COMPUTING FINITE TYPE INVARIANTS EFFICIENTLY: RESPONSE TO THE REFEREE REPORT

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We would like to thank our managing editor and referee for their time and effort in considering our manuscript. The referee's comments were very thorough with several key insights that we were glad to include. Below, we describe how we addressed each of the referee's comments.

(1) **Referee comment:** p1: The paper describes k as the "type" of a finite type invariant. I think that "degree" is the clearest term, since it comes from a vector space filtration and is analogous to the degree of a polynomial.
Author response: We agree that "degree" is synonymous to "type" for finite type invariants. To address the referee's comment, we have added a parenthetical explaining that one could equivalently call the "type" the "degree" as well. The word "type" is the widely used terminology in finite type invariant theory as in the following link: `https://en.wikipedia.org/wiki/Finite_type_invariant`

(2) **Referee comment:** p1: I think that it is confusing to number the main theorem in Theorem 4.1. I recommend Theorem 1.1, and to repeat that if you want to repeat the statement of the result in Section 4.
Author response: We have changed the name of the theorem to "Main Theorem" to avoid numbering confusion.

(3) **Referee comment:** p1 and later: The tilde notation which is defined at the bottom of the page and is used in the statement of the main theorem is both non-standard (because of the extra log factors) and outmoded. The best and most standard for asymptotics is so-called "Big O" notation, which dates back to Bachmann and Landau and was extended and popularized by Knuth, particularly in computer science. See: `https://en.wikipedia.org/wiki/Big_O_notation` The standard way to state the theorem is that the finite type invariants of any fixed degree k can be computed in time $\tilde{O}(n^{k/2})$, where the tilde diacritic allows for a polylog factor rather than just a constant factor. Likewise the standard form for the definition at the bottom of page 1 is $f(n) = \tilde{\Theta}(g(n))$.
Author response: We use the equivalence relation $\sim$ very often throughout the paper and we prefer our notation for it, which is less cumbersome than the $\tilde{O}$ notation. Also, we often use it symmetrically, whereas the $\tilde{O}$ notation is not symmetric. Please allow us to keep it.

(4) **Referee comment:**p1: Since the algorithm is a space-time tradeoff, the main theorem should state that the new algorithm uses $\tilde{O}(n^{k/2})$ space as well as time.
Second addendum: Considering that the new algorithm is a time-space tradeoff, it is also worth stating the main result in a generalized form with unequal sizes for

the two types of Gauss subdiagrams, so that you can partially trade time for space. This is the correct theoretical statement of the complexity of the algorithm. It also reflects the practical concern that in a large computation, you might well run out of computer memory before you run out of computer time.
Author response: We added a comment on space, and a space-aware version of the theorem at the very end of the paper.

(5) **Referee comment:**p1: It may be understood by the authors, but it is not explicitly stated in the paper, that it is almost impossible to get rid of log factors in robust models of computation with random access memory and growing numerical values. In fact, the paper has an omission on this point although the construction does work. Namely, there should be a statement that all calculated counts have only a polylog number of digits.
Author response: We've added footnote 2 on page 6.

(6) **Referee comment:** p1: I am not sure about the phrase "standardly believed", which according to Google Ngram is about 1/1000 as common as the phrase "commonly believed". Besides, instead of accusing people of commonly believing the wrong thing, it might be more diplomatic to say "it is easy to believe that this is the fastest possible"."
Author response: We have changed the phrasing to "commonly believed".

(7) **Referee comment:**p2: Although it's okay to refer impatient readers to refer to equation (3) for the main idea, by itself it's not that illuminating. It would be reasonable to let readers know that the main ideas are meet-in-the- middle and hyperoctrees: `https://en.wikipedia.org/wiki/Meet-in-the-middle_attack` `https://www.reddit.com/r/algorithms/comments/efgvbj/need_suggestions_how_to_improve_a_meetinthemiddle/` `https://en.wikipedia.org/wiki/Octree` `https://math.stackexchange.com/questions/644032/name-of-the-generalization-of-quadtree-and-octree`
These standard phrases could also be used in the abstract.
p4: It is cool that the authors might have independently discovered the concept of a hyperoctree and thought of this way to use it. Still, that standard concept and term would be better as the stated theme of Section 3.2 than dyadic intervals.
Addendum and Correction: I realized that the key data structure is not the same as a hyperoctree, which would not suffice to prove the main theorem. It is the same in 1 dimension. In 2 or more dimensions, it is essential to take the Cartesian product of the 1-dimensional tree structure, which yields a poset of dyadic boxes of all shapes, not just cubes. This carefully considered data structure is somewhat similar to a hyperoctree and this perhaps can be mentioned. However, the proof of and algorithm for the main theorem is even more original than I first realized.
Author response: We've added a brief comment at the bottom of page 4.

(8) **Referee comment:** p2 and later: The given definition of a Gauss diagram makes clumsy use of intervals. Really a Gauss diagram is an oriented perfect matching of the integers from 0 to 2n-1. Since it is standard to define intervals in ordered sets, including half-open intervals, I recommend the tidy notation $[0, 2n)_{\mathbb{Z}}$ for this set of integers. From the next page onward the paper often intersects an interval in the reals with the integers, which in my opinion is clumsy. It would be better to write

$(a,b)_{\mathbb{Z}}$, $[a,b)_{\mathbb{Z}}$, etc. Likewise instead of "parametrized" and "reparametrized", it would be simpler and clearer to say "numbered" and "renumbered".
Author response: We replaced all instances of $I \cap \mathbb{Z}$ with $I_{\mathbb{Z}}$, following the referee's suggestion. We also changed "parametrized" to "numbered".

(9) **Referee comment:** p2 and later: The phrase "decorated arrows" is duly visual but it is not precise. I recommend the phrase "oriented perfect matching". An "arrow" as used later is then an "edge" (of the matching).
Author response: We have added this phrasing to our definition of Gauss Diagram.

(10) **Referee comment:** p3 and later: The data used to combine two Gauss diagrams of size k and ℓ into a larger one of size $k+\ell$ is set up in a not-very-intuitive, asymmetric way in my opinion. Instead of a non-decreasing function λ, the data could be expressed as a subset S of $[0, 2(k+\ell))_{\mathbb{Z}}$, or as a partition into two subsets.
p3: I guess superimposition is okay as a term here, but.
Author response: We replaced λ with what we call a "pattern" P. We rewrote the proof of the Main Theorem to use patterns instead of gluing maps λ's.

(11) **Referee comment:**p3: It doesn't look like the paper uses $\bar{\phi}_k$ except to define ϕ_k. I think that the two sentences could be combined.
Author response: We have reorganized that paragraph to focus on ϕ_k, as suggested by the referee's remark.

(12) **Referee comment:**p4: Following Wikipedia, this is not the definition of a "lookup table", which instead refers to a precomputed array with numbered indexing: `https://en.wikipedia.org/wiki/Lookup_table` Rather, a searchable collection of key-value pairs is called an associative array: `https://en.wikipedia.org/wiki/Associative_array` Note that an associative array is not necessarily lex ordered. It can be implemented in any of several ways so that retrieving a value — and incrementally adding or removing a key-value pair — takes $\tilde{O}(1)$ time.
Author response: We have added the terminology "lexicographically-ordered associative array", based on the referee's suggestion.

(13) **Referee comment:** p4: Instead of $\bar{n}$, you can use $[0,n)_{\mathbb{Z}}$ as in the rest of the paper.
Author response: It is a bit cumbersome to write $[0,n)_{\mathbb{Z}}^{\ell}$ instead of $\bar{n}^{\ell}$, so we continued to use the $\bar{n}$ notation, but first introduced it as $\bar{n} := [0,n)_{\mathbb{Z}}$.

(14) **Referee comment:** p4: "this Theorem" → "this theorem"
Author response: Fixed the typo.

(15) **Referee comment:** p4: The "$\ll$" upper bound in the statement of Theorem 3.1 is plainly vestigial. The "$\ll$" lower bound is needed only to protect the specific meaning of the paper's specific asymptotic notation, and even for that purpose is stronger than necessary. Another formulation is to let Q be a multiset elements in $[0,n)_{\mathbb{Z}}^{\ell}$, and then say that the time and space complexity to build the table are both $\tilde{O}(q)$, while the time complexity $\tilde{O}(1)$, with a modified meaning of the tilde to allow factors of both log q and log n.

Author response: The referee is right that $\ll$ is used here for rather weak reasons. Yet rather than making the statement more general at the cost of having to redefine $\sim$, we chose to make it slightly simpler and less general (yet sufficient for our purposes).

(16) **Referee comment:** p4: I was thrown by the word "rectangle". I would call a typical shipping box a rectangular box and not a rectangle.
Author response: We have changed the phrasing to "rectangular box".

(17) **Referee comment:** p4: The opening sentence of section 3.2 uses the exact same notation $[a,b)_{\mathbb{Z}}$ (except without the subscript) that would be useful for so much else in the paper. Note that there is no need to make the integers non-negative.
Author response: We have changed the notation to $[a,b)_{\mathbb{Z}}$, to also align with comment (8) above.

(18) **Referee comment:** p5: "can be decomposed" $\rightarrow$ "decomposes uniquely".
Author response: We have changed the phrase to say "decomposes uniquely".

(19) **Referee comment:** p5: "Viewing $Q \cap R$ as the sum..." It would be better to go ahead and state Corollary 3.3 in the very nice Theorem 3.1, for one reason because it is a corollary of the proof and not the theorem. Actually, Corollary 3.3 is subtly incorrect as stated; or at a minimum, not well stated. If the weights are reciprocals of distinct primes, then you can get a denominator explosion when you add them. Of course denominators are not needed in the paper until the very end, where you cannot get a denominator explosion. I recommend stating "Corollary" 3.3 in terms of a finite-rank free abelian group rather than a rational vector space. It should also say something about the sizes of the numbers involved.
Author response: We renamed "Corollary 3.3" to be "Proposition 3.3" and rephrased it in terms of $\mathbb{Z}$-modules, for which the "complicated coefficients" issue is much milder. Rather than clarifying the issue, which would take space and add little value, we note it in footnote 1 and note that in our case, it does not arise. See also footnote 2.

(20) **Referee comment:** p6: In the second sentence of the proof of the main theorem, the paper has "$\sum_{i=1,\cdots,k}$". Of course $\sum_{i=1}^{k}$ which appeared before would be better.
Author response: We have updated the notation to $\sum_{i=1}^{k}$.

(21) **Referee comment:** p6: The variables $k = e + f$ are unfortunate, both because it is a change of variables from $k + \ell$ on page 3, and because f is commonly used as the name of a function and for other purposes. I recommend using something like $k = \ell + m$ consistently across both sections.
Author response: Our reasoning for the variable names is that we call our diagrams D, and we needed to break D into two subdiagrams E and F with respective sizes e and f. We prefer to keep the labelling as is, since introducing l or m here might lead to more confusion with other notation with dyadic intervals.

(22) **Referee comment:** p6: "that fits in the complement" $\rightarrow$ "that lies in the complement".

Author response: We have fixed the typo.

(23) **Referee comment:** p7: I highly recommend including arXiv numbers in the bibliography, particularly for [Rou] which is not published elsewhere, but also for references that are published in journals.
Author response: We have included the arXiv number of the unpublished paper [Rou].

(24) **Referee comment:** One more thing that I forgot to mention: The results in the paper make me curious to know about the complexity status of computing finite-type invariants of homology 3-spheres.
Author response: We are grateful to the referee for the question and research suggestion. We might consider this in a future paper.

Best regards,
Dror, Itai, Iva, and Nancy