

# EMERGENT VERSION OF DRINFELD'S ASSOCIATOR EQUATIONS

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ABSTRACT. Alekseev, Enriquez, and Torossian [AET], continuing Alekseev and Torossian [AT], show how from a solution of the Drinfel'd pentagon equation one may construct a solution of the Kashiwara-Vergne (KV) equations [KV]. In this paper we show that solutions of a weak version of the pentagon equations that we call *the emergent pentagon equations* are sufficient for the same task, the construction of solutions of the KV equations.

The emergent pentagon equations arise within a natural topological context of *emergent tangles*, which we discuss. Our results are adjacent to the results of [BN1, BDHLS] on the relationship between emergent tangles and the Goldman-Turaev Lie bialgebra. We hope that in time our results will play a role in relating several bodies of work, on Drinfel'd associators, Kashiwara-Vergne equations, and on expansions for classical tangles, for w-tangles, and for the Goldman-Turaev Lie bialgebra.

## 1. INTRODUCTION

### REFERENCES

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