PENTAGON AND HEXAGON EQUATIONS FOLLOWING FURUSHO

DROR BAR-NATAN AND ZSUZSANNA DANCSO

ABSTRACT. In [F] H. Furusho proves the beautiful result that of the three defining equations for associators, the pentagon implies the two hexagons (see also [W]). In this note we present a simpler proof for this theorem (although our paper is less dense, and hence only slightly shorter). In particular, we package the use of algebraic geometry and Groethendieck-Teichmuller groups into a useful and previously known principle, and, less significantly, we eliminate the use of spherical braids.

CONTENTS

1. INTRODUCTION

Associators are useful and intricate gadgets that were first introduced and studied by Drinfel'd in [Dr1] and [Dr2]. The theory was later put in the context of parenthesized (a.k.a. non-associative) braids by [LM] and [BN]. Associators arise in several different areas of mathematics, and thus constructing an associator is a task faced by many. Unfortunately, it is a very difficult task: no closed formulas are known at present.

Associators are essentially the solutions to three equations, called the pentagon and the positive and negative hexagons, which live in complicated diagrammatic spaces. Finding an associator amounts to finding a solution to this system of equations. Furusho's result states that the last two of these three equations are superfluous: a solution to the pentagon will automatically be a solution to both hexagons. This statement is quite surprising, and thus we felt that a simpler proof would be of value.

The paper is organized as follows: we first review some definitions, then present the main tool or "extension principle" for the simplified proof: a theorem from [Dr2] and [BN], and also a simpler standard fact which we call the "linearization principle". We then prove the theorem modulo a "Lie-algebraic" version which we call the "Main Lemma", followed by the

proof of the Main Lemma, and finally a side note on how one of the algebraic maps used in the proof arises from topology.

The part of the proof which depends on the extension principle is significantly different from that of [F] (and [W]), and significantly simpler: Furusho's proof uses algebraic geometry and Groethendieck-Teichmuller groups to go from the Lie algebra version to the group statement.

The proof of the main lemma is essentially Furusho's proof, except for the elimination of the use of spherical braids. Using spherical braids makes the statement of the Main Lemma look very symmetric, so in some sense prettier. However, in our opinion, it makes the combinatorial argument of the proof less transparent: it is the break of symmetry which helps the reader figure out how someone might have discovered the proof. Also, we save the trouble of having to pass back and forth between the spherical and regular chord-diagram spaces. The elimination of spherical braids is easy: the space of chord diagrams of pure spherical 5-braids and that of regular pure 4-braids only differ in taking a quotient by the center of the latter.

1.1. **Acknowledgements.** The authors would like to thank the referee for the thorough reading of the paper and detailed helpful suggestions, and Stavros Garoufalidis for an enlightening discussion of the proof of the Main Lemma.

2. Definitions

Algebraically, the space of chord diagrams of pure n-braids, $\mathcal{A}_n$, is defined to be the graded completion of the following non-associative algebra, where generation is understood over a fixed field k:

$$\left\langle t_{ij} : 1 \leq i \neq j \leq n \,\middle|\, \begin{array}{l} t_{ij} = t_{ji}, \\ [t_{ij}, t_{kl}] = 0 \text{ when } i, j, k, l \text{ are distinct (L)}, \\ [t_{ij} + t_{ik}, t_{jk}] = 0 \text{ when } i, j, k \text{ are distinct (4T)} \end{array} \right\rangle$$

Here, $4T$ is short for *four term relations*, and L stands for *locality*.

One usually thinks of $\mathcal{A}_n$ in terms of horizontal chord diagrams on n strands, where t_{ij} is represented by a chord between strands i and j, and multiplication is done by stacking diagrams:
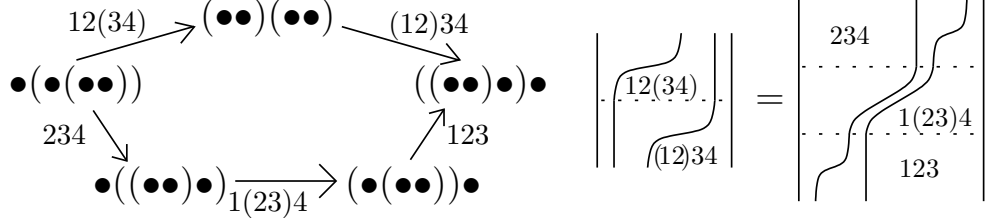

Let us denote by $\mathcal{F}_2$ the free lie algebra over a field k of characteristic 0, on two generators X and Y, and by $\mathcal{UF}_2$ its universal enveloping algebra, which is isomorphic to $k\langle\langle X, Y\rangle\rangle$, the algebra of non-commutative power series over k. By a superscript (m), as in $\mathcal{UF}_2^{(m)}$, we mean the space or object modulo degree $(m + 1)$, e.g. non-commutative polynomials of degree up to m.

There is a co-product on $\mathcal{UF}_2$, defined by $\Delta(X) = 1 \otimes X + X \otimes 1$, $\Delta(Y) = 1 \otimes Y + Y \otimes 1$. An element Φ is called *group-like* if $\Delta(\Phi) = \Phi \otimes \Phi$. An element φ is *primitive* (meaning it is a Lie-algebra element) if $\Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes 1$.

Now let us present the main characters of this story: the pentagon and hexagon equations.
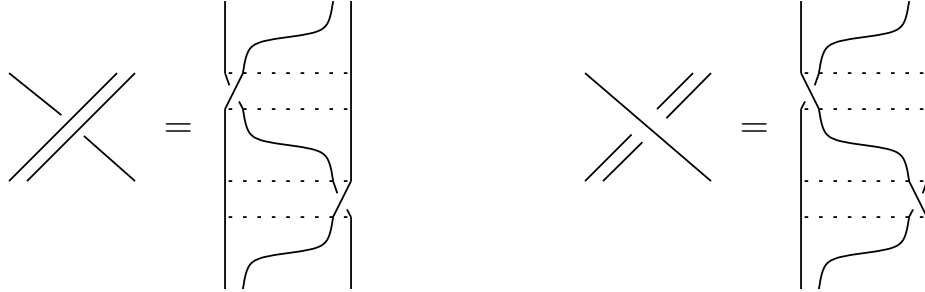
The pentagon equation originates from the fact that in a non-associative algebra, there are five ways to multiply four elements, and these are connected by a pentagon of re-associations. This fact and its parenthesized braid analogue are shown in the figure below:

The above equality of parenthesized braids implies an algebraic equation in $\mathcal{A}_4$ called the *pentagon equation*, $P(\Phi) = 1$, for $\Phi \in \mathcal{UF}_2$, where

$$P(\Phi) = \Phi(t_{13} + t_{23}, t_{34})^{-1}\Phi(t_{12}, t_{23} + t_{24})^{-1}\Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}).$$

The hexagons arise from the following equivalences of parenthesized braids:

Algebraically, the implied equations in $\mathcal{A}_3$ are $H_\pm(\Phi) = 1$, for $\Phi \in \mathcal{UF}_2$, where

$$H_\pm(\Phi) = e\left(\mp\frac{t_{13} + t_{23}}{2}\right)\Phi(t_{13}, t_{12})e\left(\pm\frac{t_{13}}{2}\right)\Phi(t_{13}, t_{23})^{-1}e\left(\pm\frac{t_{23}}{2}\right)\Phi(t_{12}, t_{23}).$$

Here, $e(x) := e^x$.

An *associator* is a group-like element of $\mathcal{UF}_2$ which satisfies the pentagon and hexagon equations.

Note that the spaces $\mathcal{A}_3$ and $\mathcal{UF}_2$ are almost isomorphic: $\mathcal{UF}_2 = \mathcal{A}_3/Z(\mathcal{A}_3)$ is the quotient of $\mathcal{A}_3$ by its center $Z(\mathcal{A}_3) = \langle t_{12} + t_{23} + t_{31}\rangle$. An associator is sometimes defined as an element of $\mathcal{A}_3$, which is equivalent, since associators are commutator-grouplike, meaning their abelianization is 1. (This follows from the pentagon equation.)

3. Theorem and Proof

In [F] Furusho proves the following surprising result:

Theorem 3.1. *If Φ is a group-like element of $\mathcal{UF}_2$ with $c_2(\Phi) = \frac{1}{24}$, where $c_2(\Phi)$ denotes the coefficient of XY in Φ, and Φ satisfies the pentagon equation $P(\Phi) = 1$ in $\mathcal{A}_4$, then Φ satisfies the hexagon equations $H_\pm(\Phi) = 1$ in $\mathcal{A}_3$, and therefore Φ is an associator.*

To prove Theorem 3.1 we need two tools, which we will call *extension* and *linearization* principles, and a major lemma, Lemma 3.2 below.

The "extension principle" is the assertion that an associator modulo degree m can be lifted to an associator modulo degree $(m+1)$. This statement is highly non-obvious; it takes up a good part of [Dr2] and the main part of [BN]. Its main application is the fact that non-trivial rational associators exist: indeed, up to degree 2 a direct computation shows that $P(\Phi) = 1$ and $H_\pm(\Phi) = 1$ have a unique non-trivial solution (namely, $\Phi^{(2)} = 1 + \frac{1}{24}[X, Y]$), and then by extension, one may construct a rational associator inductively.

The "linearization principle" is the standard fact that the pentagon $P(\Phi) = 1$ and the hexagons $H_\pm(\Phi) = 1$ can be "linearized". Precisely, this means that there exist degree-preserving linear operators $dP : \mathcal{U}F_2 \to \mathcal{A}_4$ and $dH_+ = dH_- = dH : \mathcal{U}F_2 \to \mathcal{A}_3$ so that if Φ and Φ' satisfy the pentagon equation modulo degree m, i.e. $P(\Phi) = P(\Phi') = 1$ modulo degree m, and are equal modulo degree m, i.e. $\varphi := (\Phi - \Phi')^{(m)}$ is homogeneous of degree m, then

$$P(\Phi) - P(\Phi') = dP(\varphi) \text{ modulo degree } (m + 1)$$

Likewise for the hexagons: if Φ and Φ' satisfy one of the hexagons modulo degree m, and are equal modulo degree m, then

$$H_\pm(\Phi) - H_\pm(\Phi') = dH(\varphi) \text{ modulo degree } (m + 1),$$

where φ is as above. Note that if Φ and Φ' are group-like, then φ is primitive.

To prove the theorem we use the following main lemma:

Lemma 3.2. *If $\varphi \in \mathcal{U}F_2$ is homogeneous of degree $m \geq 3$ and is primitive, and satisfies the linearized pentagon equation $dP(\varphi) = 0$, then $dH(\varphi) = 0$, in other words the linearized pentagon equation implies the linearized hexagon equation.*

Proof of Theorem 3.1 from Lemma 3.2. Let us assume that Φ is as in Theorem 3.1, in particular $P(\Phi) = 1$, and, by contradiction, that one of the hexagons fails to hold, i.e. $H_\pm(\Phi) \neq 1$. Let m be the minimal degree in which this happens.

By the simple computation in low degrees mentioned before, we know that $m \geq 3$. Note that by the minimality of m, Φ satisfies the hexagons modulo degree m and hence it is an associator modulo degree m. By extension, there exists a $\Phi' \in \mathcal{U}F_2$ which agrees with Φ modulo degree m, and which satisfies both the pentagon and the hexagon equations modulo degree $(m+1)$. Let $\varphi = (\Phi - \Phi')^{(m)}$, which is homogeneous of degree m since $\Phi = \Phi'$ modulo degree m.

By linearization, $dP(\varphi) = P(\Phi) - P(\Phi') = 0$ modulo degree $(m + 1)$, so by Lemma 3.2, $H_\pm(\Phi) - H_\pm(\Phi') = dH(\varphi) = 0$ modulo degree $(m + 1)$. Back again by linearization, as Φ' satisfies the hexagons in degree m, it follows that so does Φ, contradicting our pessimistic assumption from the beginning. $\square$

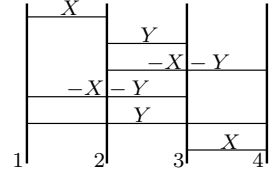## 4. Proof of the main lemma

By explicit computations,

$$dP(\varphi) = -\varphi(t_{12}, t_{23} + t_{24}) - \varphi(t_{13} + t_{23}, t_{34}) + \varphi(t_{23}, t_{34}) + \varphi(t_{12} + t_{13}, t_{24} + t_{34}) + \varphi(t_{12}, t_{23}),$$

$$dH(\varphi) = \varphi(t_{13}, t_{12}) - \varphi(t_{13}, t_{23}) + \varphi(t_{12}, t_{23}).$$

Let us start by proving two simple but necessary lemmas. Throughout this section, φ is assumed to be a primitive, homogeneous element of $\mathcal{U}F(2)$ of degree ≥ 3, as in Lemma 3.2.

Lemma 4.1. $dP(\varphi) = 0$ *implies that φ is anti-symmetric, i.e.* $\varphi(X,Y) + \varphi(Y,X) = 0$.

Proof. We use the map $q : \mathcal{A}_4 \to \mathcal{F}_2$ defined by: $t_{12} \mapsto X$, $t_{23} \mapsto Y$, $t_{13} \mapsto (-X-Y)$, $t_{14} \mapsto Y$, $t_{24} \mapsto (-X-Y)$, and $t_{34} \mapsto X$, as illustrated by the figure on the right. Since $q(dP(\varphi)) = \varphi(X,Y) + \varphi(Y,X)$, the lemma follows. $\qquad\square$

Note that by the anti-symmetry of φ, we have:

$$dP(\varphi) = \varphi(t_{12}, t_{23}) + \varphi(t_{34}, t_{13} + t_{23}) + \varphi(t_{23} + t_{24}, t_{12}) + \varphi(t_{23}, t_{34}) + \varphi(t_{12} + t_{13}, t_{24} + t_{34}),$$

$$dH(\varphi) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, t_{31}) + \varphi(t_{31}, t_{12}).$$

Lemma 4.2. *The linearized hexagon $dH(\varphi) = 0$ is equivalent to the equation*

$$(1) \qquad\qquad \varphi(X,Y) + \varphi(Y, -X-Y) + \varphi(-X-Y, X) = 0$$

in $\mathcal{UF}_2$, if φ is of degree ≥ 2.

By an abuse of notation, we shall denote the left side of equation (1) by $dH(X,Y)$.

Proof. $dH(\varphi) = 0$ implies (1) via the quotient map $\pi : \mathcal{A}_3 \to \mathcal{UF}_2$ which factors out by the center of $\mathcal{A}_3$, defined by $\pi(t_{12}) = X$, $\pi(t_{23}) = Y$, and $\pi(t_{13}) = -X - Y$.

For the other direction we apply the map $i : \mathcal{UF}_2 \to \mathcal{A}_3$ given by $i(X) = t_{12}$ and $i(Y) = t_{23}$ $i(dH(X,Y)) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, -t_{12} - t_{23}) + \varphi(-t_{12} - t_{23}, t_{12}) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, t_{13}) + \varphi(t_{13}, t_{12}) = dH(\varphi)$, where the second equality is due to the fact that $\varphi \in [\mathcal{F}_2, \mathcal{F}_2]$ (since it is primitive of degree ≥ 2), and that the element $t_{12} + t_{23} + t_{13}$ is central in $\mathcal{A}_3$. $\qquad\square$

Proof of the Main Lemma 3.2 Let us first introduce some notation. Any map of sets $f : [r] \to [s]$, where $[r] := \{1, 2, ..., r\}$ and $[s] = \{1, 2, ..., s\}$, induces a map $\bar{f} : \mathcal{A}_s \to \mathcal{A}_r$, where $\bar{f}(t_{ij}) := \sum_{\alpha \in f^{-1}(i), \beta \in f^{-1}(j)} t_{\alpha\beta}$. If either of the pre-images is empty, we understand the sum to be zero.

Now, if $\psi(X, Y) \in \mathcal{UF}_2$, denote $\psi(123) := \psi(t_{12}, t_{23}) \in \mathcal{A}_3$; and let

$$\psi((i_1...i_t)(j_1...j_u)(k_1...k_v)) := \bar{g}(\psi(123)),$$

where $g : [t + u + v] \to [3]$ is given by $g^{-1}(1) = \{i_1...i_t\}$ and similarly for 2 and 3.

The permutation group S_4 acts on $\mathcal{A}_4$ by commuting the strands (i.e. the indices). For any $\sigma \in S_4$, we know that $\sigma(dP(\varphi)) = 0$. We try to find a (small) set of permutations σ_i such that $0 = \sum_i \sigma_i dP(\varphi) = \sum_j \bar{g}_j(dH(\varphi))$, for some $g_j : [4] \to [3]$.

In fact, the following four permutations work (the notation $\sigma = x_1 x_2 x_3 x_4$ means $\sigma(i) = x_i$ for $i = 1, 2, 3, 4$):

$$\sigma_1 = id, \quad \sigma_2 = 4231, \quad \sigma_3 = 1342, \quad \sigma_4 = 4312$$

. Using the notation introduced above,

$$0 = \sum_{i=1}^{4} \sigma_i dP(\varphi) =$$
$$= \varphi(123) + \cancel{\varphi(43(12))} + \varphi((34)21) + \varphi(234) + \cancel{\varphi(1(23)4)} +$$
$$+ \varphi(423) + \cancel{\varphi(13(42))} + \varphi((31)24) + \varphi(231) + \cancel{\varphi(4(23)1)} +$$
$$+ \cancel{\varphi(134)} + \varphi(24(13)) + \cancel{\varphi((42)31)} + \varphi(342) + \varphi(1(34)2) +$$
$$+ \cancel{\varphi(431)} + \varphi(21(43)) + \cancel{\varphi((12)34)} + \varphi(312) + \varphi(4(31)2) =$$
$$= dH(123) + dH((34)21) + dH(423) + dH((31)24),$$

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where the cancellations are by the anti-symmetry of φ (Lemma 4.1).

Note that every chord appearing on the right side ends on strand 2, i.e., $dH(123) + dH((34)21) + dH(423) + dH((31)24) \in \langle t_{12}, t_{23}, t_{24} \rangle$. Also, $\langle t_{12}, t_{23}, t_{24} \rangle \cong \mathcal{F}_3 \subseteq \mathcal{A}_4$, since there are no relations in $\mathcal{A}_4$ that involve only these elements.

Note that the anti-symmetry of φ also implies the anti-symmetry of dH, i.e. $dH(X, Y) = -dH(Y, X)$, and in particular $dH(X, X) = 0$. We can now finish the proof using two projections:

Let $p_1 : \mathcal{F}_3 \to \mathcal{F}_2$ be the projection defined by $t_{12} \mapsto X$, $t_{23} \mapsto Y$, $t_{24} \mapsto X$, and apply this to the equality $0 = dH(123) + dH((34)21) + dH(423) + dH((31)24)$. We obtain $0 = dH(X, Y) + dH(X, Y) + dH(X + Y, X) + dH(X + Y, X)$, and therefore $dH(X + Y, X) = -dH(X, Y)$.

Now do the same for the projection p_2 defined by $t_{12} \mapsto X$, $t_{23} \mapsto X$, $t_{24} \mapsto Y$. We get $0 = dH(X, X) + dH(Y, X) + dH(X + Y, X) + dH(2X, Y)$, and so using the above, we arrive to $dH(2X, Y) = 2dH(X, Y)$.

This means that $dH(X, Y)$ contains only commutators that involve one X and some number of Y's, and so writing dH in a linear basis we have:
$$dH(X, Y) = \sum_{n=1}^{\infty} a_n (adY)^{n-1}(X).$$

Since $dH(X, Y) = -dH(Y, X)$, we know that $a_n = 0$ for all n except possibly $n = 2$, and $a_2 = 0$ by the assumption that φ is of degree ≥ 3. Thus, $dH(X, Y) = 0$, which is equivalent to the Main Lemma by Lemma 4.2. $\qquad \square$

A surprising moment in the proof above is the equality relating a sum of four linearized pentagon equations to a sum of linearized hexagons. As pointed out by Stavros Garoufalidis, there is a similar but more natural equality which, as the reader can check, arises from the permuto-associahedron on page 21 of [BN]:

$$dP(1234) - dP(1243) + dP(1423) - dP(4123) = dH(34(12)) - dH((23)41) + dH(241) - dH(342)$$

It is easy to verify that the rest of the proof goes through the same way using this equtlity in place of the one above.
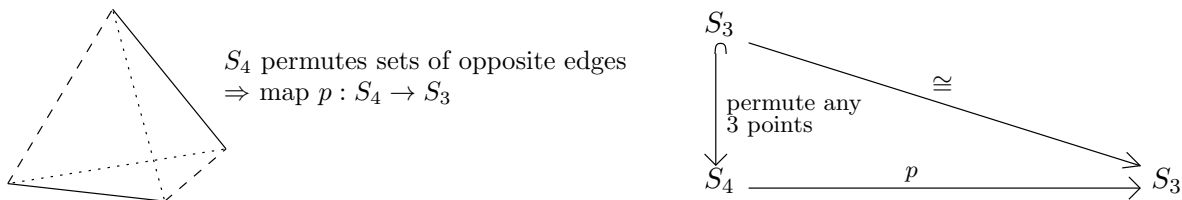
5. A SIDE NOTE

The projection q, used in the proof of Lemma 4.1, has an interesting property and a nice topological interpretation.

If we embed $\mathcal{A}_3$ in $\mathcal{A}_4$ on any three strands (i.e. by any embedding of the index set [3] into [4]), and then apply q, we get the "almost isomorphism" between $\mathcal{A}_3$ and $\mathcal{F}_2$, i.e. the composition is factoring out by the center of $\mathcal{A}_3$:

This is a braid theoretic analogue of the following fact about the symmetric group S_4. Since S_4 is isomorphic to the group of symmetries of the tetrahedron, and each element of S_4 also permutes the three pairs of opposite edges of the tetrahedron, we obtain a map $p : S_4 \to S_3$. Pre-composing this map with any embedding of S_3 into S_4 induced by an

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embedding of the set [3] into [4], we get an automorphism of S_3, namely, an isomorphism from S_3 to the group of symmetries of a face:

S_4 permutes sets of opposite edges
$\Rightarrow$ map $p : S_4 \to S_3$

$$S_3$$

permute any
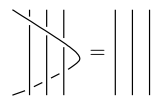3 points

$$S_4 \xrightarrow{\quad p \quad} S_3$$

$$\cong$$

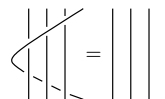Topologically, $q : \mathcal{A}_4 \to \mathcal{F}_2$ is induced by a map $\tilde{q}$, defined as follows:

$$pB_4 \xrightarrow{\tilde{q}_1} spB_4 \xrightarrow{\tilde{q}_2} pB_3/\text{full twist}$$

$$\tilde{q}$$

Here, pB_i denotes the pure braid group on i strands, and spB_4 denotes the group of pure spherical braids on four strands. Spherical braids live in $S^2 \times I$, as opposed to $D^2 \times I$, which means that one strand wrapping all the way around the others is trivial, as shown for strand 1 on the right, and similarly for all other strands. This defines the quotient map $\tilde{q}_1$ above.

For $\tilde{q}_2$, take any spherical braid on four strands, pull the last strand straight, and consider the first three strands as a braid in the complement of strand 4. The target space of this map is the group of regular pure 3-braids factored out by the full twist of the three strands: pull the 4-th strand straight on the left side of the spherical relation shown in the figure on the right, and observe that what we get is a full twist of the first 3 strands, which then has to be trivial in the image.

Note that the chord diagram map induced by factoring out by the full twist in pB_3 is exactly the quotient map $\pi : \mathcal{A}_3 \to \mathcal{F}_2$ which sends $t_{12} + t_{23} + t_{13}$ to 0.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO ONTARIO M5S 2E4, CANADA
E-mail address: drorbn@math.toronto.edu, zsuzsi@math.toronto.edu
URL: http://www.math.toronto.edu/drorbn, http://www.math.toronto.edu/zsuzsi