EMERGENT VERSION OF DRINFELD'S ASSOCIATOR EQUATIONS

DROR BAR-NATAN AND YUSUKE KUNO

Abstract. It is sometimes beneficial in and around knot theory to think of some knot strands as hard and unmoving, or *fixed*, and some other strands as *flexible*. Further to that, we consider the *emergent quotient*, in which the flexible strands are made to be nearly transparent to themselves and to each other — we don't quite decry that for flexible strands $\overcrossing = \undercrossing$ for that would reduce the flexible strands to homotopy classes in the complement of the fixed strands — yet we do mod out by relations that say that $\overcrossing = \undercrossing$ is nearly true, and so in the quotient that remains knot theory is just barely visible, or *emergent*.

We show that within this context the Drinfel'd pentagon equation for associators remains meaningful and that furthermore, solutions to the resulting emergent linearized Drinfel'd pentagon equation still lead to solutions of the linearized Kashiwara-Vergne equations.

Our results are adjacent to the results of [BN2, BDHLS] on the relationship between emergent tangles and the Goldman-Turaev Lie bialgebra and we hope that in time they will play a role in relating several bodies of work, on Drinfel'd associators, Kashiwara-Vergne equations, and on expansions for classical tangles, for w-tangles, and for the Goldman-Turaev Lie bialgebra.

1. Introduction

1.1. **Emergent Knotted Objects.** Consider some space $\mathcal{K}$ of knotted objects in a three manifold M (knots, or links, or knotted graphs, or tangles if M has a boundary, or braids if one can specify a "vertical" direction in M). If we mod out $\mathcal{K}$ by the relation $\overcrossing = \undercrossing$, knottedness is eliminated and what remains is a space of curves (or multi-curves, or graphs, etc.) regarded up to homotopy. "Emergent knotted objects" are what you get when you mod out $\mathcal{K}$ by a slightly weaker version of $\overcrossing = \undercrossing$ known in the theory of finite type invariants as the relation $\doublepoint^2 = 1$. Knottedness almost entirely disappear yet in the spaces we get, $\mathcal{K}^{em} := \mathcal{K}/(\doublepoint^2 = 1)$, we see that just a bit of knot theory begins to emerge beyond the homotopy theory that is already there.

More precisely, we consider $\mathbb{Q}\mathcal{K}$, the space of $\mathbb{Q}$-linear combinations of (some type of) knotted objects in M. As is often done when discussing finite type invariants [BN1, CDM], we extend $\mathcal{K}$ by allowing double points ($\doublepoint$) while declaring that a double point is just a short for the linear combination $\overcrossing - \undercrossing$; namely, we set $\doublepoint = \overcrossing - \undercrossing$. Then, as stated before, $\mathbb{Q}\mathcal{K}/\doublepoint$ is a space of (linear combinations of) curves modulo homotopy. We continue as in the theory of finite type invariants and also study the quotient $\mathcal{K}^{em}$ of $\mathbb{Q}\mathcal{K}$ by knots that have *two* double points, each of which representing a combination $\overcrossing - \undercrossing$. The space $\mathcal{K}^{em}$ is the space of emergent knotted objects in M.

If M is the Euclidean space $\mathbb{R}^3$ (or a ball in $\mathbb{R}^3$ when discussing tangles), homotopy theory is trivial so $\mathcal{K}/\doublepoint$ becomes trivial (if, say, we are studying n-component links, they all become equivalent in $\mathcal{K}/\doublepoint$). The space of emergent knots $\mathcal{K}^{em} = \mathcal{K}/(\doublepoint\doublepoint)$ is slightly more interesting,

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though one may easily check that two links become equivalent if and only if all of the pairwise linking numbers of their components are the same (with similar statements of other types of knotted objects). Thus in that case, $\mathcal{K}^{em}$ is not completely trivial yet nevertheless quite simple.

In [BN2, BDHLS], Bar-Natan, Dancso, Hogan, Liu, and Scherich study emergent tangles in a pole dancing studio PDS_m (an m-punctured disk cross an interval, or a room with m vertical lines removed). They find that in that case, an appropriate theory of expansions for emergent tangles leads to "homomorphic" expansions of the Goldman-Turaev Lie bialgebra (which in itself is a Lie bialgebra of curves modulo homotopy, a $(\mathcal{K}/\times)$-type object).

The subquotients $\mathsf{edk}_{m,n}$ of the Drifnel'd-Kohno Lie algebra dk_{m+n} that we use in this paper, in particular within the statement of our main theorem in Section 1.1, arise from emergent braids in a pole dancing studio. Let us explain how.

Let PB_n denote the n-strand pure braid group, and let $PB_{m,n}$ denote the kernel of the "forget the last n strands" map $PB_{m+n} \to PB_m$; elements of $PB_{m,n}$ are pure braids whose first m strands remain stationary, like m poles in a pole dancing studio (and hence we cease to call these first m strands "strands", and instead refer to them as "poles"). Hence elements of $PB_{m,n}$ can be thought of as n-strand braids in PDS_m. Thus, following the above discussion of emergent knotted objects, we define $PB^{em}_{m,n}$ to be the quotient of $\mathbb{Q}PB_{m,n}$ by the relation $\times\times = 0$, where all the strands involved in the double points belong to the last n strands in $PB_{m,n}$; namely, they are "strands" rather than "poles".

For the more algebraically-inclined, here's a fully-algebraic definition of $PB^{em}_{m,n}$: We let $\psi\colon PB_n \to PB_{m,n} \subset PB_{m+n}$ be map which sends an n-strand pure braid into an $(m+n)$-strand pure braid by adding m straight strands ("poles") on the left; the resulting braid is clearly in $PB_{m,n}$. Let $\mathcal{I}_{PB_n}$ be the augmentation ideal of $\mathbb{Q}PB_n$ (namely, those $\mathbb{Q}$-linear combinations of elements of PB_n whose sum of coefficients is 0). Let $\mathcal{I}_{ss}$ be the ideal of $\mathbb{Q}PB_{m,n}$ generated by $\psi(\mathcal{I}_{PB_n})$ (it's the ideal generated by $\times$, if both strands of the double point are strands and not poles). Finally, $PB^{em}_{m,n} \coloneqq \mathbb{Q}PB_{m,n}/\mathcal{I}^2_{ss}$.

In complete generality, if G is any group, we can define a decreasing filtration of its group ring $\mathbb{Q}G$ by powers of the augmentation ideal $\mathcal{I}_G$ of $\mathbb{Q}G$, and then we can consider the completed associated graded algebra $\mathcal{A}_G$ defined by that filtration: $\mathcal{A}_G \coloneqq \prod_{d\geqslant 0} \mathcal{I}^k_G/\mathcal{I}^{k+1}_G$. It is well known (see e.g. [Kohno, Drinfel'd]) that $\mathcal{A}_{PB_{m+n}}$ is the completed universal enveloping algebra of the Drinfel'd-Kohno Lie algebra dk_{m+n} that we alluded to before, whose generators are $\{t_{ij} = [\sigma_{ij} - 1]\colon 1 \leqslant i \neq j \leqslant m+n\}$, where σ_{ij} is the generator of PB_{m+n} in which strands $\#i$ and $\#j$ twist around each other once, in the positive direction.

It is now easy to determine $\mathcal{A}_{PB^{em}_{m,n}}$: it is the completed universal enveloping algebra of the subquotient of $\mathsf{dk}_{m,n}$ in which all the generators of the form $\{t_{ij}\colon i,j \leqslant m\}$ are removed (as the generators $\{\sigma_{ij}\colon i,j \leqslant m\}$, the pole-pole twists, are removed from $PB_{m,n}$), and in which we mod out by the square of the ideal generated by $\{t_{ij}\colon i,j > m\}$, corresponding to the emergent quotient $\times\times = 0$. Thus $\mathcal{A}_{PB^{em}_{m,n}} \cong \mathcal{U}(\mathsf{edk}_{m,n})$ with the same $\mathsf{edk}_{m,n}$ as in the previous section.

Strictly speaking, the statement $\mathcal{A}_{PB^{em}_{m,n}} \cong \mathcal{U}(\mathsf{edk}_{m,n})$ still requires a proof, which we omit. We only comment that one standard proof of $\mathcal{A}_{PB_{m+n}} \cong \mathcal{U}(\mathsf{dk}_{m+n})$ uses an "expansion" $Z\colon PB_{m+n} \to \mathcal{U}(\mathsf{dk}_{m+n})$ defined using the Kontsevich integral and/or using holonomies of the Knizhnik-Zamolodchikov connection, and whose associated graded is an isomorphism $\operatorname{gr} Z\colon \mathcal{A}_{PB_{m+n}} \to \mathcal{U}(\mathsf{dk}_{m+n})$. That proof easily restricts and descends to a proof of $\mathcal{A}_{PB^{em}_{m,n}} \cong \mathcal{U}(\mathsf{edk}_{m,n})$.

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