Products and Semi-Direct Products

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2. Some Soft Facts about Power Series and Expansions

2.1. **More About the Polynomial Algebra $\mathcal{A}(G)$.** We start with a few trivialities. For $g\in G$ we let $\bar g := g-1 \in \mathbb{Q}G$. It is clear that $\bar g \in I$ and that elements of the form $\bar g$ generate I. And as I/I^2 generates $\mathcal{A}(G)$, the classes of the $\bar g$'s in I/I^2 generate $\mathcal{A}(G)$.

Proposition 2.1. *In I/I^2, $\overline{gh} = \bar g + \bar h$ for any $g,h\in G$. In particular, $\overline{g^k} = k\bar g$ for any $k\in\mathbb{Z}$.*

Proof. In $\mathbb{Q}G$, $\overline{gh} = gh-1 = (g-1)+(h-1)+(g-1)(h-1) = \bar g+\bar h+\bar g\bar h$, and modulo I^2 the last term drops out. □

This proposition implies that if the group G is torsion then $\mathcal{A}(G)=0$ justifying Table 1 line 1. Indeed if G is torsion and $g\in G$, then $g^k=1$ for some k, hence $k\bar g = \overline{g^k} = g^k-1 = 0$, hence $\bar g = 0$, hence all the generators of $\mathcal{A}(G)$ vanish.

If x and y are elements of a group, we denote their group-commutator by $(x,y) := xyx^{-1}y^{-1}$. If a and b are elements of an algebra, we denote their algebra-commutator by $[a,b] = ab-ba$. In our context, these two notions are compatible:

Proposition 2.2. *If $x,y\in G$, then $\overline{(x,y)} \in I^2$ and in $\mathcal{A}(G)_2 = I^2/I^3$, $\overline{(x,y)} = [\bar x,\bar y]$.*

Proof. In $\mathbb{Q}G$ and since 1 is central, $[\bar x,\bar y] = [x,y] = \overline{(x,y)}yx$. Hence $\overline{(x,y)}yx \in I^2$, hence $\overline{(x,y)}(yx-1)\in I^3$, hence modulo I^3, $\overline{(x,y)} = \overline{(x,y)}yx = [\bar x,\bar y]$. □

The above proposition has a stronger variant:

Proposition 2.3. *If $x,y\in G$ are such that $\bar x\in I^m$ and $\bar y\in I^n$, then $\overline{(x,y)}\in I^{m+n}$ and in $\mathcal{A}(G)_{m+n}$, $\overline{(x,y)} = [\bar x,\bar y]$.*

Proof. Same proof, with I^2 replaced with I^{m+n} and I^3 with I^{m+n+1}. □

2.2. **Products and ~~Semi-~~ Almost Direct Products.** We aim to prove Proposition 1.5, asserting that $\mathcal{A}(G\times H)\cong\mathcal{A}(G)\otimes\mathcal{A}(H)$. As we shall see, the proof revolves around the fact that $\mathbb{Q}(G\times H) = (\mathbb{Q}G)\otimes(\mathbb{Q}H)$ is twice-filtered. Hence we start with a general fact about twice-filtered vector spaces.

Suppose a vector space V is twice-filtered. Namely, we have a pair of filtrations, $V = F'_0\supset F'_1\supset\cdots$ and $V = F''_0\supset F''_1\supset\cdots$. We write $F_{p,q} := F'_p\cap F''_q$ (see Figure 1). The associated doubly-graded space of V is defined by

$$\operatorname{gr}^2 V := \bigoplus_{p,q} V_{p,q} := \bigoplus_{p,q}\frac{F_{p,q}}{F_{p+1,q}+F_{p,q+1}}.$$

We can define an additional "diagonal" filtration on V, by setting $F_n := \sum_{p+q=n}F_{p,q}$, and hence a singly-graded associated space $\operatorname{gr}V = \oplus_n V_n := \oplus_n F_n/F_{n+1}$. If $p+q=n$, then $F_{p,q}\subset F_n$ and $F_{p+1,q}+F_{p,q+1}\subset F_{n+1}$, and hence there are maps $V_{p,q}\to V_n$, which induce a map $\alpha\colon \bigoplus_{p+q=n}V_{p,q}\to V_n$.

Lemma 2.4. *The map α is an isomorphism.*

We need a sub-lemma:

Lemma 2.5. *If $D_1\subset D_0$ and $E_1\subset E_0$ are all subsets of the same vector space, and $D_0\cap E_0\subset D_1$, then*

$$\frac{D_0+E_0}{D_1+E_1}\cong\frac{D_0}{D_1}\oplus\frac{E_0}{E_1+D_0\cap E_0}.$$

Proof. Define $\psi\colon \frac{D_0+E_0}{D_1+E_1}\to\frac{D_0}{D_1}\oplus\frac{E_0}{E_1+D_0\cap E_0}$ by $[d_0+e_0]\mapsto([d_0],[e_0])$ and verify that this map is well defined: if $d_0+e_0 = d'_0+e'_0$ with $d_0,d'_0\in D_0$ and $e_0,e'_0\in E_0$, then $d_0-d'_0\in D_0\cap E_0\subset D_1$ and $e_0-e'_0\in D_0\cap E_0$ so $\psi(d_0+e_0)=\psi(d'_0+e'_0)$, and likewise, easily $\psi((d_0+e_0)+(d_1+e_1)) = \psi(d_0+e_0)$ when $d_1\in D_1$ and $e_1\in E_1$. The construction of an inverse of ψ is even easier. □

Proof of Lemma 2.4. We study "the part of the picture where $p\geq s$" and compare it with "the part of the picture where $p\geq s+1$". Let $s\geq 0$ and set $F^s_n := \sum_{p+q=n,\,p\geq s}F_{p,q}$. Then

$$\frac{F^s_n}{F^s_{n+1}} = \frac{F^{s+1}_n+F_{s,n-s}}{F^{s+1}_{n+1}+F_{s,n-s+1}} = \frac{D_0+E_0}{D_1+E_1},$$

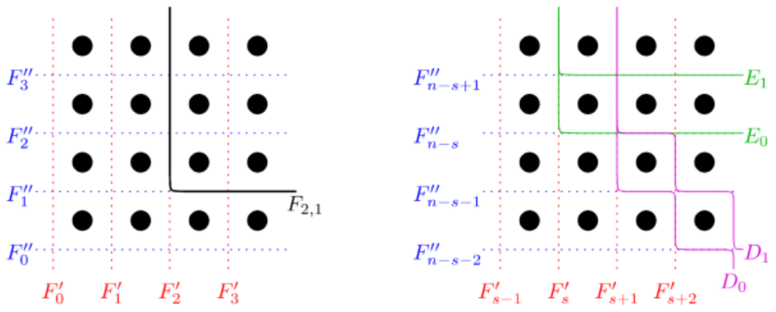

Figure 1. Subspaces of V that can be defined using F'_p and F''_q correspond to monotone subsets of the lattice $\mathbb{Z}^2_{\geq 0}$ and these are defined by their "lower / left boundary lines". For example, on the left $F_{2,1} = F'_2 \cap F''_1$ is "everything above and to the right of the solid black line", and this is the intersection of F'_2, "right of the dotted red line labeled F'_2", and F''_1, "above the dotted blue line labeled F''_1". The right half of this figure displays spaces that occurr within the proof of Lemma 2.4. lem:pain fig:lattic

where we denote $D_0 := F_n^{s+1}$, $D_1 := F_{n+1}^{s+1}$ (the Diagonal terms), and $E_0 := F_{s,n-s}$ and $E_1 := F_{s,n-s+1}$ (the Extra terms). Then $D_0 \cap E_0 = F_{s+1,n-s} \subset D_1$, and ~~hence~~ thus by Lemma 2.5, lem:D0E0

$$\frac{F_n^s}{F_{n+1}^s} \cong \frac{D_0}{D_1} \oplus \frac{E_0}{E_1 + D_0 \cap E_0} = \frac{F_n^{s+1}}{F_{n+1}^{s+1}} \oplus \frac{F_{s,n-s}}{F_{s,n-s+1} + F_{s+1,n-s}} = \frac{F_n^{s+1}}{F_{n+1}^{s+1}} \oplus V_{s,n-s}.$$

Hence by induction,

$$V_n = \frac{F_n^0}{F_{n+1}^0} \cong \frac{F_n^1}{F_{n+1}^1} \oplus V_{0,n} \cong \ldots \cong V_{n,0} \oplus \ldots \oplus V_{0,n}.$$

We leave it to the reader to verify that the above isomorphism is induced by the map α. □

Proof of Proposition 1.5. prop:products Without ~~further~~ comment we will identify G and H as commuting subgroups of $G \times H$ using the coordinate inclusions, and likewise $\mathbb{Q}G$ and $\mathbb{Q}H$ as commuting subalgebras of $\mathbb{Q}(G \times H) = \mathbb{Q}G \otimes \mathbb{Q}H$. Let I_G, I_H, and I_{GH} denote the augmentation ideals of $\mathbb{Q}G$, $\mathbb{Q}H$, and $\mathbb{Q}(G \times H)$ respectively.

Clearly, $I_{GH} = I_G \otimes \mathbb{Q}H + \mathbb{Q}G \otimes I_H$: the "$\supset$" inclusion is obvious, and the "$\subset$" inclusion follows from

$$g \otimes h - 1 \otimes 1 = (g-1) \otimes h + 1 \otimes (h-1).$$

By expanding powers it follows that $I_{GH}^n = (I_G \otimes \mathbb{Q}H + \mathbb{Q}G \otimes I_H)^n = \sum_{p+q=n} I_G^p \otimes I_H^q$, display. and therefore, taking $V = \mathbb{Q}G \otimes \mathbb{Q}H$, $F'_p = I_G^p \otimes \mathbb{Q}H$ and $F''_q = \mathbb{Q}G \otimes I_H^q$ and using notation as in the preceding discussion, $I_{GH}^n = \sum_{p+q=n} F'_p \cap F''_q = F_n$ and hence $\mathcal{A}(G \times H)_n = V_n$. Likewise

$$V_{p,q} = \frac{I_G^p \otimes I_H^q}{I_G^{p+1} \otimes I_H^q + I_G^p \otimes I_H^{q+1}} \cong \frac{I_G^p}{I_G^{p+1}} \otimes \frac{I_H^q}{I_H^{q+1}} = \mathcal{A}(G)_p \otimes \mathcal{A}(H)_q$$

and thus by Lemma 2.4, lem:pain $\mathcal{A}(G \times H)_n \cong \sum_{p+q=n} \mathcal{A}(G)_p \otimes \mathcal{A}(H)_q$. □

MORE.

Semi, bi, almost-direct quote [Ra].

2.3. **More About Expansions** $Z\colon G \to \hat{\mathcal{A}}(G)$. MORE Existance in the plain case, uniqueness, A-expansions.

MORE: Something about GT/GRT.

Quote

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