"Dimensions of Spaces of Finite Type Invariants of Virtual Knots"
1. Just the case of honest ft invts of virt round knots, to degree 5: "Finite Type invts of virtual knots" (2 paragphs).
2. Variations.
3. Justtifications: Polyak etc.
4. The code.

Dimensions of the Polyak algebra and other spaces related to virtual finite type invariants

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Abstract

FT invts of ordinary knots

Real finite type invariants have diagrammatic descriptions and relate to Lie Algebras. Analogues of the corresponding results for virtual finite type invariants exist, but are less well understood. This article collects computational results about the dimensions of the Polyak algebra and other spaces related to virtual finite type invariants. The code describes the corresponding Gauss diagram formulas for virtual finite type invariants to order five, shows that not all weight systems integrate, and motivates a class of virtual knot where only "braid-like Reidemeister moves" are permitted. In this new class of knot, the collected data suggests a conjecture.

1 Introduction

Universal finite type invariants contain the power of all finite type invariants, and are central to their study. Consideration of an algebra due to M. Polyak, M. Goussarov, and O. Viro, [3], called the *Polyak algebra*, gives access to a computable universal virtual type n invariant and is the subject of this article. We will see that the invariant itself is simply a clever change of basis on the space generated by virtual knot diagrams, see Section 2.

The outline of the paper is as follows; Section 2 introduces the Polyak algebra, and motivates the subsequent two sections which contain the main content of the paper. Section 3 deals with relations on the space of arrow diagrams and tabulates the results from code written for the purposes of computing the dimensions of these spaces. Section 4 talks about associated algebras, the significance of the data from Section 3, answers the question of whether all weight systems can be integrated, and culminates with a conjecture connecting arrow diagrams mod 6T and a new class of knot allowing only "braid-like" Reidemeister moves.

of "Weight systems"

2 The Polyak Algebra

We start with a short account of the Polyak algebra, which comes mostly from [3]. To begin, we need the following definition:

Definition 1. *The map s on a virtual knot diagram is the sum of all subdiagrams, while the map s_n is the map that truncates this sum to contain only diagrams with less than or equal to n crossings.*

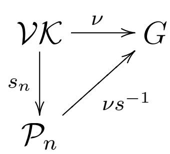
Here, a subdiagram is the obvious thing, simply a diagram obtained by virtualising some number of crossings. With this in mind, the maps s and s_n are written schematically by ⤫ → ⤫ + ⤫, thus s^{-1} exists and sends real crossings to semi-virtual crossings. For details see [3], [8].

Now, the map s_n is only well defined on diagrams, and we require a map on knots. To account for this, we need to keep track of the Reidemeister moves. In particular, we need to factor out our domain by the Reidemeister moves and our range by s of the Reidemeister relations. This gives a map from the space of virtual knots, $\mathcal{VK}$, to the so called *Polyak algebra*, $\mathcal{P}$. Of course, now s becomes an isomorphism of the space generated by knots. The truncated Polyak algebra is denoted $\mathcal{P}_n$.

The point is that any virtual type n invariant, ν, factors through the map $s_n : \mathcal{VK} \to \mathcal{P}_n$ precisely because s^{-1} sends real crossings to semi-virtual crossings and ν is of type n;

$$\nu(D) = \nu s^{-1} s(D) = \nu s^{-1} s_n(D) + \nu s^{-1}(\Sigma_{|C_i|>n} C_i) = \nu s^{-1} s_n(D)$$

We obtain a commutative diagram;



In other words, $s_n : \mathcal{VK} \to \mathcal{P}_n$ is a universal type n invariant.

Now $\mathcal{P}_n$ is finite dimensional, and so ν is determined by its values on a basis of $\mathcal{P}_n$, a finite set. Letting $\mathcal{B}$ be a basis for $\mathcal{P}_n$ we can see that $\Sigma_{D\in\mathcal{B}}(\nu s^{-1}(D)D)$ is a *Gauss Diagram Formula* for ν, see [3], [7], [8]. Thus, every virtual finite type invariant has a Gauss diagram formula.

Starting at the other end, all elements of $\mathcal{P}_n^*$ correspond to virtual finite type invariants under composition with s_n because any real crossing may be expressed as a sum of a semi-virtual crossing and the same diagram but with the specified crossing virtualized, while s_n sends semi-virtual crossings to real crossings before truncating. Further, elements of $\mathcal{P}_n$ implicitly subsume s_n of the Reidemeister moves making the composition of our functional with s_n an invariant.

Our code, [9], describes a basis of $\mathcal{P}_n$ to order five. The code also draws the actual diagrammatic formulas, see [9]. Thus all virtual type n invariants are classified in terms of their Gauss diagram formulas to order five, see Section 3.

So far we have seen the strengths of $\mathcal{P}_n$ and its connection to virtual finite type invariants, however the algebra lacks a grading. This is in contrast to the space of arrow diagrams mod 6T known as the arrow algebra $\vec{\mathcal{A}}$. The homogeneous peices, $\vec{\mathcal{A}}_n$, give the grading, and and can be thought of as being given by virtual knot diagrams mod s_n of Reidemeister three. It is exactly the graded structure that leads to Lie algebras in the real case, [2], and to Lie bialgebras and Quantum groups in the virtual case, [4]. In the real setting, functionals may be "integrated" to form finite type invariants, [10]. Our computations answer the corresponding question for $\vec{\mathcal{A}}_n$ in the virtual case, see Sections 3 and 4. The answer leads naturally to "new" Polyak algebras and the experimental evidence from our code, [9], raises an interesting conjecture, see Section 4.

3 Variations of the *Polyak Algebra* and Their Dimensions

3.1 The *Polyak algebra* relations

Given the *algebra of arrow diagrams* $\mathcal{A}$, the *Polyak algebra* described in [GPV] is obtained through taking the quotient of $\mathcal{A}$ by the relations illustrated in Figures 1, 2, and 3 which also appear in the original article. Each of the dotted arrows signifies a semivirtual crossing: $\boxtimes = \times - \times$ (respectively $\boxtimes = \times - \times$).

opposite

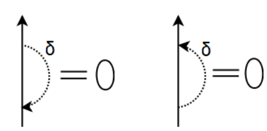


Figure 1:

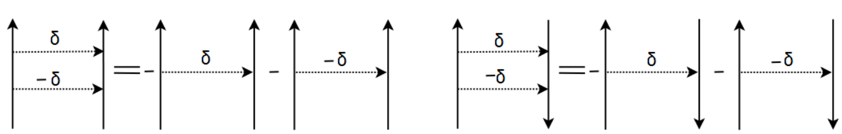


Figure 2:

3.2 Braid-like and cyclic Reidemeister moves

In making the connection between the relations above and the Reidemeister moves, we observe that each Reidemeister move can be split into two types. The two variations of the *RI* move, are with a positive or a negative crossing, while the *RII* and *RIII* moves

3

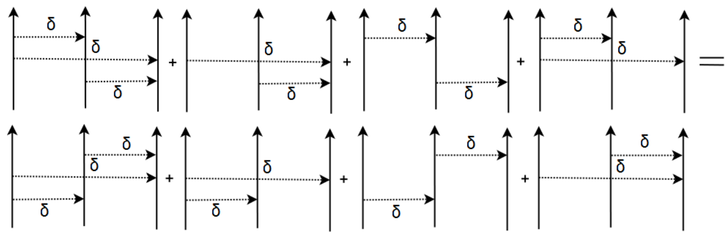


Figure 3:

can be braid-like (RII^b and $RIII^b$) or cyclic (RII^c and $RIII^c$), depending on whether all the strands go in one direction or form a cycle respectively. The Reidemeister three move can also be split into $RIII^-$ and $RIII^+$, namely $RIII$ with only positive crossings. Yet, given RII^b, $RIII^+$ implies $RIII^-$ so we need only consider $RIII^+$.

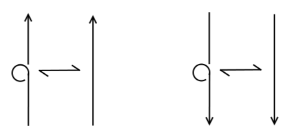


Figure 4: RI^+ and RI^-

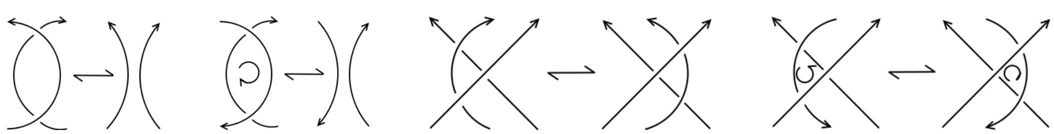


Figure 5: RII^b, RII^c, $RIII^b$ and $RIII^c$

Then, given the virtual Reidemeister three moves with two virtual crossings, we have that $RIII^b$ is equivalent to the eight-term (8T) relation in Figure 3. Furthermore, RII^b and RII^c are equivalent to the two four-term relations in Figure 2 and the two RI moves are equivalent to the relations in Figure 1. With those equivalences established, the original *Polyak algebra* $\mathcal{P}$ is $\mathcal{P} = \mathcal{A}/\{RI, RII, RIII\}$, namely the quotient of $\mathcal{A}$ by all types of *Reidemeister I, II* and *III* moves. It is of interest to consider other *Polyak algebras* obtained from taking the quotient of $\mathcal{A}$ by a subset of the six types of Reidemeister moves. Further relationships between those six moves are as follows:

1. RI, RII^b, and $RIII^c$ together imply RII^c.

2. $RIII^b$ and RII^c give us $RIII^c$.

3.3 Results

Given any variation $\mathcal{P}'$ of the *Polyak algebra* reached through the described method, we can, in parallel to the [GPV] article, consider the *truncated algebra* $\mathcal{P}'_m$ where any diagram in $\mathcal{P}'$ with more than m arrows is taken to be 0. The table below shows the calculated dimensions of several such truncated variations of the *Polyak algebra*. The truncated quotients used in the computations are listed below. TC refers to the "tails commute" relation, which diagrammatically is: The "tails commute" relation relates to

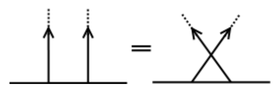


Figure 6: Tails Commute

w-knots mentioned in [1].

m	$\mathcal{P}^1_m$	$\mathcal{P}^2_m$	$\mathcal{P}^3_m$	$\mathcal{P}^4_m$	$\mathcal{P}^5_m$	$\mathcal{P}^6_m$	$\mathcal{P}^7_m$	$\mathcal{P}^8_m$	$\mathcal{P}^9_m$	$\mathcal{P}^{10}_m$	$\mathcal{P}^{11}_m$
2	9	2	6	1	7	5	2	2	5	0	0
3	36	9	13	2	22	15	9	7	27	0	1
4	175	51	25	4	89	67	51	42	245	0	5

Table 1: Computed Dimensions

$\mathcal{P}^1 = \mathcal{A}/\{RIII^b, RII^b\}$
$\mathcal{P}^2 = \mathcal{A}/\{RIII^b, RII^b, RI\}$
$\mathcal{P}^3 = \mathcal{A}/\{RIII^b, RII^b, TC\}$
$\mathcal{P}^4 = \mathcal{A}/\{RIII^b, RII^b, TC, RI\}$
$\mathcal{P}^5 = \mathcal{A}/\{RIII^b, RII^b, RII^c\}$
$\mathcal{P}^6 = Gr(\mathcal{A}/\{RIII^b, RII^b, RII^c\})$
$\mathcal{P}^7 = \mathcal{A}/\{RIII^b, RII^b, RII^c, RI\}$
$\mathcal{P}^8 = Gr(\mathcal{A}/\{RIII^b, RII^b, RII^c, RI\})$
$\mathcal{P}^9 = \mathcal{A}(\bigcirc)$
$\mathcal{P}^{10} = \mathcal{A}(\bigcirc)/\{RIII^b, RII^b, TC, RI\}$
$\mathcal{P}^{11} = \mathcal{A}(\bigcirc)/\{RIII^b, RII^b, RII^c, RI\}$

Gr above means the graded quotient, and $\mathcal{A}(\bigcirc)$ refers to the compact knots case, as opposed to long knots.

4 The Associated Graded Algebras

Definition 2. *([6]) $\vec{\mathcal{A}}_k$ is the space of arrow diagrams with k arrows modulo 6T. A degree-k weight system is an element in $\vec{\mathcal{A}}^*_k$.*

Figure 7: The 6T relation

The 6T relation is shown in figure 7. As previously remarked, $\vec{\mathcal{A}}_k$'s are of interest because of their relations to Lie bialgebras ([1],[4],[5]). Let $\mathcal{P}_n$ be any variations of the Polyak algebra (i.e. arrow diagrams mod $RII^b, RIII^b$ with possibly other relations). We have a map $\iota_n : \vec{\mathcal{A}}_n \to \mathcal{P}_n$ which takes a diagram to itself and the following commutative diagram:

$$\begin{array}{ccc} \mathcal{P}_n & \xrightarrow{\nu s^{-1} s_n} & G \\ \uparrow{\scriptstyle \iota_n} & \nearrow{\scriptstyle \nu s^{-1} s_n \iota_n} & \\ \vec{\mathcal{A}}_n & & \end{array} \quad .$$

We consider the special case where G is the base field. Since $\mathcal{P}_n^*$ is the space of invariants of at most type n ([3]), if all weight systems integrate to finite type invariants, the adjoint map $\iota_n^* : \mathcal{P}_n^* \to \vec{\mathcal{A}}_n^*$ is surjective. This is equivalent to ι_n being injective.

To determine whether this is the case we consider the filtration

$$0 = \Pi_{n,n+1} \subseteq \Pi_{n,n} \subseteq \cdots \subseteq \Pi_{n,1} \subseteq \Pi_{n,0} = \mathcal{P}_n$$

where each $\Pi_{n,k}$ is comprised of diagrams of degree $\geq k$ and $\leq n$. The associated graded algebra

$$gr(\mathcal{P}_n) = \bigoplus_{k=1}^{n} \Pi_{n,k}/\Pi_{n,k+1}$$

is isomorphic to $\mathcal{P}_n$ as a vector space. For each k, by mapping each diagram in $\vec{\mathcal{A}}_k$ to itself, we get a surjective map

$$\mu_{n,k} : \vec{\mathcal{A}}_k \twoheadrightarrow \Pi_{n,k}/\Pi_{n,k+1} \subseteq gr(\mathcal{P}_n),$$

so, for all $k \leq n$,

$$\dim(\vec{\mathcal{A}}_k) \geq \dim(\Pi_{n,k}/\Pi_{n,k+1}), \tag{1}$$

which implies

$$\sum_{k=1}^{n} \dim\vec{\mathcal{A}}_k \geq \dim(gr(\mathcal{P}_n)) = \dim\mathcal{P}_n. \tag{2}$$

We have the following proposition.

Proposition 1.

$$\sum_{k=1}^{n} \dim\vec{\mathcal{A}}_k = \dim(gr(\mathcal{P}_n)) = \dim\mathcal{P}_n$$

if and only if the map $\iota_k : \vec{\mathcal{A}}_k \to \mathcal{P}_k$ *is injective for all* $k \leq n$.

Proof. If $\iota_k : \vec{\mathcal{A}}_k \to \mathcal{P}_k$ is injective for each k, since the image of this map is exactly $\Pi_{k,k}/\Pi_{k,k+1} \cong \Pi_{n,k}/\Pi_{n,k+1}$, (1) will be an equality, and so will be (2). Conversely if (2) is an equality, (1) will have to be an equality, for each k. Since $\mu_{n,k}$ is surjective, it must be injective. Since $\Pi_{k,k}/\Pi_{k,k+1} \cong \Pi_{n,k}/\Pi_{n,k+1}$, $\mu_{k,k}$ and ι_k are injective. □

The injectivity of $\iota_k : \vec{\mathcal{A}}_k \to \mathcal{P}_k$ is equivalent to $\vec{\mathcal{A}}_k$ being isomorphic to the degree-k homogeneous subspace of $gr(\mathcal{P}_n)$. The dimensions of $\vec{\mathcal{A}}_k$ for $k = \{1, 2, 3, 4\}$ are 2, 7, 27 and 139, respectively ([1]). From the table above we know that (2) is an equality if we take $\mathcal{P}_n$ to be the quotient by only RII^b and $RIII^b$, and not RII^c. This suggests that not all weight systems of arrow diagrams modulo 6T integrate to finite type invariants which respect RII^c. Alternatively, this suggests that if our study of finite type invariants is motivated by arrow diagrams mod 6T, we should focus on virtual knot diagrams modulo only RII^b and $RIII^b$. A challenge is to come up with topological interpretations of such objects.

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