ or $R2^c$, namely a braid-like or cyclic Reidemeister two move depending on whether all



-> "The Method"

Figure 3: The image of R3 under s.

the strands point in the same direction or not. Similarly, he relation in Figure 3 corresponds to $R3^b$ or $R3^c$. These moves are shown in the Figures 2 and 5 below.

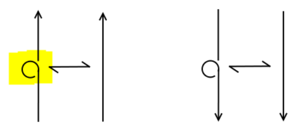


Figure 4: RI^+ and RI^-

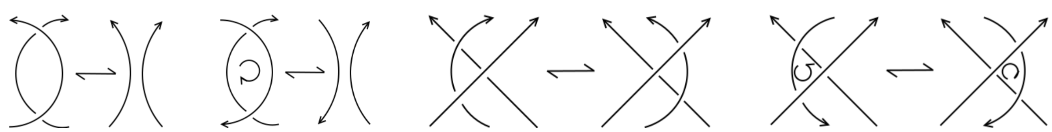


Figure 5: $R2^b$, $R2^c$, $R3^b$ and $R3^c$

The Reidemeister three move can also be split into $R3^-$ and $R3^+$, namely $R3$ with only positive crossings. Yet, given $R2^b$, $R3^+$ implies $R3^-$ so we need only consider $R3^+$. Further relationships between those six moves are as follows:

1. $R1$, $R2^b$, and $R3^c$ together imply $R2^c$.
2. $R3^b$ and $R2^c$ give us $R3^c$.

It is of interest to consider the spaces of finite type invariants on these different quotient spaces of virtual knot diagrams. We have computed the dimensions of some of these spaces in the table below. For the original code see [10]. The notation for the table is as follows: If $\mathcal{W}_m$ refers to the quotient space of virtual knot diagrams $\mathcal{VKD}/\{R3^b, R2^b\}$ where we set diagrams with more than m real crossings to be 0, then $\mathcal{V}_m$ is the space of finite type invariants on $\mathcal{W}_m$. The superscripts in the table indicate further relations that have been included in the quotient: "1" refers to the R1 move and leads to framing independence, "2c" refers to the $R2^c$ move, "no3b" means not including the $R3^b$ relation in the quotient, o refers to the round knot case and tc refers to the "tails commute" relation, illustrated in figure 6, which deals with the w-knots case described in [1]. In each box of the table ,the first number gives the non-graded and the second number gives the graded dimension.

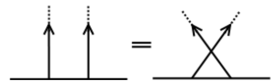


Figure 6: Tails Commute

Overcrossings Commute

m	$\mathcal{V}_m$	$\mathcal{V}_m^{tc}$	$\mathcal{V}_m^{2c}$	$\mathcal{V}_m^{o,no3b}$	$\mathcal{V}_m^{2c,no3b}$
2	9, 7	6, 4	7, 5	5, 4	12, 10
3	36, 27	13, 7	22, 15	27, 22	108, 96
4	175, 139	25, 12	89, 67	245, 218	1440, 1332
5	?	297	?	?	?

Table 3: Computed Dimensions without Framing Independence

Comparison of the obtained results with those in [1] suggests that Conjecture 1. can be extended to all studied classes of knots:

Conjecture 2.1. *For all considered classes of virtual knot diagrams, every weight system integrates.*

Even more generally, it is reasonable to conjecture, based on the wide range of supporting results, that using any set of meaningful knot theoretic relations, every weight system integrates. The additivity of dimensions between the spaces of finite type invariants and the associated algebras described in the previous section is also observable for the considered variations, which further supports the conjecture.

3 Interpretation

Let $\mathcal{W}_n$ be any variation of the space of virtual knot diagrams (i.e. the image under s of $\mathcal{VKD}/R$ where R contains $R2^b$, $R3^b$ and possibly other relations). We consider the sequence of subsets

$$0 = \Pi_{n,n+1} \subseteq \Pi_{n,n} \subseteq \cdots \subseteq \Pi_{n,1} \subseteq \Pi_{n,0} = \mathcal{W}_n$$

where each $\Pi_{n,k}$ is comprised of diagrams of degree $\geq k$ and $\leq n$. Note for $m, n \geq k$, $\Pi_{n,k}/\Pi_{n,k+1} \cong \Pi_{m,k}/\Pi_{m,k+1}$. For each k, we define

$$\mathcal{G}_k := \Pi_{n,k}/\Pi_{n,k+1},$$

m	$\mathcal{V}_m^1$	$\mathcal{V}_m^{1,tc}$	$\mathcal{V}_m^{1,2c}$	$\mathcal{V}_m^{1,tc,o}$	$\mathcal{V}_m^{1,2c,o}$	$\mathcal{V}_m^{1,2c,no3b}$
2	2, 2	1, 1	2, 2	0, 0	0, 0	2, 2
3	9, 7	2, 1	9, 7	0, 0	1, 1	30, 28
4	51, 42	4, 2	51, 42	0, 0	5, 4	450, 420
5	?	?	?	?	?	8258, 7808

Table 4: Computed Dimensions with Framing Independence

where n is some number greater than or equal to k. We have a function

$$\iota_k : \mathcal{G}_k \longrightarrow \mathcal{W}_k$$

which maps each diagram in $\mathcal{G}_k$ to itself, so every diagram in $\mathcal{W}_k$ with k arrows has a preimage.

$$\sum_{k=1}^{n} \dim(\mathcal{G}_k) = \dim(\bigoplus_{k=1}^{n} \mathcal{G}_k) \geq \dim(\bigcup_{k=1}^{n} \mathcal{W}_{n,k}) = \dim(\mathcal{W}_n). \tag{1}$$

If $\mathcal{F}$ is the base field we can draw the following commutative diagram:

$$\begin{array}{ccc} \mathcal{W}_n & \xrightarrow{\nu s^{-1} s_n} & \mathcal{F} \\ \uparrow{\scriptstyle \iota_n} & \nearrow{\scriptstyle \nu s^{-1} s_n \iota_n} & \\ \mathcal{G}_n & & \end{array} .$$

Since $\mathcal{W}_n^*$ is the space of invariants of at most type n ([3]), if all weight systems integrate to finite type invariants, the adjoint map $\iota_n^* : \mathcal{W}_n^* \to \mathcal{G}_n^*$ is surjective. This is equivalent to ι_n being injective.

Definition 1. *A weight system is an element of* $\mathcal{G}_k^*$

Therefore all weight systems integrate to finite type invariants if all ι_n are injective. We have the following proposition.

Proposition 3.1. *If the map* $\iota_k : \mathcal{G}_k \to \mathcal{W}_k$ *is injective for all* $k \leq n$*, then given* $m \leq n$*,*

$$\dim(\mathcal{W}_m) = \sum_{k=1}^{m} \dim(\mathcal{G}_k).$$

In particular $\dim(\mathcal{W}_n) = \sum_{k=1}^{n} \dim(\mathcal{G}_k)$.

Proof. The proof is by induction. Since there are no relations equating diagrams of degree 0 and diagrams of degree 1, $\mathcal{G}_1 = \mathcal{W}_1$, and $\dim(\mathcal{G}_1) = \dim(\mathcal{W}_1)$. Suppose, as our inductive hypothesis,

$$\sum_{k=1}^{m-1} \dim(\mathcal{G}_k) = \dim(\mathcal{W}_{m-1}).$$

Since $\iota_m : \mathcal{G}_m \to \mathcal{W}_m$ is injective, and the image contains exactly the diagrams with m arrows,*as a vector space*, $\mathcal{W}_m$ is a direct sum of $\mathcal{W}_{m-1}$ and $\mathcal{G}_m$. Therefore

$$\dim(\mathcal{W}_m) = \dim(\mathcal{W}_{m-1}) + \dim(\mathcal{G}_m) = \sum_{k=1}^{m} \dim(\mathcal{G}_k)$$

By looking at the tables we have evidence suggesting that all weight systems integrate. Since the contrary would be too much coincidence, we conjecture the following:

Conjecture 3.2. *Weight systems of* $\mathcal{G}_n^{\{R2^b,R3^b\}}$, $\mathcal{G}_n^{\{R2^b,R2^c,R3^b\}}$ *and* $\mathcal{G}_n^{\{R1,R2^b,R2^c,R3^b\}}$ *integrate to corresponding finite type invariants.*

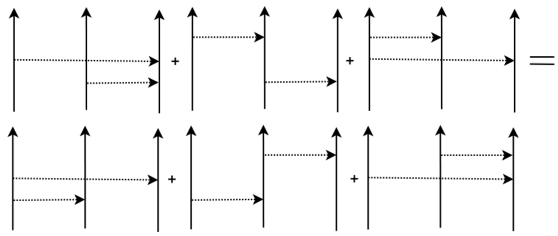


Figure 7: The 6T relation

In partiuclar, when $\mathcal{W}_n$ is the space of virtual knot diagrams modulo $R2^b$ and $R3^b$, $\mathcal{G}_n$ is the algebra $\vec{\mathcal{A}}_n$ of arrow diagrams mod 6T ([7]). The relation 6T is shown in diagram 7.

The spaces $\vec{\mathcal{A}}_n$ are of particular interest because of their relations to Lie bialgebras ([1],[4],[6]). The dimensions of $\vec{\mathcal{A}}_k$ for $k = \{1, 2, 3, 4\}$ are 2, 7=9-2, 27=36-9 and 139=175-36, respectively ([1]). Therefore, (1) is an equality, so Conjecture 3.2 suggests that all weight systems of $\vec{\mathcal{A}}_n$ integrate to finite type invariants of virtual knot diagrams modulo $R2^b$ and $R3^b$. Note, however, that since 27>22 and 139>89, the column $\mathcal{V}_m^{2c}$ and Proposition 3.1 together tell us that not all weight systems (functionals) of arrow diagrams modulo 6T integrate to finite type invariants which respect $R2^c$. This suggests that if our study of finite type invariants is entirely motivated by $\vec{\mathcal{A}}_n$, we should focus on virtual knot diagrams modulo only $R2^b$ and $R3^b$. A challenge is to come up with topological interpretations of such objects.

4 The Code

Technique/method/Algorithm

The results of this paper come from a computer program, [10], so, how does this code work? Firstly, the program finds all arrow diagrams of fixed order, followed by all relations which are generated from the initial relations we impose, finally we are left with linear algebra and it remains to find the rank of a sparse matrix. We collect the number of diagrams and number of relations in table 5. The two numbers for each space and dimension are collected in boxes with the top number representing the number of diagrams and the bottom number representing the number of relations.

m	$\mathcal{V}_m$	$\mathcal{V}_m^1$	$\mathcal{V}_m^{tc}$	$\mathcal{V}_m^{1,tc}$	$\mathcal{V}_m^{2c}$	$\mathcal{V}_m^{2c,gr}$	$\mathcal{V}_m^{1,2c}$	$\mathcal{V}_m^{1,2c,gr}$	$\mathcal{V}_m^{o}$	$\mathcal{V}_m^{1,tc,o}$	$\mathcal{V}_m^{1,2c,o}$
2	14	4	14	4	?	?	?	?	?	?	?
	6	6	9	9	?	?	?	?	?	?	?
3	134	44	134	44	?	?	?	?	?	?	?
	126	126	189	189	?	?	?	?	?	?	?
4	1814	620	1814	620	?	?	?	?	?	?	?
	2646	2646	3969	3969	?	?	?	?	?	?	?
5	?	11148	?	?	?	?	?	?	?	?	?
	?	63126	?	?	?	?	?	?	?	?	?

Table 5: Number of Diagrams and Relations

In fact, the spaces of arrow algebras modulo the specified relations are all isomorphic to their dual spaces and the elements correspond to the Gauss diagram formulas for virtual finite type invariants. Thus, by computing the basis of the relevant space, the code computes a maximal set of linearly independent Gauss diagram formulas for virtual finite type invariants.

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