Emergent version of Drinfeld's associator equations

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Abstract

We present a step-by-step approach to the theory of Drinfeld associators, which is based on the Vassiliev invariants of braids on disks with various number of punctures. We focus on the degree one invariants, the first step of our approach which captures the knotting phenomenon of the braids. The corresponding topological objects are called emergent braids. As an application, we give another proof for a result of Alekseev and Torossian on the embedding of the Grothendieck-Teichmüller Lie algebra into the Kashiwara-Vergne Lie algebra.

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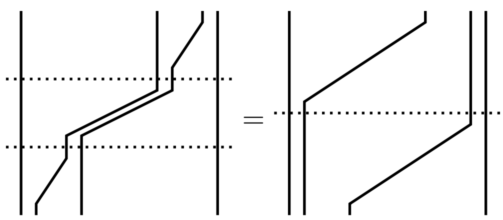
1 Introduction

Two objectives of the paper. (1) Foundations for the category of mixed braids and its formality. (2) To show that the first step (emergent quotient) is already useful: it is related to KV.

Remark/Excuse: (1) is not complete and hopefully to be continued. (2) is not completely satisfactory and hopefully to be continued.

A Drinfeld associator [8] is a group-like formal power series $\Phi = \Phi(x, y)$ in two non-commutative variables satisfying a pentagon equation and two hexagon equations which take values in the algebra of horizontal chord diagrams on several vertical strands.

As was initially pointed out in [8], the Drinfeld associators have a topological nature, and they play important roles in knot theory. A choice of Φ gives rise to universal Vassiliev invariants (Kontsevich invariant) of various knotted objects in 3-space. In [5] the category **PaB** of parenthesized braids was introduced, and it was shown that one can identify the set of 1-formality isomorphisms of this category with the set of Drinfeld associators (with coupling constant 1). In more detail, Φ, regarded as a formal sum of horizontal chord diagrams by a substitution of variables, is the value for the parenthesized braid  which corresponds to the associativity constraint in braided monoidal category. In this formulation, the pentagon equation for Φ comes from the following equality of parenthesized braids:

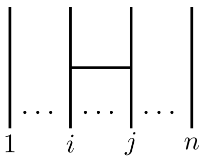


The other equations (the two hexagons) arise similarly.

Drinfeld's associator equations are "hard to solve". By a remarkable result by Furusho [10], the two hexagon equations are actually a consequence of the pentagon equation. However, it is still hard. All known solutions are obtained by some transcendental method, for instance the monodromy of the Knizhnik-Zamolodchikov connection. It is also known that one can construct a Drinfeld associator iteratively [5], but with no explicit example thus constructed. We do not know an example of a rational Drinfeld associator whose coefficients are determined in all degrees.

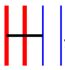
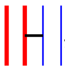
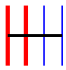
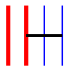
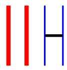
In this paper, we present a step-by-step approach to Drinfeld's associator equations. We consider these equations in certain subquotients (quotients of a subalgebra) of the algebra of horizontal chord diagrams.

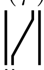
To explain our subquotients in more detail, let us fix some notation. Recall that the Drinfeld-Kohno Lie algebra dk_n, also known as the Lie algebra of infinitesimal Lie algebra, is the graded Lie algebra generated by degree one elements t_{ij} for $1 \leq i < j \leq n$ subject to some relations. Pictorially these generators are the horizontal chord diagrams on n vertical lines which connect the two lines corresponding to their indices:

$t_{ij} =$  .

The pentagon equation for Φ takes values in $\widehat{U}(\mathsf{dk}_4)$, the degree completion of the universal enveloping algebra of dk_4.

Now let $\mathsf{dk}_{2,2}$ be the Lie subalgebra of dk_4 generated by five elements t_{13}, t_{23}, t_{14}, t_{24} and t_{34}. If we draw the first two strands in red and the last two strands in blue, these generators look as follows:

$t_{13} =$ , $t_{23} =$ , $t_{14} =$ , $t_{24} =$ , $t_{34} =$ .

There are no chords between red strands. Since any Φ is of the form $\Phi = \exp(\varphi)$, where φ is a Lie series without linear term, we can regard the value for  determined by Φ as a series in t_{13} and t_{23}, two of the three generators in dk_3. From this simple observation, we see that the pentagon equation for Φ actually takes values in $\widehat{U}(\mathsf{dk}_{2,2})$. In this space, there is no t_{12} which plays the role of detecting the braiding phenomenon between the first and second strands. Thus, we are naturally led to consider the braids on a punctured disk, where we view the red colored strands as the Cartesian product of the punctures and the interval. The crossing changes between blue strands defines the Vassiliev filtration on the algebra spanned by such braids. For the corresponding space of chord diagrams, this filtration amounts to counting the number of chords connecting blue strands. Let c be the Lie ideal of $\mathsf{dk}_{2,2}$

generated by t_{34} and let $\mathsf{c}^{(\mathsf{k})}$ the kth commutator ideal ($\mathsf{c}^{(1)} = \mathsf{c}$, $\mathsf{c}^{(2)} = [\mathsf{c}, \mathsf{c}]$ and so on). Our subquotients are defined to be $\widehat{U}(\mathsf{dk}_{2,2}/\mathsf{c}^{(\mathsf{k})})$ for any $k \geq 1$. In this quotient, we will consider the pentagon equation for a given Φ.

It is possible to view our pentagon equation valued in $\widehat{U}(\mathsf{dk}_{2,2}/\mathsf{c}^{(\mathsf{k})})$ as a piece of the set of equations required for a 1-formality isomorphism for braids on disks with various number of punctures. For precise formulation, we need to introduce the category **PaMB** of parenthesized mixed braids and its truncated version $\mathbf{PaMB}^{/k}$ to which Sections 2 and 4 are devoted.

central definition: 1-formality isomorphisms for these categories. the exposition is incomplete in the sense that we do not know a complete set of relations.

When $k = 1$, the braids in $\mathbf{PaMB}^{/1}$ are considered only up to homotopy, and thus one cannot detect any braiding phenomenon. In Section 3, we consider the case $k = 2$, which amounts to the degree one Vassiliev invariants of braids on punctured disks. We call the corresponding topological objects emergent braids. This will be the focus of the present paper.

One motivation for studying emergent braids lies in its relationship with the Kashiwara-Vergne theory. In [4], Alekseev and Torossian introduced a graded Lie algebra krv_2 called the Kashiwara-Vergne Lie algebra. Its elements are pairs $(u(x,y), v(x,y))$ of two Lie polynomials satisfying some conditions (see Section 5.3 for more precise definition). The Lie algebra krv_2 describes the infinitesimal deformations to solutions to the Kashiwara-Vergne equations. Alekseev and Torossian showed that there is a Lie algebra embedding

$$\nu : \mathsf{grt}_1 \hookrightarrow \mathsf{krv}_2$$

TODO: say more about KV and grt_1.

from the Grothendieck-Teichmüller Lie algebra, and in addition proved that any Drinfeld's associator Φ gives rise to a solution to the Kashiwara-Vergne equations. We will mainly work in the infinitesimal setting and give another proof for the embedding above.

the conjecture $\mathsf{grt}_1 = \mathsf{lie}(\sigma_3, \sigma_5, \ldots)$. the conjecture $\mathsf{krv}_2 = \mathbb{Q}t \oplus \nu(\mathsf{grt}_1)$.

Let us state a main application of our approach to the Kashiwara-Vergne theory. We define $\mathsf{grt}_1^{\mathrm{em}}$ to be the space of Lie polynomials $\varphi = \varphi(x,y)$ satisfying the linearization of a certain pentagon equation valued in $\mathsf{dk}_{2,2}/\mathsf{c}^{(2)}$ and an additional technical condition $[x, \varphi(y,x)] + [y, \varphi(x,y)] = 0$ (for more precise definition, see Section 4.3). By construction, we have a natural embedding $\mathsf{grt}_1 \hookrightarrow \mathsf{grt}_1^{\mathrm{em}}$. An element $(u(x,y), v(x,y)) \in \mathsf{krv}_2$ is called symmetric if $v(x,y) = u(y,x))$. The space of symmetric elements in krv_2 forms a Lie subalgebra denoted by $\mathsf{krv}_2^{\mathrm{sym}}$ (see [4, Section 8]). For a Lie series $\varphi = \varphi(x,y) \in \mathsf{lie}_2$, set

$$\nu^{\mathrm{em}}(\varphi) := (\varphi(y,x), \varphi(x,y)).$$

Theorem 1.1. (i) *For any $\varphi \in \mathsf{grt}_1^{\mathrm{em}}$, we have $\nu^{\mathrm{em}}(\varphi) \in \mathsf{krv}_2^{\mathrm{sym}}$.*

(ii) *The map $\nu^{\mathrm{em}} : \mathsf{grt}_1^{\mathrm{em}} \to (\mathsf{krv}_2^{\mathrm{sym}})_{\geq 2}$ is a graded $\mathbb{Q}$-linear isomorphism.*

It turns out that the space $\mathsf{grt}_1^{\mathrm{em}}$ has a Lie algebra structure. Note that it is not known whether $\mathsf{krv}_2^{\mathrm{sym}}$ coincides with krv_2 or not [4, Remark 8.10].

The proof of Theorem 1.1 will be given in Section 6. Our proof uses an interpretation of the Kashiwara-Vergne theory in terms of surface topology [1, 2], which will be explained in Section 5.

The idea of emergent braids stems from an ongoing joint work of the first-named author with Dancso, Hogan, Liu and Scherich [6]. Recent works by Alekseev, Naef and Ren [3] and Naef and Betancourt [15] study the same quotient $\mathsf{dk}_{2,2}/\mathsf{c}^{(2)}$ and closely related topics to ours.

Acknowledgements

The authors thank... The first-named author is supported by... The second-named author is supported by...

Notation

- Throughout this paper we work over the rationals $\mathbb{Q}$, though all of our argument holds true over any field of characteristic zero.

- For a nonnegative integer n, let ass_n be the free associative algebra on n free generators. When we need to specify generators, we write $\mathsf{ass}_n = \mathsf{ass}(x_1, \dots, x_n)$ for example.

- The algebra ass_n has a structure of Hopf algebra whose coproduct, antipode and augmentation are given on generators by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, $\iota(x_i) = -x_i$ and $\varepsilon(x_i) = 1$. We also use the notation $\overline{a} = \iota(a)$ for the antipode.

- We denote by $\mathsf{lie}_n = \mathsf{lie}(x_1, \dots, x_n)$ the free Lie algebra on n free generators $x_1, \dots, x_n$. One can identify lie_n with the space of primitive elements in ass_n, namely $\mathsf{lie}_n = \{a \in \mathsf{ass}_n \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}$. It holds that $\iota(a) = -a$ for any $a \in \mathsf{lie}_n$.

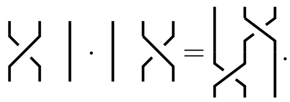
- Let $\mathcal{C}$ be a groupoid or more generally a category, and O, O' objects in $\mathcal{C}$. We denote by $\mathcal{C}(O, O')$ the set of morphisms in $\mathcal{C}$ from O to O'.

2 Mixed braids and chord diagrams

We introduce the notion of mixed braids. Then we define the notion of mixed chord diagrams as the corresponding associated graded object.

2.1 Mixed braids

For a nonnegative integer l, let B_l be Artin's braid group on l strands. Our convention about the product of B_l is as follows: the product $\beta\beta'$ of two braids β and β' is the braid obtained by placing β' above β. For example,

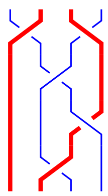


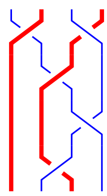
Definition 2.1. Fix nonnegative integers m and n. A mixed braid of type (m, n) is an element of B_{m+n} equipped with a coloring of its strands with either red or blue such that

- there are m red colored strands which we draw slightly thicker and n blue colored strands which we draw slightly thinner, and
- if we forget all the blue colored strands and view the rest as an element in B_m, we are left with the trivial m-braid.

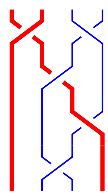
A blue colored strand in a mixed braid is simply called a strand, and a red colored strand is called a pole.

Example 2.2. In the following three pictures, the first two pictures are mixed braids of type $(2, 2)$. Observe that their underlying braids on $2+2 = 4$ strands are the same. However, the picture on the right is not a mixed braid.





non-example:

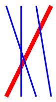


We denote by $B_{m,n}$ the set of mixed braids of type (m, n). One can construct the product of two mixed braids β, β' of the same type when the coloring of the strands of β at the top matches that of β' at the bottom. In this manner, the set $B_{m,n}$ forms a groupoid. Its set of objects is the set $W_{m,n}$ of words of length $m+n$ consisting of m red (slightly bigger) bullets ● and n blue (slightly smaller) bullets •. When $o \in W_{m,n}$, the word o is called of type (m, n). For $o, o' \in W_{m,n}$, we denote by $B_{m,n}(o, o')$ the set of mixed braids whose bottom and top ends match o and o', respectively. For example, the leftmost picture in Example 2.2 is an element in $B_{2,2}$(●●••, •●●•).

Definition 2.3. Let $m, n \geq 0$ and let $o, o' \in W_{m,n}$. A mixed permutation (of type (m, n)) from o to o' is a permutation σ of $m + n$ letters such that

- for any $1 \leq i \leq m + n$, the ith letter of o and the $\sigma(i)$th letter of o' have the same color, and
- if we forget all the blue letters in o and o' and view the restriction of σ to the red bullets as a permutation of m letters, then it is trivial.

Alternatively, a mixed permutation is a mixed braid whose over/under information at each crossing of strands is lost. For example,



is a mixed permutation from ${\color{red}\bullet}{\color{blue}\bullet\bullet\bullet}$ to ${\color{blue}\bullet\bullet\bullet}{\color{red}\bullet}$ given by $1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 1$, and $4 \mapsto 3$.

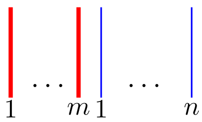
For $o, o' \in W_{m,n}$, we denote by $\mathfrak{S}_{m,n}(o, o')$ the set of mixed permutations from o to o'. The set $\mathfrak{S}_{m,n} = \bigsqcup_{o,o' \in W_{m,n}} \mathfrak{S}_{m,n}(o, o')$ of all mixed permutations of type (m, n) naturally forms a groupoid. The forgetful map

$$\pi : B_{m,n} \to \mathfrak{S}_{m,n}$$

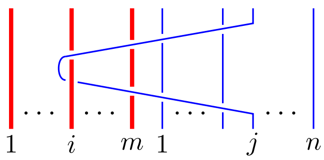
is a homomorphism of groupoids.

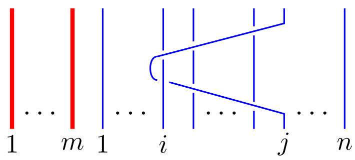
Let $o^{\mathrm{std}}_{m,n} := \underbrace{{\color{red}\bullet \cdots \bullet}}_{m}\underbrace{{\color{blue}\bullet \cdots \bullet}}_{n} \in W_{m,n}$, then the set $B^{\mathrm{std}}_{m,n} := B_{m,n}(o^{\mathrm{std}}_{m,n}, o^{\mathrm{std}}_{m,n})$ forms a group with respect to the groupoid structure of $B_{m,n}$. One can regard $B^{\mathrm{std}}_{m,n}$ a subgroup of B_{m+n} in a natural way.

The trivial permutation of degree $m + n$ defines the mixed permutation

$1_{m,n} :=$  $\in \mathfrak{S}_{m,n}(o^{\mathrm{std}}_{m,n}, o^{\mathrm{std}}_{m,n})$.

Then, $P_{m,n} := \pi^{-1}(1_{m,n})$ is a normal subgroup of $B^{\mathrm{std}}_{m,n}$. We call $P_{m,n}$ the mixed pure braid group of type (m, n). In fact, Lambropoulou [11, Sections 2 and 3] introduced the same group with the same notation and gave its explicit presentation. In particular, $P_{m,n}$ is generated by the following elements α_{ij}, where $1 \le i \le m$, $1 \le j \le n$, and τ_{ij}, where $1 \le i < j \le n$:

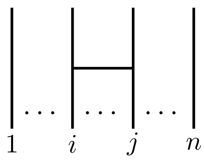
$\alpha_{ij} =$  ,

$\tau_{ij} =$  .

Collecting all types of mixed braids and mixed permutations we consider the groupoids $B_{{\color{red}\bullet}{\color{blue}\bullet}} := \bigsqcup_{m,n \ge 0} B_{m,n}$ and $\mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}} := \bigsqcup_{m,n \ge 0} \mathfrak{S}_{m,n}$. Both of them have $W_{{\color{red}\bullet}{\color{blue}\bullet}} := \bigsqcup_{m,n \ge 0} W_{m,n}$ as the set of objects. We define the category **MB** of mixed braids as a $\mathbb{Q}$-linear extension of the groupoid $B_{{\color{red}\bullet}{\color{blue}\bullet}}$ fibered over $\mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}$, following the treatment in [5, Section 2.2.1]. Its set of objects is $W_{{\color{red}\bullet}{\color{blue}\bullet}}$. Let $o, o' \in W_{{\color{red}\bullet}{\color{blue}\bullet}}$. If the types of o and o' are different, there is no morphism from o to o'. If not, then morphisms from o to o' are pairs $(\sum_j c_j\beta_j, \sigma)$, where $\sigma \in \mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}(o, o')$ and $\sum_j c_j\beta_j$ is a $\mathbb{Q}$-linear combination of mixed braids such that $\pi(\beta_j) = \sigma$ for all j. Thus when the types of o and o' are the same, the set of morphisms from o to o' decomposes as $\mathbf{MB}(o,o') = \bigsqcup_{\sigma \in \mathfrak{S}_{\textcolor{red}{\bullet}\textcolor{blue}{\bullet}}(o,o')} \mathbf{MB}(o,o')_\sigma$, where the subscript σ stands for consisting of elements which have σ as the second entry. The composition in $\mathbf{MB}$ is naturally induced from the composition in $B_{\textcolor{red}{\bullet}\textcolor{blue}{\bullet}}$.

2.2 Mixed version of the Drinfeld-Kohno Lie algebra

Let n be a nonnegative integer. Recall that the Drinfeld-Kohno Lie algebra, which we denote by dk_n, is the graded Lie algebra generated by degree one elements $t_{ij} = t_{ji}$ for $1 \le i \ne j \le n$ subject to the commutation relation $[t_{ij}, t_{kl}] = 0$ for distinct indices i, j, k, l, and the 4T relation $[t_{ij} + t_{jk}, t_{ik}] = 0$ for distinct indices i, j, k. In a diagrammatic language, dk_n is the Lie algebra of horizontal chord diagrams on n vertical lines, and the generator t_{ij} corresponds to the chord diagram consisting of a single chord connecting the ith and jth lines:

$t_{ij} =$ 

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For every $n > 0$, there is a semi-direct product decomposition

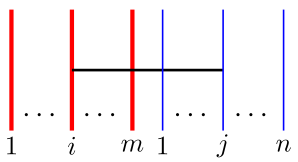
$$\mathsf{dk}_n = \mathsf{dk}_{n-1} \ltimes \mathsf{lie}(t_{1n}, \dots, t_{(n-1)n}). \tag{1}$$

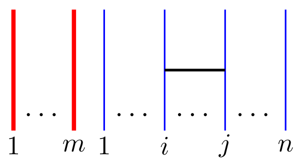
It is known that the universal enveloping algebra of dk_n is isomorphic to the associated graded of the group algebra of the pure braid group on n strands with respect to the powers of the augmentation ideal. With this in mind, we introduce a variant of dk_n corresponding to the group $P_{m,n}$.

TODO: reference? Kohno "Série de Poincaré-Koszul..." or, Fresse's textbook...

Definition 2.4. For $m, n \ge 0$, let $\mathsf{dk}_{m,n}$ be the graded Lie algebra generated by degree one elements a_{ij} for $1 \le i \le m$, $1 \le j \le n$ and $c_{ij} = c_{ji}$ for $1 \le i \ne j \le n$, subject to the commutation and 4T relations among them, where we regard $a_{ij} = t_{i(m+j)}$ and $c_{ij} = t_{(m+i)(m+j)}$ as the corresponding generators of dk_{m+n}.

Diagrammatically, the generators of $\mathsf{dk}_{m,n}$ are horizontal chord diagrams with a single chord on m vertical red lines and n vertical blue lines:

$a_{ij} =$ 

, $c_{ij} =$ 

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Remark 2.5. We have $\mathsf{dk}_{0,n} = \mathsf{dk}_n$.

The semi-direct product decomposition (1) generalizes to $\mathsf{dk}_{m,n}$:

Lemma 2.6. *There is a semi-direct product decomposition of Lie algebra*

$$\mathsf{dk}_{m,n} = \mathsf{dk}_{m,n-1} \ltimes \mathsf{lie}(a_{1n}, \ldots, a_{mn}, c_{1n}, \ldots, c_{(n-1)n}).$$

Proof. We simply write $\mathsf{lie}(a,c) = \mathsf{lie}(a_{1n}, \ldots, a_{mn}, c_{1n}, \ldots, c_{(n-1)n})$. First we describe the Lie action ρ of $\mathsf{dk}_{m,n-1}$ on $\mathsf{lie}(a,c)$ that is used in forming the semi-direct product $\mathsf{dk}_{m,n-1} \ltimes \mathsf{lie}(a,c)$. It is specified by the value on generators of $\mathsf{dk}_{m,n-1}$: for $1 \le i \le m$, $1 \le j \le n-1$, $1 \le k \le m$ and $1 \le l \le n-1$,

$$\rho(a_{ij})(a_{kn}) = \begin{cases} 0 & (i \neq k) \\ -[c_{jn}, a_{kn}] & (i = k) \end{cases}, \quad \rho(a_{ij})(c_{ln}) = \begin{cases} 0 & (j \neq l) \\ -[a_{in}, c_{ln}] & (j = l) \end{cases},$$

and for $1 \le i \neq j \le n-1$, $1 \le k \le m$ and $1 \le l \le n-1$,

$$\rho(c_{ij})(a_{kn}) = 0, \qquad \rho(c_{ij})(c_{ln}) = \begin{cases} 0 & (l \notin \{i,j\}) \\ -[c_{jn}, c_{ln}] & (i = l) \end{cases}.$$

Note that these formulas are compatible with the Lie bracket in $\mathsf{dk}_{m,n}$. For example, we have $[a_{kj}, a_{kn}] = -[c_{jn}, a_{kn}]$ by the 4T relation, and this matches the value $\rho(a_{kj})(a_{kn}) = -[c_{jn}, a_{kn}]$. Now we define the map $\mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n-1} \ltimes \mathsf{lie}(a,c)$ by

$$a_{ij} \mapsto \begin{cases} (a_{ij}, 0) & (j \le n-1) \\ (0, a_{in}) & (j = n) \end{cases}, \qquad c_{ij} \mapsto \begin{cases} (c_{ij}, 0) & (j \le n-1) \\ (0, c_{in}) & (j = n) \end{cases}.$$

Then one can check that this map is a Lie algebra isomorphism. □

added a new remark

Remark 2.7. By Lemma 2.6, we inductively see that the map $\mathsf{dk}_{m,n} \to \mathsf{dk}_{m+n}$ defined by $a_{ij} \mapsto t_{i(m+j)}$ and $c_{ij} \mapsto t_{(m+i)(m+j)}$ is an injective Lie homomorphism.

We show how the Lie algebra $\mathsf{dk}_{m,n}$ and the group $P_{m,n}$ are related. On the one hand, let $\mathcal{A}_{m,n} = U(\mathsf{dk}_{m,n})$ be the universal enveloping algebra of $\mathsf{dk}_{m,n}$. It is an associative $\mathbb{Q}$-algebra generated by the same generators a_{ij} and c_{ij} as those of $\mathsf{dk}_{m,n}$, subject to the same relations as those of $\mathsf{dk}_{m,n}$, where we regard bracket symbol as commutator: $[a,b] = ab - ba$. On the other hand, the powers of the augumentation ideal $I = IP_{m,n}$ define a decreasing filtration of $\mathbb{Q}P_{m,n}$. Thus one can construct the associated graded $\operatorname{gr}\mathbb{Q}P_{m,n}$ of the filtered algebra $\mathbb{Q}P_{m,n}$.

Proposition 2.8. *There is a canonical isomorphism of graded $\mathbb{Q}$-algebras $\operatorname{gr}\mathbb{Q}P_{m,n} \cong \mathcal{A}_{m,n}$, through which the class of $\alpha_{ij} - 1$ corresponds to a_{ij} and the class of $\tau_{ij} - 1$ to c_{ij}.*

Proof. The proof is similar to the proof of the isomorphism $\operatorname{gr}\mathbb{Q}P_n \cong U(\mathsf{dk}_n)$ given in [9, Theorem 10.0.4], so we just give a sketch. We start with the fact that there is a semi-direct product decomposition

$$P_{m,n} \cong P_{m,n-1} \ltimes F_{m+n-1},$$

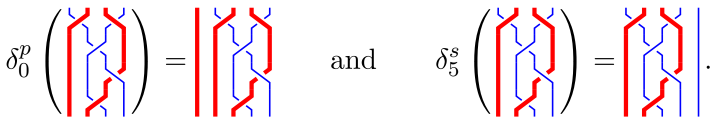
where F_{m+n-1} is the free group generated by α_{in}, $1 \leq i \leq m$ and τ_{in}, $1 \leq i \leq n-1$ (see [11, Section 3]). Here, the action of $P_{m,n-1}$ on F_{m+n-1} is by conjugation and hence is trivial on the abelianization of F_{m+n-1}. Applying [9, Proposition 8.5.7], one has $\operatorname{gr}\mathbb{Q}P_{m,n} \cong (\operatorname{gr}\mathbb{Q}P_{m,n-1})\,\sharp\,(\operatorname{gr}\mathbb{Q}F_{m+n-1})$, where $\sharp$ denotes the semi-direct product of Hopf algebras. Note that $\operatorname{gr}\mathbb{Q}F_{m+n-1}$ is naturally isomorphic to ass_{m+n-1}, and Lemma 2.6 implies that there is an isomorphism $\mathcal{A}_{m,n-1}\sharp\mathsf{ass}_{m+n-1} \cong \mathcal{A}_{m,n}$. Hence we can prove $\operatorname{gr}\mathbb{Q}P_{m,n} \cong \mathcal{A}_{m,n}$ by induction on n. One can check that this isomorphism maps the class of $\alpha_{ij} - 1$ to a_{ij} and the class of $\tau_{ij} - 1$ to c_{ij}. □

TODO: do we need the normalized expression for elements in $\mathcal{A}^{\mathrm{em}}_{m,n}$? In particular, do we need the formula $\exp(x_1 + x_2) = \exp(x_1)\exp(x_2) + \exp(x_1)\exp(x_2)E(x)_{12}$? Here, $E(x) = ((1 - e^{-x})/x) - 1$. Recall the classical case?

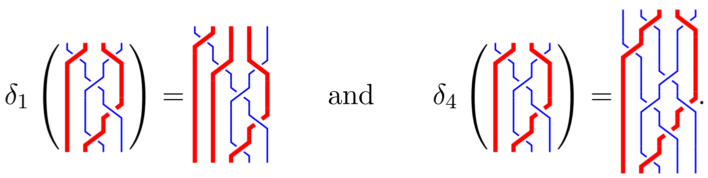
2.3 Operadic structure and coface maps

There are naturally defined operations on mixed braids. Let $\beta \in B_{m,n}$ be a mixed braid.

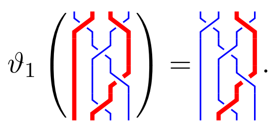
- Extension operations. We denote by $\delta_0^p(\beta)$ (resp. $\delta_0^s(\beta)$) be the mixed braid of type $(m+1, n)$ (resp. of type $(m, n+1)$) obtained from β by adding a red (resp. blue) straight strand on the left. Similarly, we define $\delta_{m+n+1}^p(\beta)$ (resp. $\delta_{m+n+1}^s(\beta)$) by adding a red strand (resp. blue strand) on the right. For example,



- Cabling operations. For $1 \leq i \leq m+n$, let $\delta_i(\beta)$ be the mixed braid obtained from β by doubling its ith strand, where we count strands at the bottom end of β. The two strands newly created inherits the color of the original strand. For example,



- Changing a pole to a strand. For $1 \leq i \leq m$, let $\vartheta_i(\beta)$ be the mixed braid obtained from β by changing the ith red strand to a blue strand. For example,



The operations defined above have counterparts in $\mathsf{dk}_{m,n}$.

TODO: want diagrammatic explanations?

- Extension operations. Let $\delta_0 = \delta_0^p : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m+1,n}$ (resp. $\delta_{m+n+1} = \delta_{m+n+1}^s : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n+1}$) be the Lie homomorphism defined by $a_{ij} \mapsto a_{(i+1)j}$ and $c_{ij} \mapsto c_{ij}$ (resp. $a_{ij} \mapsto a_{ij}$ and $c_{ij} \mapsto c_{ij}$).
- Cabling operations. For $1 \le k \le m$, we define the Lie homomorphism $\delta_k : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m+1,n}$ by

$$\delta_k(a_{ij}) = \begin{cases} a_{ij} & (1 \le i \le k-1) \\ a_{kj} + a_{(k+1)j} & (i = k) \\ a_{(i+1)j} & (k+1 \le i \le m) \end{cases}, \qquad \delta_k(c_{ij}) = c_{ij}.$$

  For $1 \le k \le n$, we define $\delta_{m+k} : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n+1}$ by

$$\delta_{m+k}(a_{ij}) = \begin{cases} a_{ij} & (1 \le j \le k-1) \\ a_{ik} + a_{i(k+1)} & (j = k) \\ a_{i(j+1)} & (k+1 \le j \le n) \end{cases}$$

  and

$$\delta_{m+k}(c_{ij}) = \begin{cases} c_{ij} & (j < k) \\ c_{ik} + c_{i(k+1)} & (j = k) \\ c_{i(j+1)} & (i < k < j) \\ c_{k(j+1)} + c_{(k+1)(j+1)} & (i = k) \\ c_{(i+1)(j+1)} & (k < i) \end{cases}.$$

- Changing a pole to a strand. For the sake of simplicity we only introduce this operation applied to the last pole. Let $\vartheta_m : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m-1,n+1}$ be the Lie homomorphism defined by

$$\vartheta_m(a_{ij}) = \begin{cases} a_{i(j+1)} & (i < m) \\ c_{1(j+1)} & (i = m) \end{cases}, \qquad \vartheta_m(c_{ij}) = c_{(i+1)(j+1)}.$$

Using these operations, we define coface maps and a differential on $\mathsf{dk}_{m,n}$.

Definition 2.9. For $0 \le k \le m+n+1$, we define the map $d_k = d_k^{m,n} : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n+1}$ as follows:

$$d_k = \begin{cases} \vartheta_{m+1} \circ \delta_k & (0 \le k \le m) \\ \delta_k & (m+1 \le k \le m+n+1) \end{cases}.$$

Furthermore, we set $d^{m,n} := \sum_{k=0}^{m+n+1} (-1)^k d_k : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n+1}$.

The family of maps $\{d^{m,n}\}_n$ is indeed a differential.

Lemma 2.10. *We have* $d^{m,n+1} \circ d^{m,n} = 0 : \mathsf{dk}_{m,n} \to \mathsf{dk}_{m,n+2}$.

Proof. The proof is straightforward by using the relation $d_i \circ d_j = d_{j+1} \circ d_i$ for $i \le j$, which can be checked directly. □

3 Emergent braids and chord diagrams

In this section, we introduce the notion of emergent braids and chord diagrams.

3.1 Emergent braids

The group $B^{\mathrm{std}}_{m,n}$ acts on its normal subgroup $P_{m,n}$ by conjugation, and this extends linearly to an action on the group algebra $\mathbb{Q}P_{m,n}$. We denote by J the two-sided ideal of $\mathbb{Q}P_{m,n}$ generated by $\tau_{ij}-1$, $1 \leq i < j \leq n$. The powers J^l, $l \geq 0$, define a $B^{\mathrm{std}}_{m,n}$-invariant decreasing filtration of $\mathbb{Q}P_{m,n}$.

Definition 3.1. For each $k \geq 1$ we set $\mathbb{Q}P^{/k}_{m,n} := \mathbb{Q}P_{m,n}/J^k$. In particular, the algebra of emergent pure braids of type (m,n) is defined to be

$$\mathbb{Q}P^{\mathrm{em}}_{m,n} := \mathbb{Q}P^{/2}_{m,n} = \mathbb{Q}P_{m,n}/J^2.$$

Remark 3.2. Why "emergent"? In primary school language, "Dror has an emergent knowledge of the French language" means "Dror knows French just a bit better than nothing at all". In a similar way, $\mathbb{Q}P^{/1}_{m,n}$ means "no braiding phenomenon yet", for in $\mathbb{Q}P^{/1}_{m,n}$ the blue strands are fully transparent to each other, and $\mathbb{Q}P^{/2}_{m,n}$ is "emergent braiding", for after moding out by J^2 just a whiff of braiding remains.

The ideal J of $\mathbb{Q}P_{m,n} = \mathbf{MB}(o^{\mathrm{std}}_{m,n}, o^{\mathrm{std}}_{m,n})_{1_{m,n}}$ and its powers extend to a multiplicative filtration of the $\mathbb{Q}$-linear category $\mathbf{MB}$ in the following way. Let $o, o' \in W_{m,n}$ for some $m, n \geq 0$ and let $\sigma \in \mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}(o,o')$. One can take mixed braids $\beta \in B_{m,n}(o^{\mathrm{std}}_{m,n}, o)$ and $\beta' \in B_{m,n}(o^{\mathrm{std}}_{m,n}, o')$ such that $\sigma = \pi(\beta)^{-1}\pi(\beta')$. Then, the map $\mathbb{Q}P_{m,n} \to \mathbf{MB}(o,o')_\sigma, u \mapsto \beta^{-1}u\beta'$ is a $\mathbb{Q}$-linear isomorphism. Since the ideal J is $B^{\mathrm{std}}_{m,n}$-invariant, it follows that the subspaces $J^l_\sigma := \beta^{-1}J^l\beta'$, $l \geq 0$, are independent of the choice of β and β'. The collection $\{J^l_\sigma\}_{l\geq 0, \sigma\in\mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}}$ is multiplicative in the sense that $J^l_\sigma \cdot J^{l'}_{\sigma'} \subset J^{l+l'}_{\sigma\sigma'}$ holds for any $l, l' \geq 0$ whenever σ and σ' are composable.

For each $k \geq 1$, we define the $\mathbb{Q}$-linear category $\mathbf{MB}^{/k}$ as follows. The set of objects is $W_{{\color{red}\bullet}{\color{blue}\bullet}}$. For $o, o' \in W_{{\color{red}\bullet}{\color{blue}\bullet}}$, the set of morphisms from o to o' is

$$\mathbf{MB}^{/k}(o,o') := \begin{cases} \bigsqcup_{\sigma\in\mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}(o,o')} \dfrac{\mathbf{MB}(o,o')_\sigma}{J^k_\sigma} & \text{if } o \text{ and } o' \text{ have the same type,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The composition in $\mathbf{MB}^{/k}$ is induced from the composition in $\mathbf{MB}$. Our main focus is on the case $k=2$: we set $\mathbf{EB} := \mathbf{MB}^{/2}$.

3.2 Emergent version of the Drinfeld-Kohno Lie algebra

Let $\mathsf{c} = \mathsf{c}_{m,n}$ be the Lie ideal of $\mathsf{dk}_{m,n}$ generated by c_{ij} for $1 \le i \ne j \le n$.

Definition 3.3. The emergent version of the Drinfeld-Kohno Lie algebra of type (m, n) is the quotient Lie algebra $\mathsf{edk}_{m,n} := \mathsf{dk}_{m,n}/[\mathsf{c}, \mathsf{c}]$.

Remark 3.4. Similarly, for each $k \ge 1$ one can define the quotient Lie algebra $\mathsf{dk}^{/k}_{m,n} := \mathsf{dk}_{m,n}/\mathsf{c}^{(k)}$, where $\mathsf{c}^{(k)}$ is the Lie ideal of $\mathsf{dk}_{m,n}$ inductively defined by $\mathsf{c}^{(1)} = \mathsf{c}$ and $\mathsf{c}^{(k)} = [\mathsf{c}^{(k-1)}, \mathsf{c}]$. One has $\mathsf{edk}_{m,n} = \mathsf{dk}^{(2)}_{m,n}$.

In what follows we describe the structure of the Lie algebra $\mathsf{edk}_{m,n}$.

Lemma 3.5. *We have a $\mathbb{Q}$-linear graded direct sum decomposition*

$$\mathsf{edk}_{m,n} \cong \mathsf{edk}_{m,n-1} \oplus \left(\mathsf{lie}_m(a_{1n}, \dots, a_{mn}) \oplus \bigoplus_{i=1}^{n-1} \mathsf{ass}_m(a_{1n}, \dots, a_{mn})[-1] \right).$$

Here, $\mathsf{ass}_m(a_{1n}, \dots, a_{mn})[-1]$ is the degree shift of $\mathsf{ass}_m(a_{1n}, \dots, a_{mn})$ by -1: the constant term has degree 1, the generators $x_1, \dots, x_m$ have degree 2, and so on.

Proof. Let c_0 be the Lie ideal of $\mathsf{lie}(a, c)$ generated by c_{in}, $1 \le i \le n-1$. Through the semi-direct decomposition of Lemma 2.6 the ideal $[\mathsf{c}_{m,n}, \mathsf{c}_{m,n}]$ corresponds to $[\mathsf{c}_{m,n-1}, \mathsf{c}_{m,n-1}] \oplus [\mathsf{c}_0, \mathsf{c}_0]$ in $\mathsf{dk}_{m,n-1} \oplus \mathsf{lie}(a, c)$, because $\mathsf{c}_{m,n} = \mathsf{c}_{m,n-1} \oplus \mathsf{c}_0$ and $[\mathsf{c}_{m,n-1}, \mathsf{c}_0] \subset [\mathsf{c}_0, \mathsf{c}_0]$. Thus we obtain

$$\mathsf{edk}_{m,n} \cong \mathsf{edk}_{m,n-1} \oplus \big(\mathsf{lie}(a, c)/[\mathsf{c}_0, \mathsf{c}_0]\big)$$

as a $\mathbb{Q}$-linear space. By the Lazard elimination theorem [7, Chap II §2.9, Proposition 10], we have the following $\mathbb{Q}$-linear direct sum decomposition

$$\mathsf{lie}(a, c) \cong \mathsf{lie}(a_{1n}, \dots, a_{mn}) \oplus \mathsf{lie}(\{\mathrm{ad}_w(c_{in})\}_{w,i}).$$

Here, $\mathsf{lie}(\{\mathrm{ad}_w(c_{in})\}_{w,i})$ is the free Lie algebra generated by all elements of the form $\mathrm{ad}_w(c_{in}) = \mathrm{ad}_{w_1} \cdots \mathrm{ad}_{w_\lambda}(c_{in})$, where $1 \le i \le n-1$ and $w = w_1 \cdots w_\lambda$ with $w_1, \dots, w_\lambda \in \{a_{1n}, \dots, a_{mn}\}$ runs over all associative words in $a_{1n}, \dots, a_{mn}$ (including the empty word). Hence

$$\mathsf{lie}(a, c)/[\mathsf{c}_0, \mathsf{c}_0] \cong \mathsf{lie}(a_{1n}, \dots, a_{mn}) \oplus \bigoplus_{i=1}^{n-1} \bigoplus_{w} \mathbb{Q}\, \mathrm{ad}_w(c_{in}).$$

This proves the lemma. □

Repeated use of Lemma 3.5 yields a $\mathbb{Q}$-linear graded direct sum decomposition

$$\mathsf{edk}_{m,n} \cong \bigoplus_{i=1}^{n} (\mathsf{lie}_m)_i \oplus \bigoplus_{1 \le i < j \le n} (\mathsf{ass}_m[-1])_{ij}, \tag{2}$$

where the meaning of the components $(\mathsf{lie}_m)_i$ and $(\mathsf{ass}_m[-1])_{ij}$ is as follows:

$$(\mathsf{lie}_m)_i \ni u(x_1,\dots,x_m)_i \mapsto u(a_{1i},\dots,a_{mi}) \in \mathsf{edk}_{m,n},$$
$$(\mathsf{ass}_m[-1])_{ij} \ni w(x_1,\dots,x_m)_{ij} \mapsto \mathrm{ad}_{w_j}(c_{ij}) \in \mathsf{edk}_{m,n}.$$

Here, $u = u(x_1,\dots,x_m) \in \mathsf{lie}_m$, $w = w(x_1,\dots,x_m) \in \mathsf{ass}_m$ and we write $w_j = w(a_{1j},\dots,a_{mj}) \in \mathsf{ass}(a_{1j},\dots,a_{mj})$.

Example 3.6. (i) $\mathsf{edk}_{2,1} \cong \mathsf{lie}_2$.

(ii) $\mathsf{edk}_{1,2} \cong (\mathsf{lie}_1)_1 \oplus (\mathsf{lie}_1)_2 \oplus (\mathsf{ass}_1[-1])_{12} \cong \mathbb{Q}x_1 \oplus \mathbb{Q}x_2 \oplus \mathsf{ass}(x)[-1]$.

(iii) $\mathsf{edk}_{2,2} \cong \mathsf{lie}(x,y)_1 \oplus \mathsf{lie}(x,y)_2 \oplus (\mathsf{ass}(x,y)[-1])_{12}$.

In order to describe the Lie bracket on $\mathsf{edk}_{m,n}$ in view of the direct sum decomposition (2), we need to recall the partial differential operators on lie_m with respect to the generators $x_1,\dots,x_m$. Let $a \in \mathsf{lie}_m$. Viewed as an element in ass_m, it is uniquely written as

$$a = \sum_{i=1}^m (\partial_i a) x_i = \sum_{i=1}^m x_i(\partial^i a),$$

where $\partial_i a, \partial^i a \in \mathsf{ass}_m$. Furthermore, we have $\partial^i a = \iota(\partial_i a)$. The operator $\partial_i : \mathsf{lie}_m \to \mathsf{ass}_m$ satisfies the following formula: for any $u, v \in \mathsf{lie}_m$,

$$\partial_i([u,v]) = u(\partial_i v) - v(\partial_i u). \tag{3}$$

The following proposition describes the Lie bracket on $\mathsf{edk}_{m,n}$.

Proposition 3.7. *Let $u = u(x_1,\dots,x_m), v = v(x_1,\dots,x_m) \in \mathsf{lie}_m$ and $w = w(x_1,\dots,x_m), w' = w(x_1,\dots,x_m) \in \mathsf{ass}_m$.*

(i) *For any $1 \le j \le n$, we have $[u_j, v_j] = [u,v]_j$. For any $1 \le j < k \le n$,*

$$[u_j, v_k] = \left(\sum_{i=1}^m (\partial_i v) x_i \iota(\partial_i u)\right)_{jk}. \tag{4}$$

(ii) *Let $1 \le i \le n$ and $1 \le j < k \le n$. If $i \notin \{j,k\}$, we have $[u_i, w_{jk}] = 0$. Furthermore, we have $[u_k, w_{jk}] = (uw)_{jk}$ and $[u_j, w_{jk}] = -(wu)_{jk}$.*

(iii) *We have $[w_{ij}, w'_{kl}] = 0$ for any $1 \le i < j \le n$ and $1 \le k < l \le n$.*

We need a lemma.

Lemma 3.8. *For $w = w(x_1,\dots,x_m) \in \mathsf{ass}_m$ and $1 \le j \ne k \le n$, we have*

$$\mathrm{ad}_{w_k}(c_{jk}) = \mathrm{ad}_{\overline{w}_j}(c_{jk}).$$

Proof. We may assume that w is a monomial of degree $d \geq 1$. So let $w = x_{i_1} \cdots x_{i_d}$. If $d = 1$, the formula holds true since $[a_{i_1 k}, c_{jk}] = -[a_{i_1 j}, c_{jk}]$. Let $d \geq 2$ and assume that the formula holds true in degrees less than d. Set $w' = x_{i_2} \cdots x_{i_d}$. Using the inductive assumption, we compute

$$\begin{aligned} &\mathrm{ad}_{a_{i_1 k} \cdots a_{i_d k}}(c_{jk}) = \mathrm{ad}_{a_{i_1 k}} \mathrm{ad}_{a_{i_2 k} \cdots a_{i_d k}}(c_{jk}) = \mathrm{ad}_{a_{i_1 k}} \mathrm{ad}_{\overline{a_{i_2 j} \cdots a_{i_d j}}}(c_{jk}) \\ &= (-1)^{d-1} \sum_{p=2}^{d} \mathrm{ad}_{a_{i_d j}} \cdots \mathrm{ad}_{[a_{i_1 k}, a_{i_p j}]} \cdots \mathrm{ad}_{a_{i_1 j}}(c_{jk}) + \mathrm{ad}_{\overline{a_{i_2 j} \cdots a_{i_d j}}}([a_{i_1 k}, c_{jk}]). \end{aligned}$$

Since $[a_{i_1 k}, a_{i_p j}] = -\delta_{i_1 i_p}[c_{jk}, a_{i_p j}] \in \mathsf{c}$, the first term vanishes in $\mathsf{edk}_{m,n}$. Therefore, $\mathrm{ad}_{a_{i_1 k} \cdots a_{i_d k}}(c_{jk})$ is equal to

$$\mathrm{ad}_{\overline{a_{i_2 j} \cdots a_{i_d j}}}([a_{i_1 k}, c_{jk}]) = -\mathrm{ad}_{\overline{a_{i_2 j} \cdots a_{i_d j}}}[a_{i_1 j}, c_{jk}] = \mathrm{ad}_{\overline{a_{i_1 j} \cdots a_{i_d j}}}(c_{jk}).$$

This completes the proof. □

Proof of Proposition 3.7. First of all, the formula $[u_j, v_j] = [u, v]_j$ in (i) is clear. In what follows, we will use this formula without mentioning explicitly.

(iii) Since the expressions w_{ij} and w'_{kl} viewed as elements in $\mathsf{dk}_{m,n}$ are in the ideal c, their commutator lies in $[\mathsf{c}, \mathsf{c}]$. Therefore $[w_{ij}, w'_{kl}] = 0 \in \mathsf{edk}_{m,n}$.

(ii) To prove $[u_i, w_{jk}] = 0$ when $i \notin \{j, k\}$, it is sufficient to consider the case where u is of degree 1 and w is a monomial. So we may assume that $u = x_q$ for some $1 \leq q \leq m$ and $w = x_{i_1} \cdots x_{i_d}$. We compute

$$\begin{aligned} [u_i, w_{jk}] &= [a_{qi}, \mathrm{ad}_{w_k}(c_{jk})] \\ &= \sum_{p=1}^{d} \mathrm{ad}_{a_{i_1 k}} \cdots \mathrm{ad}_{[a_{qi}, a_{i_p k}]} \cdots \mathrm{ad}_{a_{i_d k}}(c_{jk}) + \mathrm{ad}_{w_k}([a_{qi}, c_{jk}]). \end{aligned}$$

The first term vanishes since $[a_{qi}, a_{i_p k}] = -\delta_{q i_p}[c_{ik}, a_{i_p k}] \in \mathsf{c}$. The second term vanishes as well, since $[a_{qi}, c_{jk}] = 0$ by the commutation relation.

To prove the other two formulas, we first prove that

$$[u_k, \mathrm{ad}_{w_k}(c_{jk})] = \mathrm{ad}_{(uw)_k}(c_{jk}) \tag{5}$$

for any $1 \leq i \leq n$ and $1 \leq j \neq k \leq n$. We may assume that u is homogeneous and proceed by induction on $\deg u$. When $\deg u = 1$, we have $[u_k, w_{jk}] = \mathrm{ad}_{u_k} \mathrm{ad}_{w_k}(c_{jk}) = \mathrm{ad}_{(uw)_k}(c_{jk}) = (uw)_{jk}$. Let $\deg u > 2$ and assume that the formula holds true in degrees less than $\deg u$ and that u is of the form $u = [u', u'']$. We compute

$$\begin{aligned} [u_k, \mathrm{ad}_{w_k}(c_{jk})] &= [[u'_k, \mathrm{ad}_{w_k}(c_{jk})], u''_k] + [u'_k, [u''_k, \mathrm{ad}_{w_k}(c_{jk})]] \\ &= [\mathrm{ad}_{(u'w)_k}(c_{jk}), u''_k] + [u'_k, \mathrm{ad}_{(u''w)_k}(c_{jk})] \\ &= -\mathrm{ad}_{(u''u'w)_k}(c_{jk}) + \mathrm{ad}_{(u'u''w)_k}(c_{jk}) \\ &= \mathrm{ad}_{(uw)_k}(c_{jk}). \end{aligned}$$

In the second and third lines, we have used the inductive assumption.

Equation (5) shows that $[u_k, w_{jk}] = (uw)_{jk}$ for $j < k$. To prove $[u_j, w_{jk}] = -(wu)_{jk}$ we compute

$$[u_j, w_{jk}] = [u_j, \mathrm{ad}_{\overline{w}_j}(c_{jk})] = \mathrm{ad}_{u_j \overline{w}_j}(c_{jk}) = \mathrm{ad}_{w_k \overline{u}_k}(c_{jk}) = -\mathrm{ad}_{w_k u_k}(c_{jk}).$$

Here, we have used Lemma 3.8 in the first and third equalities, formula (5) in the second equality, and the fact that $\overline{u}_k = -u_k$ in the last equality.

(i) It remains to prove formula (4). Setting $\Phi(u,v) := \sum_{i=1}^m (\partial_i v) x_i \iota(\partial_i u)$, let us prove that $[u_j, v_k] = \Phi(u,v)_{jk}$ for any homogeneous elements $u, v \in \mathsf{lie}_m$. We use the induction on the bidegree $(\deg u, \deg v)$. Since $[a_{i_1 j}, a_{i_2 k}] = \delta_{i_1 i_2}[a_{i_2 k}, c_{jk}] = \delta_{i_1 i_2}(x_{i_2})_{jk}$, the case $(\deg u, \deg v) = (1,1)$ is done. We first increase $\deg u$. Let $\deg u > 1$ and assume that $u = [u', u'']$ for some $u', u'' \in \mathsf{lie}_m$ satisfying $[u'_j, v_k] = \Phi(u', v)_{jk}$ and $[u''_j, v_k] = \Phi(u'', v)_{jk}$. On the one hand, using these assumptions we compute

$$\begin{aligned} [u_j, v_k] &= [[u'_j, v_k], u''_j] + [u'_j, [u''_j, v_k]] \\ &= [\Phi(u', v)_{jk}, u''_j] + [u'_j, \Phi(u'', v)_{jk}] \\ &= \big(\Phi(u', v)u'' - \Phi(u'', v)u'\big)_{jk}. \end{aligned}$$

In the last line, we have used (ii). On the other hand, using (3) and the fact that ι acts as minus the identity on lie_m, we see that $\Phi(u,v) = \Phi([u', u''], v) = \Phi(u', v)u'' - \Phi(u'', v)u'$. Hence we conclude that $[u_j, v_k] = \Phi(u,v)_{jk}$. A similar argument works for increasing $\deg v$. This completes the proof. □

Let $\mathcal{A}^{\mathrm{em}}_{m,n} = U(\mathsf{edk}_{m,n})$ be the universal enveloping algebra of $\mathsf{edk}_{m,n}$. It is the quotient of $\mathcal{A}_{m,n}$ by the span of monomials in a_{ij} and c_{ij} which contain at least two generators of type c_{ij}.

The following proposition is a consequence of Proposition 2.8.

Proposition 3.9. *There is a canonical isomorphism of graded $\mathbb{Q}$-algebras $\mathrm{gr}\,\mathbb{Q}P^{\mathrm{em}}_{m,n} \cong \mathcal{A}^{\mathrm{em}}_{m,n}$, through which the class of $\alpha_{ij} - 1$ (resp. $\tau_{ij} - 1$) corresponds to a_{ij} (resp. c_{ij}).*

3.3 Description of operadic operations on $\mathsf{edk}_{m,n}$

The operadic operations introduced in Section 2.3 naturally induces operations on emergent braids and chord diagrams. Let us describe the operations on $\mathsf{edk}_{m,n}$ in view of the direct sum decomposition (2). In what follows, let $u = u(x_1, \dots, x_m) \in \mathsf{lie}_m$ and $w = w(x_1, \dots, x_m) \in \mathsf{ass}_m$.

First, we have

$$\delta_0(u_i) = u(x_2, \dots, x_{m+1})_i, \qquad \delta_0(w_{ij}) = w(x_2, \dots, x_{m+1})_{ij}, \tag{6}$$

and for $1 \leq k \leq m$,

$$\begin{aligned}\delta_k(u_i) &= u(x_1, \ldots, x_k + x_{k+1}, \ldots, x_{m+1})_i, \\ \delta_k(w_{ij}) &= w(x_1, \ldots, x_k + x_{k+1}, \ldots, x_{m+1})_{ij}.\end{aligned} \tag{7}$$

Second, we describe the cabling operations with respect to blue strands. Let $R : \mathsf{lie}_m \to \mathsf{ass}_m$ be the unique $\mathbb{Q}$-linear map satisfying $R(x_i) = 0$ for $i = 1, \ldots, m$ and for any $a, b \in \mathsf{lie}_m$,

$$R([a,b]) = [R(a), b] + [a, R(b)] + \sum_{i=1}^{m} \left((\partial_i b) x_i \iota(\partial_i a) - (\partial_i a) x_i \iota(\partial_i b) \right). \tag{8}$$

Lemma 3.10. *For $1 \leq k \leq n$, we have the following:*

$$\delta_{m+k}(u_i) = \begin{cases} u_i & (i < k) \\ u_k + u_{k+1} + R(u)_{k(k+1)} & (i = k) \\ u_{i+1} & (i > k) \end{cases},$$

and

$$\delta_{m+k}(w_{ij}) = \begin{cases} w_{ij} & (j < k) \\ w_{ik} + w_{i(k+1)} & (j = k) \\ w_{i(j+1)} & (i < k < j) \\ w_{k(j+1)} + w_{(k+1)(j+1)} & (i = k) \\ w_{(i+1)(j+1)} & (k < i) \end{cases}.$$

Proof. We will prove the formula $\delta_{m+k}(u_k) = u_k + u_{k+1} + R(u)_{k(k+1)}$ and $\delta_{m+k}(w_{ik}) = w_{ik} + w_{i(k+1)}$ only. The proof of the other formulas is rather straightforward, so we omit it.

First we prove that $\delta_{m+k}(u_k) = u_k + u_{k+1} + R(u)_{k(k+1)}$. This is true in degree one, since $\delta_{m+k}(a_{ik}) = a_{ik} + a_{i(k+1)}$. Assume that $\deg u > 1$, we have $u = [a, b]$ for some homogeneous elements a, b, and

$$\delta_{m+k}(a_k) = a_k + a_{k+1} + R(a)_{k(k+1)}, \quad \delta_{m+k}(b_k) = b_k + b_{k+1} + R(b)_{k(k+1)}$$

for some $R(a), R(b) \in \mathsf{ass}_m$. Then, we have

$$\begin{aligned}\delta_{m+k}(u_k) &= [\delta_{m+k}(a_k), \delta_{m+k}(b_k)] \\ &= [a_k + a_{k+1} + R(a)_{k(k+1)}, b_k + b_{k+1} + R(b)_{k(k+1)}].\end{aligned}$$

Computing the right hand side using Proposition 3.7, we obtain

$$u_k + u_{k+1} + \Big([R(a), b] + [a, R(b)] + \sum_{i=1}^{m} \left((\partial_i b) x_i \iota(\partial_i a) - (\partial_i a) x_i \iota(\partial_i b) \right) \Big)_{k(k+1)}.$$

This completes the proof.

Next we show that $\delta_{m+k}(w_{ik}) = w_{ik} + w_{i(k+1)}$. We have

$$\begin{aligned}\delta_{m+k}(w_{ik}) &= \mathrm{ad}_{w(a_{1k}+a_{1(k+1)},\dots,a_{mk}+a_{m(k+1)})}(c_{ik} + c_{i(k+1)}) \\ &= \mathrm{ad}_{w(a_{1k}+a_{1(k+1)},\dots,a_{mk}+a_{m(k+1)})}(c_{ik}) \\ &\qquad + \mathrm{ad}_{w(a_{1k}+a_{1(k+1)},\dots,a_{mk}+a_{m(k+1)})}(c_{i(k+1)}).\end{aligned}$$

Since $a_{j(k+1)}$ and c_{ik} commute and the Lie bracket of a_{jk} and $a_{l(k+1)}$ lies in c, the first term is equal to $\mathrm{ad}_{w(a_{1k},\dots,a_{mk})}(c_{ik}) = w_{ik}$. Similarly, the second terms is equal to $w_{i(k+1)}$. This completes the proof. $\square$

Finally, we describe the map ϑ_m.

Lemma 3.11. *We have the following:*

$$\begin{aligned}\vartheta_m(u_i) &= u(x_1,\dots,x_{m-1},0)_{i+1} + \big((\partial_m u)(x_1,\dots,x_{m-1},0)\big)_{1(i+1)}, \\ \vartheta_m(w_{ij}) &= w(x_1,\dots,x_{m-1},0)_{(i+1)(j+1)}.\end{aligned}$$

Proof. The proof of the first formula is similar to the proof of the formula for $\delta_{m+k}(u_k)$ in Lemma 3.10. We denote by $H(u)$ the right hand side of the formula. We first check that the formula holds true in degree one. Now let $a, b \in \mathsf{lie}_m$ and assume that $\vartheta_m(a_i) = H(a)$ and $\vartheta_m(b_i) = H(b)$. Then, by a direct computation using Proposition 3.7 and formula (3), we verify that $\vartheta_m([a,b]_i) = [H(a), H(b)]$ is equal to $H([a,b])$. Since this is straightforward, we omit the detail.

To prove the second formula, modulo $[\mathsf{c},\mathsf{c}]$ we compute

$$\begin{aligned}\vartheta_m(w_{ij}) &= \mathrm{ad}_{w(a_{1(j+1)},\dots,a_{(m-1)(j+1)},c_{1(j+1)})}(c_{(i+1)(j+1)}) \\ &= \mathrm{ad}_{w(a_{1(j+1)},\dots,a_{(m-1)(j+1)},0)}(c_{(i+1)(j+1)}) \\ &= w(x_1,\dots,x_{m-1},0)_{(i+1)(j+1)}.\end{aligned}$$ $\square$

4 Homomorphic expansions for mixed braids

In [5], the category **PaB** of parenthesized braids was introduced, and it was shown that the Drinfeld associators give rise to formality isomorphisms (homomorphic expansions) for this category. In this section, we extend this formalism to mixed braids.

4.1 Parenthesized mixed braids and chord diagrams

We need some notation. Let $\mathbf{Par}_{\textcolor{red}{\bullet}\textcolor{blue}{\bullet}} = \bigsqcup_{m,n\geq 0} \mathbf{Par}_{m,n}$ be the set of parenthesized words in two letters $\textcolor{red}{\bullet}$ and $\textcolor{blue}{\bullet}$, where $\mathbf{Par}_{m,n}$ is the subset consisting of parenthesized words with m red bullets and n blue bullets. For example, $(\textcolor{red}{\bullet}\textcolor{red}{\bullet})\textcolor{blue}{\bullet} \in \mathbf{Par}_{2,1}$ and $\textcolor{red}{\bullet}(\textcolor{red}{\bullet}(\textcolor{blue}{\bullet}\textcolor{red}{\bullet})) \in \mathbf{Par}_{3,1}$. For $O \in \mathbf{Par}_{m,n}$, let

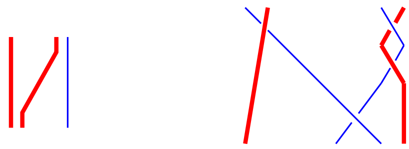
$f(O) = \overline{O} \in \mathbf{Par}_{m,0}$ be the parenthesized word in ${\color{red}\bullet}$ obtained by forgetting all the blue bullets in O, and let $p(O) = o \in W_{m,n}$ be the word obtained by forgetting the parenthesization of O. For example, if $O = {\color{red}\bullet}({\color{red}\bullet}({\color{blue}\bullet}{\color{red}\bullet}))$, then $\overline{O} = {\color{red}\bullet}({\color{red}\bullet}{\color{red}\bullet})$ and $o = {\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}$.

First we define the category **PaMB** of parenthesized mixed braids. The set of objects is $\mathbf{Par}_{{\color{red}\bullet}{\color{blue}\bullet}}$. Let $O, O' \in \mathbf{Par}_{{\color{red}\bullet}{\color{blue}\bullet}}$ with $f(O) = \overline{O}$, $f(O') = \overline{O'}$, $p(O) = o$ and $p(O') = o'$. Then the set of morphisms from O to O' is

$$\mathbf{PaMB}(O, O') := \begin{cases} \mathbf{MB}(o, o') & \text{if } \overline{O} = \overline{O'}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The composition is defined using that of **MB**. Note that there are no morphisms from O to O' unless $\overline{O} = \overline{O'}$. For example, we have no morphism from $({\color{red}\bullet}{\color{red}\bullet}){\color{red}\bullet}$ to ${\color{red}\bullet}({\color{red}\bullet}{\color{red}\bullet})$. When we draw pictures of morphisms in **PaMB**, which are represented by linear combinations of mixed braids, we use the same convention used for **PaB** in [5]. Namely, we draw the bottom and top ends of mixed braids so that their distances respect their "distances" in the parenthesization of the source and domain of the morphism.

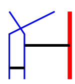
Example 4.1. In the following two pictures, the first one shows a morphism from $({\color{red}\bullet}{\color{red}\bullet}){\color{blue}\bullet}$ to ${\color{red}\bullet}({\color{red}\bullet}{\color{blue}\bullet})$, and the second one from ${\color{red}\bullet}({\color{blue}\bullet}({\color{blue}\bullet}{\color{red}\bullet}))$ to $({\color{blue}\bullet}{\color{red}\bullet})({\color{blue}\bullet}{\color{red}\bullet})$.



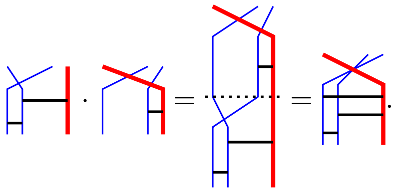
Next we define the category **PaMCD**. The set of objects is the same as the set of objects of **PaMB**, namely $\mathbf{Par}_{{\color{red}\bullet}{\color{blue}\bullet}}$. The set of morphisms from O to O' is

$$\mathbf{PaMCD}(O, O') := \begin{cases} \mathcal{A}_{m,n} \times \mathfrak{S}_{{\color{red}\bullet}{\color{blue}\bullet}}(o, o') & \text{if } \overline{O} = \overline{O'}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Here, in the first case, (m, n) is the type of $o \in W_{m,n}$. By definition, a morphism in **PaMCD** is of the form (u, σ), where $u \in \mathcal{A}_{m,n}$ and σ is a mixed permutation of type (m, n). Recall that u is expressed as a linear combination of words in a_{ij} and c_{ij}, which are interpreted as horizontal chords. We draw u on the picture of σ so that a_{ij} (resp. c_{ij}) becomes a chord connecting the ith red strand and jth blue strand (resp. the ith and jth blue strands), where we count the strands at the bottom. We also express the information on the parenthesization using the distances between endpoints. For example,

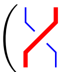
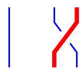
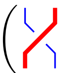
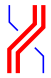


is a morphism from (●●)● to ●(●●) which corresponds to $\big(c_{12}a_{12},$  $\big)$. In this view point, the composition in **PaMCD** is given by stacking of diagrams. For example, one has



More formally, the composition of (composable) morphisms (u, σ) and (u', σ') is given by $(u\sigma(u'), \sigma\sigma')$. Here, through the restriction to the blue bullets, σ induces a permutation of $\{1, \ldots, n\}$ and hence acts on $\mathcal{A}_{m,n}$.

The operadic operations to mixed braids and chord diagrams introduced in Section 2.3 extends naturally to their parenthesized enhancements **PaMB** and **PaMCD**, with an extra care for parenthesizations. For the extension operations, we draw the ends of the added strand outer-most in the picture. For the cabling operations, we draw the ends of the two newly created strands closest to each other. For example,

$\delta_0^s\Big($  $\Big) =$  and $\delta_1\Big($  $\Big) =$ .

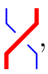
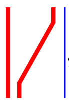
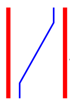
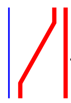
To compare the categories **PaMB** and **PaMCD**, we need to consider their completions. On the one hand, **PaMB** is filtered. By the same argument used for the ideal J in Section 3.1, the augmentation ideal $I = IP_{m,n}$ and its powers extend naturally to a multiplicative filtration of the $\mathbb{Q}$-linear category **MB** and hence of **PaMB**. Therefore, one can define the I-adic completion $\widehat{\mathbf{PaMB}}$ and the associated graded $\operatorname{gr}\mathbf{PaMB}$. On the other hand, **PaMCD** is graded. The grading comes from the grading of the algebra $\mathcal{A}_{m,n}$. Thus one can define the degree completion $\widehat{\mathbf{PaMCD}}$, where $\mathcal{A}_{m,n}$ is replaced with its degree completion $\widehat{\mathcal{A}}_{m,n}$. The operadic operations on **PaMB** and **PaMCD** extends to their completions.

The isomorphism $\operatorname{gr}\mathbb{Q}P_{m,n} \cong \mathcal{A}_{m,n}$ proven in Proposition 2.8 extends naturally to a canonical isomorphism

$$\operatorname{gr}\mathbf{PaMB} \cong \mathbf{PaMCD} \tag{9}$$

of graded $\mathbb{Q}$-linear categories which is the identity on the objects. Moreover, one checks that this isomorphism respects all the operadic operations.

Proposition 4.2. *The category* **PaMB** *is generated by the following morphisms, their inverses, and their images by repeated applications of the operadic operations in* **PaMB***:*

$\sigma^+_{\mathrm{ps}} =$ , $\sigma^-_{\mathrm{ps}} =$ , $\alpha_{\mathrm{pps}} :=$ , $\alpha_{\mathrm{psp}} :=$ , $\alpha_{\mathrm{spp}} :=$ . (10)

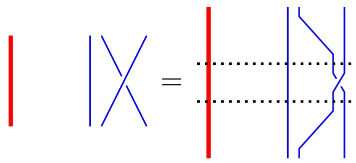
Sketch of proof. We say that a parenthesized mixed braid is basic if it is one of the morphisms listed in the statement of the proposition. Let $m, n \geq 0$ and $\overline{O}^*_m \in \mathbf{Par}_{m,0}$. Joining $\overline{O}^*_m$ and the left-nested parenthesization of n blue bullets, we obtain a parenthesized word $O^*_{m,n} \in \mathbf{Par}_{m,n}$. For example, if $\overline{O}^*_4 = (\bullet\bullet)(\bullet\bullet)$ then $O^*_{4,3} = ((\bullet\bullet)(\bullet\bullet))((\bullet\bullet)\bullet)$.

Let $m, n \geq 0$ and $O, O' \in \mathbf{Par}_{m,n}$ such that $\overline{O}^*_m := \overline{O} = \overline{O'}$. Given any parenthesized mixed braids $\beta \in \mathbf{PaMB}(O^*_{m,n}, O)$ and $\beta' \in \mathbf{PaMB}(O^*_{m,n}, O')$, any morphism ξ from O to O' decomposes as $\xi = \beta^{-1}(\beta\xi\beta'^{-1})\beta'$. Therefore, to prove the proposition it is sufficient to show the following: given any parenthesization $\overline{O}^*_m \in \mathbf{Par}_{m,0}$,

(i) any parenthesized mixed braid from $O^*_{m,n}$ to itself decomposes into a product of basic morphisms, and

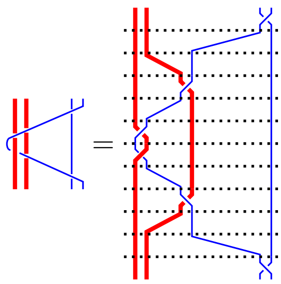
(ii) for any $O \in \mathbf{Par}_{m,n}$ with $\overline{O} = \overline{O}^*_m$, there is morphism from $O^*_{m,n}$ to O which decomposes into a product of basic morphisms.

To prove (i), note that the underlying mixed braid of a parenthesized one lies in the group $B^{\text{std}}_{m,n}$ introduced in Section 2.1. Since this group is generated by α_{ij}'s and the simple braids among blue strands [11, Section 4], it is sufficient to deal with these generators. We give two sample computations:



$$= \delta^p_0(\alpha_{\text{sss}}) \circ \delta^p_0(\delta^s_0(\sigma_{\text{ss}})) \circ \delta^p_0(\alpha^{-1}_{\text{sss}}),$$

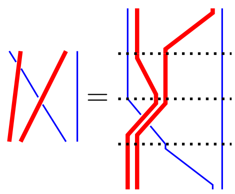
where we set $\sigma_{\text{ss}} = \vartheta_1(\sigma^+_{\text{ps}})$ and $\alpha_{\text{sss}} = \vartheta_1(\vartheta_2(\alpha_{\text{pps}}))$, and



$$= \xi \circ \delta^s_4(\delta^p_3(\sigma^+_{\text{ps}})) \circ \delta^s_4(\delta^p_3(\sigma^-_{\text{ps}})) \circ \xi^{-1},$$

where $\xi = \delta_1(\delta^p_0(\sigma^{-1}_{\text{ss}})) \circ \delta_1(\vartheta_2(\alpha^{-1}_{\text{pps}})) \circ \delta^s_4(\alpha_{\text{pps}}) \circ \delta^s_4(\delta^p_0(\sigma^-_{\text{ps}})) \circ \delta^s_4(\alpha^{-1}_{\text{psp}})$.

For (ii), we give one example:



$$= \delta_1(\vartheta_2(\alpha^{-1}_{\text{pps}})) \circ \delta^s_4(\delta_2(\sigma^+_{\text{ps}})) \circ \delta^s_4(\alpha^{-1}_{\text{spp}}) \circ \vartheta_1(\delta_1(\alpha_{\text{pps}})).$$

□

In a similar fashion to the definition of **PaMB**, for each $k \geq 1$ we define the parenthesized version of the category $\mathbf{MB}^{/k}$, which we denote by $\mathbf{PaMB}^{/k}$. Likewise, for each $k \geq 1$ we define the $\mathbb{Q}$-linear category $\mathbf{PaMCD}^{/k}$. The isomorphism (9) descends to an isomorphism $\operatorname{gr}\mathbf{PaMB}^{/k} \cong \mathbf{PaMCD}^{/k}$. For $k = 2$, we use the special notation $\mathbf{PaEB} = \mathbf{PaMB}^{/2}$ and $\mathbf{PaECD} = \mathbf{PaMCD}^{/2}$.

4.2 Homomorphic expansions for mixed braids

Here comes the concept of homomorphic expansions for the category **PaMB**:

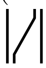
Definition 4.3. A homomorphic expansion for **PaMB** is a functor $Z^{\mathrm{mb}} : \mathbf{PaMB} \to \widehat{\mathbf{PaMCD}}$ which is the identity on the objects, preserves the filtrations, induces the identity at the associated graded, respects all the operadic operations, and is group-like.

The group-like condition in the above definition means that for each mixed braid β of type (m, n) one has $Z^{\mathrm{mb}}(\beta) = (\exp(u), \pi(\beta))$, where u is an element in $\widehat{\mathsf{dk}}_{m,n}$, the degree completion of $\mathsf{dk}_{m,n}$.

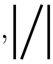
For each $k \geq 1$, we can formulate the concept of a homomorphic expansion for $\mathbf{PaMB}^{/k}$: it is defined to be a functor $Z^{\mathrm{mb}/k} : \mathbf{PaMB}^{/k} \to \widehat{\mathbf{PaMCD}}^{/k}$ satisfying the same conditions required for Z^{mb} in Definition 4.3.

Homomorphic expansions for **PaMB** exist by the following

Proposition 4.4. *Any Drinfeld associator gives rise to a homomorphic expansion for* **PaMB** *and consequently to a homomorphic expansion for* $\mathbf{PaMB}^{/k}$ *for any* $k \geq 1$.

Proof. Let Φ be a Drinfeld associator. It is of the form $\Phi = \exp(\varphi)$, where $\varphi = \varphi(x, y) \in \widehat{\mathsf{lie}}_2$ is a Lie series without linear term. As shown in [5, Proposition 3.4], Φ extends to a functor $Z^{\mathrm{pb}} : \mathbf{PaB} \to \widehat{\mathbf{PaCD}}$, where the target is the degree completion of the category of parenthesized (horizontal) chord diagrams. The category **PaB** is generated (in the same sense as in Proposition 4.2) by the elements $\sigma =$  and $\alpha =$  (see [5, Claim 2.6]). The functor Z^{pb} is specified by the values on these generators:

Unlucky crash of notation. We are using σ for permutations!

$$Z^{\mathrm{pb}}(\sigma) := \left(\exp\left(\frac{1}{2}t_{12}\right), \right.$$  $$\left.\right), \qquad Z^{\mathrm{pb}}(\alpha) := \left(\Phi(t_{12}, t_{23}), \right.$$  $$\left.\right).$$

There are functors $\mathbf{PaMB} \to \mathbf{PaB}$ and $\widehat{\mathbf{PaMCD}} \to \widehat{\mathbf{PaCD}}$ obtained by forgetting the colors of poles and strands. These functors are faithful. For the former, this follows from the fact that $P_{m,n}$ is a subgroup of P_{m+n}, the pure braid group of $m + n$ strands. For the latter, this follows from the injectivity of the map $\mathsf{dk}_{m,n} \to \mathsf{dk}_{m+n}$; see Remark 2.7.

We will show that Z^{pb} induces a functor $Z^{\mathrm{mb}} : \mathbf{PaMB} \to \widehat{\mathbf{PaMCD}}$ which is the identity on the objects. In view of the faithfulness of the forgetful functors above, it is sufficient to prove the following claim:

Claim. Let β be a mixed braid with m poles and n strands which represents a morphism in $\mathbf{PaMB}$. Let β^0 be the parenthesized braid obtained by forgetting the colors of β and write $Z^{\mathrm{pb}}(\beta^0) = (B, \pi(\beta^0))$, where $B \in \exp(\widehat{\mathsf{dk}}_{m+n})$ and $\pi(\beta^0)$ is the parenthesized permutation induced by β^0. Then, B lies in $\exp(\widehat{\mathsf{dk}}_{m,n})$, where we view $\mathsf{dk}_{m,n}$ as a Lie subalgebra of dk_{m+n}.

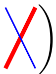
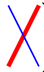
Proof of the claim. Basically, this is because there is no crossing between the strands in β^0 which were poles of β. More details are as follows.

Step 1. Assume that β is one of the elements in (10). Then, β^0 is either $\sigma^{\pm}$ or α. If $\beta^0 = \sigma^{\pm}$, then $B = \exp\left(\pm\frac{1}{2}t_{12}\right) = \exp\left(\pm\frac{1}{2}a_{11}\right) \in \exp(\widehat{\mathsf{dk}}_{1,1})$. If $\beta^0 = \alpha$, there are three possibilities: (i) $\beta = \alpha_{\mathrm{pps}}$; (ii) $\beta = \alpha_{\mathrm{psp}}$; (iii) $\beta = \alpha_{\mathrm{spp}}$. Since φ has no linear term, we have $\varphi(-t_{13} - t_{23}, t_{23}) = \varphi(t_{12}, t_{23}) = \varphi(t_{12}, -t_{12} - t_{13})$. Hence, we have $B = \Phi(-a_{11} - a_{21}, a_{21})$ in case (i), $B = \Phi(a_{11}, a_{21})$ in case (ii), and $B = \Phi(a_{11}, -a_{11} - a_{21})$ in case (iii). In all cases, we obtain that $B \in \exp(\widehat{\mathsf{dk}}_{2,1})$.

Step 2. By Proposition 4.2 we can decompose β into a product of basic morphisms (in the sense of the proof of Proposition 4.2). By Step 1, any basic morphism is sent by Z^{pb} to a morphism in $\widehat{\mathbf{PaCD}}$ whose first component lies in $\exp(\widehat{\mathsf{dk}}_{m,n})$. Hence the same is true for β, and the claim follows. □

The functor Z^{pb} is filtration-preserving, induces the identity at the associated graded, and respects all the operadic operations in $\mathbf{PaB}$. Therefore, the induced functor Z^{mb} satisfies all the required properties for a homomorphic expansion for $\mathbf{PaMB}$. This completes the proof of Proposition 4.4. □

Are there any other ways to obtain homomorphic expansions for $\mathbf{PaMB}$? By Proposition 4.2, any Z^{mb} is specified by values on the basic morphisms in (10). For the first and second elements, the group-like condition for Z^{mb} implies that $Z^{\mathrm{mb}}(\sigma^+_{\mathrm{ps}}) = \big(\exp(\lambda a_{11}),$  $\big)$ and $Z^{\mathrm{mb}}(\sigma^-_{\mathrm{ps}}) = \big(\exp(\mu a_{11}),$  $\big)$ for some $\lambda, \mu \in \mathbb{Q}$. Applying the operation ϑ_1 to the first equation, we obtain $Z^{\mathrm{mb}}($  $) = \big(\exp(\lambda c_{12}),$  $\big)$ and $Z^{\mathrm{mb}}\big((\tau_{11},$  $)\big) = Z^{\mathrm{mb}}($  $) = \big(\exp(2\lambda c_{12}),$  $\big) = \big(1 + 2\lambda c_{12},$  $\big)$. Since $\mathrm{gr}\, Z^{\mathrm{mb}}$ is the identity, we obtain $\lambda = 1/2$. Similarly, we obtain $\mu = -1/2$. In summary, we have

$$Z^{\mathrm{mb}}(\sigma^+_{\mathrm{ps}}) = \left(\exp\left(\frac{1}{2}a_{11}\right),\right.$$  $$\left.\right), \quad Z^{\mathrm{mb}}(\sigma^-_{\mathrm{ps}}) = \left(\exp\left(-\frac{1}{2}a_{11}\right),\right.$$  $$\left.\right). \tag{11}$$

For the other three morphisms in (10), we introduce the notation

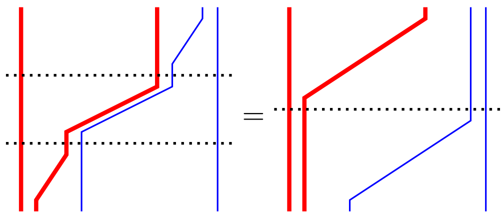
$$\Phi_{\mathrm{pps}} := Z^{\mathrm{mb}}(\alpha_{\mathrm{pps}}), \quad \Phi_{\mathrm{psp}} := Z^{\mathrm{mb}}(\alpha_{\mathrm{psp}}), \quad \Phi_{\mathrm{spp}} := Z^{\mathrm{mb}}(\alpha_{\mathrm{spp}}).$$

By the group-like condition for Z^{mb}, the $\widehat{\mathcal{A}}_{2,1}$-component of these three elements lie in $\exp(\widehat{\mathsf{dk}}_{2,1}) \subset \widehat{\mathcal{A}}_{2,1}$ so that one can write $\Phi_{\mathrm{pps}} = \exp(\varphi_{\mathrm{pps}})$, $\Phi_{\mathrm{psp}} = \exp(\varphi_{\mathrm{psp}})$, and $\Phi_{\mathrm{spp}} = \exp(\varphi_{\mathrm{spp}})$ for some $\varphi_{\mathrm{pps}}, \varphi_{\mathrm{psp}}, \varphi_{\mathrm{spp}} \in \widehat{\mathsf{dk}}_{2,1}$.

To obtain a well-defined functor Z^{mb}, the values Φ_{pps}, Φ_{psp} and Φ_{spp} have to satisfy several equations coming from equalities among parenthesized mixed braids. To give a complete description, we need to know the presentation of the category **PaMB** in terms of the generators in (10). We do not pursue this issue in the present paper. We focus on the following ppss-pentagon equality

TODO: do we want to explain other 5-gons? pssp, psps,...

TODO: should we write on solving the psss 5-gon?



as morphisms from ((●●)•)• to ●(●(••)). It can be written as

$$d^4(\alpha_{\mathrm{pps}})d^2(\alpha_{\mathrm{pps}})d^0(\alpha_{\mathrm{pps}}) = d^1(\alpha_{\mathrm{pps}})d^3(\alpha_{\mathrm{pps}}).$$

Applying Z^{mb}, we obtain

$$d^4(\Phi_{\mathrm{pps}})d^2(\Phi_{\mathrm{pps}})d^0(\Phi_{\mathrm{pps}}) = d^1(\Phi_{\mathrm{pps}})d^3(\Phi_{\mathrm{pps}}). \tag{12}$$

We will be interested in the linearization of this equation, which takes the following form:

$$d^4(\varphi_{\mathrm{pps}}) + d^2(\varphi_{\mathrm{pps}}) + d^0(\varphi_{\mathrm{pps}}) = d^1(\varphi_{\mathrm{pps}}) + d^3(\varphi_{\mathrm{pps}}), \tag{13}$$

namely, $d^{2,1}(\varphi_{\mathrm{pps}}) = 0$. We call this the linearized ppss-pentagon equation.

4.3 Emergent pentagon and (doubled) hexagon equations

From now on, we consider the case $k = 2$, namely $\mathbf{PaMB}^{/2} = \mathbf{PaEB}$, and focus on the linearized ppss-pentagon equation (13) in $\mathsf{edk}_{2,2}$.

Proposition 4.5. *Let $\varphi = \varphi(x, y) \in \mathsf{lie}_2$. Then, $\varphi_1 = \varphi(a_{11}, a_{21}) \in \mathsf{edk}_{2,1}$ satisfies $d^{2,1}(\varphi_1) = 0 \in \mathsf{edk}_{2,2}$ if and only if φ satisfies the following two equations:*

$$\varphi(y, 0) - \varphi(x + y, 0) = 0, \tag{14}$$
$$(\partial_y\varphi)(x, y) + (\partial_y\varphi)(y, 0) - (\partial_y\varphi)(x + y, 0) - R(\varphi) = 0. \tag{15}$$

Proof. Using formulas (6), (7) and Lemmas 3.10, 3.11, we obtain that

$$\begin{aligned}
d_0(\varphi_1) &= \varphi(y,0)_2 + (\partial_y\varphi)(y,0)_{12},\\
d_1(\varphi_1) &= \varphi(x+y,0)_2 + (\partial_y\varphi)(x+y,0)_{12},\\
d_2(\varphi_1) &= \varphi(x,y)_2 + (\partial_y\varphi)(x,y)_{12},\\
d_3(\varphi_1) &= \varphi(x,y)_1 + \varphi(x,y)_2 + R(\varphi)_{12},\\
d_4(\varphi_1) &= \varphi(x,y)_1.
\end{aligned}$$

The assertion follows from this. □

Remark 4.6. Equation (14) says that the coefficient of x in φ is zero. One can check that in degree one, solutions to equation (15) are scalar multiples of x. Hence, one concludes that there are no solution to $d^{2,1}(\varphi_1) = 0$ in degree one, and equation (14) is redundant in degrees at least two.

Recall the definition of the Grothendieck-Teichmüller Lie algebra grt_1 by Drinfeld [8]. It is the space of $\psi \in \mathsf{lie}_2$ which satisfy the following relations:

$$\begin{aligned}
&\psi(x,y) = -\psi(y,x),\\
&\psi(x,y) + \psi(y,-x-y) + \psi(-x-y,x) = 0,\\
&\psi(t_{12},t_{2(34)}) + \psi(t_{(12)3},t_{34}) = \psi(t_{23},t_{34}) + \psi(t_{1(23)},t_{(23)4}) + \psi(t_{12},t_{23}).
\end{aligned} \tag{16}$$

Here, the last equation (16), called the pentagon equation, takes place in dk_4 and $t_{2(34)} = t_{23} + t_{24}$, etc. It is known that nontrivial elements in grt_1 have degrees at least three.

Reverse the order? Should we first mention this?

The pentagon equation admits the following interpretation in terms of a certain differential on dk_n. In fact, the differential on $\mathsf{edk}_{m,n}$ is induced from this differential. Assume that $\psi \in \mathsf{lie}_2$ has degree at least two. Then, it can be considered as an element in dk_3 by the substitution $\psi \mapsto \psi(t_{12}, t_{23})$. There are maps $d_i : \mathsf{dk}_3 \to \mathsf{dk}_4$ for $0 \le i \le 4$ defined in terms of extension and cabling operations, and ψ is a solution to the pentagon equation if and only if $d^3(\psi) = \sum_{i=0}^{4}(-1)^i d_i(\psi) = 0$. Furthermore, through the isomorphism

$$\mathsf{dk}_3 \cong \mathbb{Q}(t_{12}+t_{13}+t_{23}) \oplus \mathsf{lie}(t_{13},t_{23}),$$

$\psi(t_{12},t_{23})$ corresponds to $\psi(-t_{13}-t_{23},t_{23}) \in \mathsf{lie}(t_{13},t_{23})$. Now, identify $\mathsf{lie}(t_{13},t_{23})$ with $\mathsf{dk}_{2,1} = \mathsf{lie}(a_{11},a_{21}) \cong \mathsf{edk}_{2,1}$ by $t_{13} \mapsto a_{11}$ and $t_{23} \mapsto a_{21}$. Since the coface maps on dk_3 and $\mathsf{edk}_{2,1}$ are compatible with this identification, it follows that any $\psi = \psi(x,y) \in \mathsf{lie}_2$ with $d^3(\psi) = 0$ satisfies $d^{2,1}(\psi(-x-y,y)) = 0$.

We now want to introduce the emergent version of the Grothendieck-Teichmüller Lie algebra as the space of solutions to the linearized ppss-pentagon equation. For a technical reason, we put an additional condition coming from the following fact proved by Drinfeld [8, equation (5.19)]: any

$\psi \in \mathsf{grt}_1$ satisfies $[x, \psi(-x-y, x)] + [y, \psi(-x-y, y)] = 0$, and hence $\varphi(x, y) = \psi(-x-y, y)$ satisfies

$$[x, \varphi(y, x)] + [y, \varphi(x, y)] = 0. \tag{17}$$

Should we explain how this equation can be thought of as a doubled 6-gon (under a symmetry assumption $\Phi_{\mathrm{spp}}(x,y) = \Phi_{\mathrm{pps}}(y,x)^{-1}$)? Or, just introduce it without explaining reasons?

Definition 4.7. Let

$$\mathsf{grt}_1^{\mathrm{em}} := \{\varphi \in \mathsf{lie}_2 \mid \varphi \text{ satisfies equations (14), (15) and (17)}\}.$$

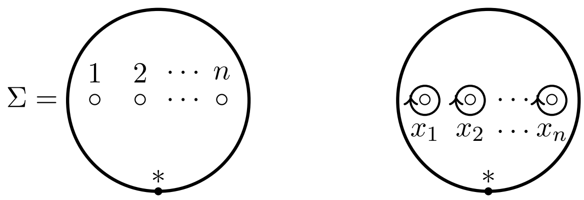
By definition, we have the injection $\mathsf{grt}_1 \hookrightarrow \mathsf{grt}_1^{\mathrm{em}}, \psi(x, y) \mapsto \psi(-x-y, y)$.

Remark 4.8. A computer experiment shows that up to degree 17, the space of solutions to equations (14) and (15) coincides with grt_1. (Recall from Remark 4.6 that (14) is automatically satisfied in degrees at least two.) We do not know if one can eliminate condition (17) from the definition of $\mathsf{grt}_1^{\mathrm{em}}$.

5 Loop operations and Kashiwara-Vergne theory

In this section, we explain an interpretation of the Kashiwara-Vergne Lie algebras in terms of surface topology [1, 2].

Fix a positive integer n and let Σ be an n-punctured disk, that is, a closed unit disk in $\mathbb{R}^2$ with n distinct points in the interior removed. Choose a basepoint $*$ in the boundary of Σ and let $\pi = \pi_1(\Sigma, *)$. Since the group π is free of rank n, the associated graded quotient of the group algebra $\mathbb{Q}\pi$ with respect to the powers of the augmentation ideal is canonically isomorphic to the free associative algebra $\mathsf{ass}_n = \mathsf{ass}(x_1, \ldots, x_n)$, where the generator x_i corresponds to the homology class of the loop around the ith puncture:



Let A be an associative $\mathbb{Q}$-algebra. The trace space $|A|$ is defined to be $A/[A, A]$. We denote by $|\ | : A \to |A|$ the natural projection. For instance, the space $|\mathbb{Q}\pi|$ is naturally identified with the set of homotopy classes of free loops in Σ, and $\mathsf{tr}_n := |\mathsf{ass}_n|$ is the space of cyclic words in $x_1, \ldots, x_n$.

5.1 Loop operations on a punctured disk

We briefly recall several loop operations on $\mathbb{Q}\pi$ and $|\mathbb{Q}\pi|$. Our focus is on their associated graded operations on ass_n and tr_n, which are actually needed for us. For more details about the loop operations themselves, see [13, 12, 2].

TODO: I will add figures for η and δ^f.

The main cast in the sequel are the (associated graded operations of) the homotopy intersection form [13] and the framed Turaev cobracket [1, 2]. The former is a $\mathbb{Q}$-linear map $\eta : \mathbb{Q}\pi^{\otimes 2} \to \mathbb{Q}\pi$ defined in terms of intersections of two based loops in Σ, and the latter is a $\mathbb{Q}$-linear map $\delta^f : |\mathbb{Q}\pi| \to |\mathbb{Q}\pi|^{\otimes 2}$ defined in terms of self-intersections of a free loop in Σ. The operation δ^f depends on the choice of a framing on Σ. Here, we choose the blackboard framing associated with the inclusion $\Sigma \subset \mathbb{R}^2$. We also use a based loop version of δ^f: a certain map $\mu_r^f : \mathbb{Q}\pi \to |\mathbb{Q}\pi| \otimes \mathbb{Q}\pi$ introduced in [2, Section 2.3]. We set $\mu^f := (\varepsilon \otimes \mathrm{id}) \circ \mu_r^f : \mathbb{Q}\pi \to \mathbb{Q}\pi$, where the map $\varepsilon : |\mathbb{Q}\pi| \to \mathbb{Q}$ is induced from the augmentation map of $\mathbb{Q}\pi$. The maps μ^f and η are related by the following formula: for any $a, b \in \mathbb{Q}\pi$,

$$\mu^f(ab) = a\mu^f(b) + \mu^f(a)b + \eta(a,b). \tag{18}$$

In fact, the operation μ^f recovers μ_r^f and δ^f. The map μ_r^f coincides with the following composition

$$\begin{aligned}\mathbb{Q}\pi \xrightarrow{\Delta} \mathbb{Q}\pi \otimes \mathbb{Q}\pi &\xrightarrow{\mathrm{id}\otimes\mu^f} \mathbb{Q}\pi \otimes \mathbb{Q}\pi \\ &\xrightarrow{\mathrm{id}\otimes((\iota\otimes\mathrm{id})\circ\Delta)} \mathbb{Q}\pi \otimes \mathbb{Q}\pi \otimes \mathbb{Q}\pi \xrightarrow{(|\ |\circ\mathrm{mult})\otimes\mathrm{id}} |\mathbb{Q}\pi| \otimes \mathbb{Q}\pi. \end{aligned} \tag{19}$$

Here, Δ and ι are the coproduct and antipode on $\mathbb{Q}\pi$ defined by $\Delta(\gamma) = \gamma\otimes\gamma$ and $\iota(\gamma) = \gamma^{-1}$ for $\gamma \in \pi$, and in the last step we use the multiplication map in the algebra $\mathbb{Q}\pi$. Furthermore, for any $a \in \mathbb{Q}\pi$ we have

$$\delta^f(|a|) = \mathrm{Alt} \circ (\mathrm{id} \otimes |\ |) \circ \mu_r^f(a) + |a| \wedge \mathbf{1}. \tag{20}$$

Here, $\mathrm{Alt}(a \otimes b) = a \otimes b - b \otimes a$ and $\mathbf{1}$ is the class of the unit in $\mathbb{Q}\pi$.

Remark 5.1. (i) We give several comments about proofs of the formulas above. First, one can derive formula (18) by applying $(\varepsilon \otimes \mathrm{id})$ to the first equation in [2, Proposition 2.9 (i)]. A formula similar to (18) was proved in [12, (3.3)] for a variant of the map μ^f. Second, the decomposition (19) of the map μ_r^f follows directly from the defining formula of μ_r^f. See [2, Section 2.3, formula (13)]. Finally, formula (20) can be found in [2, Proposition 2.9 (ii)].

(ii) The map δ^f is a refinement of the Turaev cobracket [17], which is a Lie cobracket on the quotient space $|\mathbb{Q}\pi|/\mathbb{Q}\mathbf{1}$. Turaev [16] also introduced essentially the same operations as η and (an unframed version of) μ^f.

5.2 The associated graded operations

All the loop operations in the previous section descend to the associated graded operations on ass_n and tr_n. We review their explicit formulas. For more details, see [2, Section 3]. The associated graded operation of η,

$$\eta_{\mathrm{gr}} : {\mathsf{ass}_n}^{\otimes 2} \to \mathsf{ass}_n,$$

is a map of degree -1 and given by $\eta_{\mathrm{gr}}(1,v)=\eta_{\mathrm{gr}}(u,1)=0$ and

$$\eta_{\mathrm{gr}}(a_1\cdots a_l, b_1\cdots b_m) = -a_1\cdots a_{l-1}\mathfrak{z}(a_l,b_1)b_2\cdots b_m, \tag{21}$$

where $l,m\geq 1$, the elements $a_1,\ldots,a_l,b_1,\ldots,b_m$ are of degree 1, and $\mathfrak{z}$ is defined by $\mathfrak{z}(x_i,x_j)=\delta_{ij}x_i$. The associated graded operation of μ^f,

$$\mu^f_{\mathrm{gr}} : \mathsf{ass}_n \to \mathsf{ass}_n,$$

is a map of degree -1 and given by the formula

$$\mu^f_{\mathrm{gr}}(a_1\cdots a_m) = -\sum_{j=1}^{m-1} a_1\cdots a_{j-1}\mathfrak{z}(a_j,a_{j+1})a_{j+2}\cdots a_m, \tag{22}$$

where $a_1,\ldots,a_m$ are elements of degree 1. The associated graded version of the relations (18), (19) and (20) holds true. First, for any $a,b\in\mathsf{ass}_n$

$$\mu^f_{\mathrm{gr}}(ab) = a\mu^f_{\mathrm{gr}}(b) + \mu^f_{\mathrm{gr}}(a)b + \eta_{\mathrm{gr}}(a,b). \tag{23}$$

Of course, one can directly check this from formulas (21) and (22). Second, the associated graded operation $\mu^f_{r,\mathrm{gr}}$ decomposes as

$$\begin{aligned}\mathsf{ass}_n &\xrightarrow{\Delta} \mathsf{ass}_n\otimes\mathsf{ass}_n \xrightarrow{\mathrm{id}\otimes\mu^f_{\mathrm{gr}}} \mathsf{ass}_n\otimes\mathsf{ass}_n \\ &\xrightarrow{\mathrm{id}\otimes((\iota\otimes\mathrm{id})\circ\Delta)} \mathsf{ass}_n\otimes\mathsf{ass}_n\otimes\mathsf{ass}_n \xrightarrow{(|\ |\circ\mathrm{mult})\otimes\mathrm{id}} \mathsf{tr}_n\otimes\mathsf{ass}_n.\end{aligned} \tag{24}$$

Conversely, we have $\mu^f_{\mathrm{gr}} = (\varepsilon\otimes\mathrm{id})\circ\mu^f_{r,\mathrm{gr}}$. Finally, for any $a\in\mathsf{ass}_n$

$$\delta^f_{\mathrm{gr}}(|a|) = \mathrm{Alt}\circ(\mathrm{id}\otimes|\ |)\circ\mu^f_{r,\mathrm{gr}}(a). \tag{25}$$

Note that the term $|a|\wedge\mathbf{1}$ in (20) does not contribute to the associated graded operation, since it is of filtration degree zero.

Lemma 5.2. *For any $a,b\in\mathsf{lie}_n$, we have $\eta_{\mathrm{gr}}(a,b) = -\sum_{i=1}^n(\partial_i a)x_i\iota(\partial_i b)$.*

Proof. This follows from $a=\sum_{i=1}^n(\partial_i a)x_i$ and $b=\sum_{i=1}^n x_i\iota(\partial_i b)$. □

Proposition 5.3. *The map R coincides with the restriction of μ^f_{gr} to lie_n.*

Proof. We have $\mu^f_{\mathrm{gr}}(x_i)=0$ for $i=1,\ldots,n$. Let $a,b\in\mathsf{lie}_n$. Equation (23) and Lemma 5.2 shows that $\mu^f_{\mathrm{gr}}([a,b]) = \mu^f_{\mathrm{gr}}(ab)-\mu^f_{\mathrm{gr}}(ba)$ is equal to

$$[\mu^f_{\mathrm{gr}}(a),b] + [a,\mu^f_{\mathrm{gr}}(b)] + \sum_{i=1}^n\left((\partial_i b)x_i\iota(\partial_i a) - (\partial_i a)x_i\iota(\partial_i b)\right).$$

Therefore, the map μ^f_{gr} restricted to lie_n satisfies the same recursive formula in (8) as the map R. This proves the proposition. □

5.3 Kashiwara-Vergne Lie algebras

We recall the definition of the Kashiwara-Vergne Lie algebras [4, 1].

We begin with some preliminary materials. Let $\mathsf{tder}_n = {\mathsf{lie}_n}^{\oplus n}$. The grading on lie_n makes tder_n a graded $\mathbb{Q}$-vector space. For $\tilde{u} = (u_1, \ldots, u_n) \in \mathsf{tder}_n$, let $\rho(\tilde{u})$ be a derivation on lie_n defined by $\rho(\tilde{u})(x_i) = [x_i, u_i]$ for $i = 1, \ldots, n$. The space tder_n has a structure of graded Lie algebra whose Lie bracket is given by $[\tilde{u}, \tilde{v}] = \tilde{w} = (w_1, \ldots, w_n)$ with $w_i = [u_i, v_i] + \rho(\tilde{u})(v_i) - \rho(\tilde{v})(u_i)$ for $i = 1, \ldots, n$, and the map $\tilde{u} \mapsto \rho(\tilde{u})$ is a Lie algebra homomorphism to the derivation Lie algebra of lie_n. Through this homomorphism, tder_n acts on lie_n, ass_n, tr_n and their tensor products. Elements of tder_n are called tangential derivations. The space sder_n of special derivations is defined to be the set of $\tilde{u} \in \mathsf{tder}_n$ annihilating the element $x_0 = \sum_{i=1}^n x_i$, i.e., $\rho(\tilde{u})(x_0) = 0$. It forms a Lie subalgebra of tder_n. The divergence cocycle [4] is a Lie 1-cocycle defined by the following formula:

$$\mathrm{div} : \mathsf{tder}_n \to \mathsf{tr}_n, \quad \tilde{u} \mapsto \sum_{i=1}^n |x_i(\partial_i u_i)|.$$

Definition 5.4. (i) The Kashiwara-Vergne Lie algebra krv_n is the space consisting of $\tilde{u} \in \mathsf{sder}_n$ such that $\mathrm{div}(\tilde{u}) = \sum_{i=0}^n |f_i(x_i)|$ for some formal power series $f_0(s), f_1(s), \ldots, f_n(s) \in \mathbb{Q}[[s]]$.

(ii) Let krv_n^0 be the space of $\tilde{u} \in \mathsf{sder}_n$ such that $\mathrm{div}(\tilde{u}) \in \bigoplus_{i=1}^n \mathbb{Q}|x_i|$.

In the definition of krv_n, the functions $f_i(s)$ actually agree with each other modulo the linear part [1, Proposition 8.5]. In particular, if $n = 2$ and $\tilde{u} \in \mathsf{krv}_2$ is of degree ≥ 3, then there exists an $f(s) \in \mathbb{Q}[[s]]_{\geq 2}$ such that

$$\mathrm{div}(\tilde{u}) = |f(x_1) + f(x_2) - f(x_1 + x_2)|. \tag{26}$$

We have the following sequence of inclusions of graded Lie algebras:

$$\mathsf{tder}_n \supset \mathsf{sder}_n \supset \mathsf{krv}_n \supset \mathsf{krv}_n^0.$$

The Lie algebras sder_n, krv_n and krv_n^0 have the following characterizations in terms of (the associated graded of) the loop operations.

Theorem 5.5. *Let $\tilde{u} = (u_1, \ldots, u_n) \in \mathsf{tder}_n$. Then, the following three conditions are equivalent:*

(i) $\tilde{u} \in \mathsf{sder}_n$;

(ii) $\partial_j u_i = \partial^i u_j$ *for any* $i, j \in \{1, \ldots, n\}$;

(iii) $\rho(\tilde{u})$ *commutes with* η_{gr}, *i.e.,* $\rho(\tilde{u}) \circ \eta_{\mathrm{gr}} = \eta_{\mathrm{gr}} \circ (\rho(\tilde{u}) \otimes \mathrm{id} + \mathrm{id} \otimes \rho(\tilde{u}))$.

Proof. The following computation proves the equivalence (i) $\Leftrightarrow$ (ii):

$$\rho(\tilde{u})(x_0) = \sum_{i=1}^{n}[x_i, u_i] = \sum_{i=1}^{n} x_i u_i - \sum_{j=1}^{n} u_j x_j = \sum_{i,j=1}^{n} \big(x_i(\partial_j u_i)x_j - x_i(\partial^i u_j)x_j\big).$$

To prove the equivalence (ii) $\Leftrightarrow$ (iii), note that the map η_{gr} is a Fox pairing [13]. This means that η_{gr} satisfies

$$\begin{cases} \eta_{\mathrm{gr}}(ab, c) = a\eta_{\mathrm{gr}}(b, c) + \varepsilon(b)\eta_{\mathrm{gr}}(a, c), \\ \eta_{\mathrm{gr}}(a, bc) = \eta_{\mathrm{gr}}(a, b)c + \varepsilon(b)\eta_{\mathrm{gr}}(a, c) \end{cases}$$

for any $a, b, c \in \mathsf{ass}_n$. Thanks to this property, the condition (iii) is equivalent to the commutativity of $\rho(\tilde{u})$ and η_{gr} on generators of ass_n, namely

(iv) $u(\eta_{\mathrm{gr}}(x_i, x_j)) = \eta_{\mathrm{gr}}(u(x_i), x_j) + \eta_{\mathrm{gr}}(x_i, u(x_j))$ for any $i, j \in \{1, \dots, n\}$.

Now we compute $u(\eta_{\mathrm{gr}}(x_i, x_j)) = u(\mathfrak{z}(x_i, x_j)) = \delta_{ij} u(x_i) = \delta_{ij}[x_i, u_i]$ and

$$\begin{aligned} \eta_{\mathrm{gr}}(u(x_i), x_j) + \eta_{\mathrm{gr}}(x_i, u(x_j)) &= \eta_{\mathrm{gr}}([x_i, u_i], x_j) + \eta_{\mathrm{gr}}(x_i, [x_j, u_j]) \\ &= x_i(\partial_j u_i)x_j - u_i\mathfrak{z}(x_i, x_j) \\ &\quad + \mathfrak{z}(x_i, x_j)u_j - x_i(\partial^i u_j)x_j \\ &= x_i(\partial_j u_i - \partial^i u_j)x_j + \delta_{ij}[x_i, u_i]. \end{aligned}$$

Hence the condition (iv) is equivalent to (ii). This completes the proof. □

Remark 5.6. The equivalence (i) $\Leftrightarrow$ (iii) in Theorem 5.5 is a special case of (the infinitesimal version of) more general results [13, Lemmas 6.2 and 6.3], [14, Theorem 2.31].

Theorem 5.7. *Let* $\tilde{u} = (u_1, \dots, u_n) \in \mathsf{tder}_n$.

(i) $\tilde{u} \in \mathsf{krv}_n \iff \rho(\tilde{u})$ *commutes with* η_{gr} *and* δ^f_{gr}.

(ii) $\tilde{u} \in \mathsf{krv}^0_n \iff \rho(\tilde{u})$ *commutes with* η_{gr} *and* μ^f_{gr}.

Proof. First note that in (ii) one can replace μ^f_{gr} with $\mu^f_{r,\mathrm{gr}}$, since $\mu^f_{r,\mathrm{gr}}$ is recovered from μ^f_{gr} and vice versa.

In [2, Theorem 8.21], it was shown that the Kashiwara-Vergne groups KRV_n (resp. KRV^0_n) is isomorphic to the group of tangential automorphisms of (the completion of) ass_n that commute with the operations η_{gr} and δ^f_{gr} (resp. η_{gr} and $\mu^f_{r,\mathrm{gr}}$). As krv_n and krv^0_n are the Lie algebras of KRV_n and KRV^0_n, respectively, the assertions (i) and (ii) follow from this result. □

6 Proof of the main result

In this section, we prove Theorem 1.1 in the introduction. When $n = 2$, we use the letters x, y for generators of lie_2 instead of x_1, x_2.

6.1 KV equations from emergent associator equations

We prove the first statement of Theorem 1.1.

We simply write $u = \rho(\tilde{u})$ for $\tilde{u} \in \mathsf{tder}_n$. Furthermore, we do this abbreviation for the action of $\tilde{u}$ on tensor products of ass_n and tr_n. For instance, $\tilde{u}$ acts on $\mathsf{tr}_n \otimes \mathsf{ass}_n$ as $\rho(\tilde{u}) \otimes \mathrm{id} + \mathrm{id} \otimes \rho(\tilde{u})$, and we denote it by u.

Lemma 6.1. *Let $\tilde{u} \in \mathsf{sder}_n$. Then, $d_{\tilde{u}} := \mu^f_{\mathrm{gr}} \circ u - u \circ \mu^f_{\mathrm{gr}}$ is a derivation on ass_n. Furthermore, the map $\mathcal{D}_{\tilde{u}} := \mu^f_{r,\mathrm{gr}} \circ u - u \circ \mu^f_{r,\mathrm{gr}}$ from ass_n to $\mathsf{tr}_n \otimes \mathsf{ass}_n$ satisfies the following property: for any $a, b \in \mathsf{ass}_n$,*

$$\mathcal{D}_{\tilde{u}}(ab) = \mathcal{D}_{\tilde{u}}(a)(1 \otimes b) + (1 \otimes a)\mathcal{D}_{\tilde{u}}(b).$$

Proof. We abbreviate $\mu = \mu^f_{\mathrm{gr}}$ and $\eta = \eta_{\mathrm{gr}}$. Let $a, b \in \mathsf{ass}_n$. We compute

$$\begin{aligned}
\mu(u(ab)) &= \mu(u(a)b + au(b)) \\
&= \mu(u(a))b + u(a)\mu(b) + \eta(u(a), b) \\
&\quad + \mu(a)u(b) + a\mu(u(b)) + \eta(a, u(b)), \\
u(\mu(ab)) &= u(\mu(a)b + a\mu(b) + \eta(a, b) \\
&= u(\mu(a))b + \mu(a)u(b) + u(a)\mu(b) + au(\mu(b)) + u(\eta(a, b)).
\end{aligned}$$

Since $\tilde{u} \in \mathsf{sder}_n$, we have $\eta(u(a), b) + \eta(a, u(b)) = u(\eta(a, b))$ by Theorem 5.5. Hence we see that $d_{\tilde{u}}$ is a derivation on ass_n.

The map $\mu^f_{r,\mathrm{gr}}$ decomposes as shown in (24). Since the derivation u commutes with the Hopf algebra operations on ass_n, the second assertion follows from the first assertion. □

Proposition 6.2. *Let $\tilde{u} \in \mathsf{sder}_n$ and assume that there is some $c \in \mathsf{ass}_n$ such that $\mu^f_{\mathrm{gr}}(u(x_i)) = [x_i, c]$ for all $i = 1, \ldots, n$. Then, $\tilde{u} \in \mathsf{krv}_n$.*

Proof. In view of Theorem 5.7, it is enough to prove that u commutes with δ^f_{gr}. A straightforward computation using (24) shows that $\mu^f_{r,\mathrm{gr}}(u(x_i)) = |\iota(c')| \otimes [x_i, c'']$ for all $i = 1, \ldots, n$, where we write $\Delta(c) = c' \otimes c''$ using the Sweedler notation.

Let $a = a_1 \cdots a_m \in \mathsf{ass}_n$ be a product of m elements of degree 1. Note that $\mathcal{D}_{\tilde{u}}(a_i) = |\iota(c')| \otimes [a_i, c'']$ since $\mu^f_{\mathrm{gr}}(a_i) = 0$. By Lemma 6.1, we have

$$\begin{aligned}
\mathcal{D}_{\tilde{u}}(a) &= \sum_{i=1}^{m} (1 \otimes a_1 \cdots a_{i-1})\mathcal{D}_{\tilde{u}}(a_i)(1 \otimes a_{i+1} \cdots a_m) \\
&= \sum_{i=1}^{m} |\iota(c')| \otimes a_1 \cdots a_{i-1}[a_i, c'']a_{i+1} \cdots a_m \\
&= |\iota(c')| \otimes [a, c''].
\end{aligned}$$

Since $|[a, c'']| = 0$, we obtain $(\delta^f_{\mathrm{gr}} \circ u - u \circ \delta^f_{\mathrm{gr}})(|a|) = 0$ by (25). This completes the proof. □

Proof of Theorem 1.1 (i). Let $\varphi = \varphi(x, y) \in \mathsf{grt}_1^{\mathrm{em}}$. Then φ satisfies equation (15) and the tangential derivation $\nu^{\mathrm{em}}(\varphi) = (\varphi(y, x), \varphi(x, y))$ is special. In particular, by Theorem 5.5 we have

$$\partial_y \varphi = \partial^y \varphi = \iota(\partial_y \varphi). \tag{27}$$

Put $f(s) := -(\partial_y \varphi)(s, 0) \in \mathbb{Q}[[s]]$. We will show that $\nu^{\mathrm{em}}(\varphi) \in \mathsf{sder}_2$ satisfies the assumption of Proposition 6.2. By Proposition 5.3, one may replace μ_{gr}^f with R. We first compute

$$\begin{aligned} R(\nu^{\mathrm{em}}(\varphi)(y)) &= R([y, \varphi(x, y)]) \\ &= [y, R(\varphi)] + (\partial_y \varphi) y - y \iota(\partial_y \varphi) \\ &= [y, R(\varphi) - \partial_y \varphi] \\ &= [y, (\partial_y \varphi)(y, 0) - (\partial_y \varphi)(x + y, 0)] \\ &= [y, f(x + y)]. \end{aligned}$$

Here, we have used formula (8) in the second line, equation (27) in the third line, equation (15) in the fourth line, and the fact that y commutes with any power series in y in the last line. Similarly, we compute

$$\begin{aligned} R(\nu^{\mathrm{em}}(\varphi)(x)) &= R([x, \varphi(y, x)]) \\ &= [x, R(\varphi(y, x))] + \partial_x(\varphi(y, x)) x - x \iota(\partial_x(\varphi(y, x))) \\ &= [x, R(\varphi)(y, x) - (\partial_y \varphi)(y, x)] \\ &= [x, (\partial_y \varphi)(x, 0) - (\partial_y \varphi)(y + x, 0)] \\ &= [x, f(x + y)]. \end{aligned}$$

This completes the proof. □

6.2 Symmetric Kashiwara-Vergne Lie algebra

Recall from [4, Section 8] that the symmetric part of the Kashiwara-Vergne Lie algebra $\mathsf{krv}_2^{\mathrm{sym}}$ is the invariant Lie subalgebra of krv_2 by the involution $(u(x, y), v(x, y)) \mapsto (v(y, x), u(y, x))$. In this section, we prove the second statement of Theorem 1.1.

Lemma 6.3. *Let $\varphi = \varphi(x, y) \in \mathsf{lie}_2$ be an element of degree at least two. Then, $R(\varphi)(0, y) = R(\varphi)(x, 0) = (\partial_y \varphi)(0, y) = 0$.*

Proof. Notice that φ seen as an element of ass_2 is a linear combination of words which contain at least one x and at least one y. Formula (22) implies that $R(\varphi)$ is a linear combination of words with the same property. Hence $R(\varphi)(0, y) = R(\varphi)(x, 0) = 0$. Similarly we have $(\partial_y \varphi)(0, y) = 0$, since $\partial_y \varphi$ is a linear combination of words which contain at least one x. □

Proof of Theorem 1.1 (ii). Let $\tilde{u} = (\varphi(y,x), \varphi(x,y)) \in \mathsf{krv}_2^{\mathrm{sym}}$ be homogeneous of degree at least two.

Step 1. We first consider the case where $\tilde{u} \in \mathsf{krv}_2^0$. By Theorem 5.7 (ii), $\tilde{u}$ commutes with $\mu_{\mathrm{gr}}^f = R$. Hence

$$0 = \tilde{u}(R(y)) = R(\tilde{u}(y)) = R([y, \varphi]) = [y, R(\varphi) - \partial_y \varphi].$$

Therefore, we have $R(\varphi) - \partial_y \varphi \in \mathbb{Q}[[y]]_{\geq 1}$. By Lemma 6.3, we obtain $R(\varphi) - \partial_y \varphi = 0$. Furthermore, $(\partial_y \varphi)(x, 0) = R(\varphi)(x, 0) = 0$. Therefore, we obtain equation (15) for φ. Hence $\varphi \in \mathsf{grt}_1^{\mathrm{em}}$ and $\tilde{u} = \nu^{\mathrm{em}}(\varphi)$.

Step 2. We next consider the general case. Let $l = \deg \varphi$. If l is even, then $\mathrm{div}(\tilde{u}) = 0$ by [4, Proposition 4.5]. Hence $\tilde{u} \in \mathsf{krv}_2^0$, and $\tilde{u}$ is in the image of ν^{em} by Step 1. Assume that l is odd (and ≥ 3). Recall that the Drinfeld-Ihara generator $\sigma_l \in \mathsf{grt}_1$ satisfies the property

$$\mathrm{div}(\nu(\sigma_l)) = |x^l + y^l - (x+y)^l|$$

(see [4, Proposition 4.10]). Thus there exists a constant $c \in \mathbb{Q}$ such that $\tilde{u} - c\nu(\sigma_l)$ has the vanishing divergence, i.e., $\tilde{u} - c\nu(\sigma_l) \in \mathsf{krv}_2^0$. From Step 1, we obtain that $\tilde{u} - c\nu(\sigma_l)$ is in the image of ν^{em}. Let $\psi_l(x,y) = \sigma_l(-x-y, y) \in \mathsf{grt}_1^{\mathrm{em}}$. Then $\nu(\sigma_l) = \nu^{\mathrm{em}}(\psi_l)$. Therefore, $\tilde{u} = (\tilde{u} - c\nu(\sigma_l)) + c\nu(\sigma_l)$ is in the image of ν^{em}. This completes the proof. □

References

[1] A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem, *Adv. Math.* **326** (2018), 1–53.

[2] A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem in higher genera, preprint, arXiv:1804.09566v3

[3] A. Alekseev, F. Naef and M. Ren, Poisson brackets and coaction maps of regularized holonomies of the KZ equation, preprint, arXiv:2409.08894

[4] A. Alekseev and C. Torossian, The Kashiwara-Vergne conjecture and Drinfeld's associators, *Ann. of Math.* (2) **175** (2012), no. 2, 415–463.

[5] D. Bar-Natan, On associators and the Grothendieck-Teichmuller group, I, *Selecta Math. (N.S.)* **4** (1998), no. 2, 183–212.

[6] D. Bar-Natan, Z. Dancso, T. Hogan, J. Liu, and N. Scherich, Goldman-Turaev formality from the Kontsevich integral, in preparation. See also talks `http://drorbn.net/ld22` and `http://drorbn.net/ge24`

[7] N. Bourbaki, *Groupes et algébres de Lie*, Hermann, Paris, 1971–71.

[8] V. G. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), *Leningrad Math. J.* **2** (1991) no. 4, 829–860.

[9] B. Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups, Part I*, Math. Surveys Monogr. **217**, American Mathematical Society, Providence, RI, 2017.

[10] H. Furusho, Pentagon and hexagon equations, *Ann. of Math.* (2) **171** (2010), no. 1, 545–556.

[11] S. Lambropoulou, Braid structures in knot complements, handlebodies and 3-manifolds, in: *Knots in Hellas '98 (Delphi), Ser. Knots Everything* **24**, 274–289 (2000).

[12] G. Massuyeau, Formal descriptions of Turaev's loop operations, *Quantum Topol.* **9** (2018), no. 1, 39–117.

[13] G. Massuyeau and V. Turaev, Fox pairings and generalized Dehn twists, *Ann. Inst. Fourier (Grenoble)* **63** (2013), no. 6, 2403–2456.

[14] F. Naef, Poisson brackets in Kontsecvich's "Lie world", *J. Geom. Phys.* **155** (2020), 103741, 13 pp.

[15] R. Navarro Betancourt and F. Naef, A functorial approach to the Kashiwara-Vergne problem, in preparation. more detail?

[16] V. Turaev, Intersections of loops in two-dimensional manifolds, *Mat. Sb.* **106/148** (1978), no. 4, 566–588.

[17] V. Turaev, Skein quantization of Poisson algebras of loops on surfaces, *Ann. Sci. École Norm. Sup.* (4) **24** (1991), no. 6, 635–704.