EMERGENT VERSION OF DRINFELD'S ASSOCIATOR EQUATIONS

DROR BAR-NATAN AND YUSUKE KUNO

ABSTRACT. It is sometimes beneficial in and around knot theory to think of some knot strands as hard and unmoving, or *fixed*, and some other strands as *flexible*. Further to that, we consider the *emergent quotient*, in which the flexible strands are made to be nearly transparent to themselves and to each other — we don't quite decry that for flexible strands $\asymp = \nасимп$ for that would reduce the flexible strands to homotopy classes in the complement of the fixed strands — yet we do mod out by relations that say that $\asymp = \nасимп$ is nearly true, and so in the quotient that remains knot theory is just barely visible, or *emergent*.

We show that within this context the Drinfel'd pentagon equation for associators remains meaningful and that furthermore, solutions to the resulting emergent linearized Drinfel'd pentagon equation still lead to solutions of the linearized Kashiwara-Vergne equations.

Our results are adjacent to the results of [BN2, BDHLS] on the relationship between emergent tangles and the Goldman-Turaev Lie bialgebra and we hope that in time they will play a role in relating several bodies of work, on Drinfel'd associators, Kashiwara-Vergne equations, and on expansions for classical tangles, for w-tangles, and for the Goldman-Turaev Lie bialgebra.

1. INTRODUCTION

1.1. Emergent Knotted Objects. Consider some space $\mathcal{K}$ of knotted objects in a three manifold M (knots, or links, or knotted graphs, or tangles if M has a boundary, or braids if one can specify a "vertical" direction in M). If we mod out $\mathcal{K}$ by the relation $\asymp = \nасимп$, knottedness is eliminated and what remains is a space of curves (or multi-curves, or graphs, etc.) regarded up to homotopy. "Emergent knotted objects" are what you get when you mod out $\mathcal{K}$ by a slightly weaker version of $\asymp = \nасимп$ known in the theory of finite type invariants as the relation $\times^2 = 1$. Knottedness almost entirely disappear yet in the spaces we get, $\mathcal{K}^{em} := \mathcal{K}/(\times^2 = 1)$, we see that just a bit of knot theory begins to emerge beyond the homotopy theory that is already there.

More precisely, we consider $\mathbb{Q}\mathcal{K}$, the space of $\mathbb{Q}$-linear combinations of (some type of) knotted objects in M. As is often done when discussing finite type invariants [BN1, CDM], we extend $\mathcal{K}$ by allowing double points ($\times$) while declaring that a double point is just a short for the linear combination $\asymp - \nасимп$; namely, we set $\times = \asymp - \nасимп$. Then, as stated before, $\mathbb{Q}\mathcal{K}/\times$ is a space of (linear combinations of) curves modulo homotopy. We continue as in the theory of finite type invariants and also study the quotient $\mathcal{K}^{em}$ of $\mathbb{Q}\mathcal{K}$ by knots that have *two* double points, each of which representing a combination $\asymp - \nасимп$. The space $\mathcal{K}^{em}$ is the space of emergent knotted objects in M.

If M is the Euclidean space $\mathbb{R}^3$ (or a ball in $\mathbb{R}^3$ when discussing tangles), homotopy theory is trivial so $\mathcal{K}/\times$ becomes trivial (if, say, we are studying n-component links, they all become equivalent in $\mathcal{K}/\times$). The space of emergent knots $\mathcal{K}^{em} = \mathcal{K}/(\times\times)$ is slightly more interesting,

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though one may easily check that two links become equivalent if and only if all of the pairwise linking numbers of their components are the same (with similar statements of other types of knotted objects). Thus in that case, $\mathcal{K}^{em}$ is not completely trivial yet nevertheless quite simple.

In [BN2, BDHLS], Bar-Natan, Dancso, Hogan, Liu, and Scherich study emergent tangles in a pole dancing studio (a punctured disk cross an interval, or a room with several vertices lines removed). They find that in that case, an appropriate theory of expansions for emergent tangles leads to "homomorphic" expansions of the Goldman-Turaev Lie bialgebra (which in itself is a Lie bialgebra of curves modulo homotopy, a $(\mathcal{K}/\times)$-type object).

The subquotients $\mathsf{edk}_{m,n}$ of the Drifnel'd-Kohno Lie algebra dk_{m+n} that we use in this paper, in particular within the statement of our main theorem in Section 1.1, arise from emergent braids in a pole dancing studio. Let us explain how.

Let PB_n denote the n-strand pure braid group, and let $PB_{m,n}$ denote the kernel of the "forget the last n strands" map $PB_{m+n} \to PB_m$; elements of $PB_{m,n}$ are pure braids whose first m strands remain stationary, like n poles in a pole dancing studio. Hence elements of $PB_{m,n}$ can be thought of as n-strands braids in an m-pole pole dancing studio.

References

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[BN2]   Dror Bar-Natan, *Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef,* talk given in *Algebra, Topology and the Grothendieck-Teichmüller group*, Les Diablerets, September 2022. Handout and video at `http://drorbn.net/ld22`.
[BDHLS] Dror Bar-Natan, Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich, *Goldman-Turaev formality from the Kontsevitch integral,* in preparation.
[CDM]   S. Chmutov, S. Duzhin, and J. Mostovoy, *Introduction to Vassiliev Knot Invariants,* Cambridge University Press, Cambridge UK, 2012.