Pensieve header: Finding Z^w starting from a non-even associator, to test if cap is always $\nu^{1/4}$. Continues WKOSession.nb of pensieve://Projects/WKO4/.

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Duflo"];
<< ../WKO4/FreeLie.m;
<< ../WKO4/AwCalculus.m;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, d△, dη, dσ, El, Es, hA,
  hm, hS, h△, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, t△, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

In[ ]:=
```
DK2Es[s___][ℰ_] := El[ℰ // αMap[s], CWS[0]] // Γ;
```

In[ ]:=
```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φs[3, 1, 1, 2] = 1/7; Φs[3, 1, 1, 1, 1, 2] = 1/13;
Φ0 = DKS[3, Φs];
SeriesSolve[Φ0, (Φ0^σ[3,2,1] ≡ -Φ0) ∧ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@{6}
```

Out[ ]=

$$\mathrm{DKS}\Big[0,\ \frac{1}{24}\,t_{13}t_{23},\ \frac{1}{7}\,t_{13}t_{13}t_{23},\ -\frac{7\,t_{13}t_{23}t_{23}t_{23}}{5760}+\frac{7\,t_{13}t_{13}t_{23}t_{23}}{5760}-\frac{t_{13}t_{13}t_{13}t_{23}}{1440},$$
$$-\frac{181\,t_{13}t_{13}t_{23}t_{13}t_{23}}{4368}-\frac{181\,t_{13}t_{23}t_{13}t_{23}t_{23}}{2184}+\frac{155\,t_{13}t_{13}t_{23}t_{23}t_{23}}{2184}-\frac{155\,t_{13}t_{13}t_{13}t_{23}t_{23}}{2184}+$$
$$\frac{1}{13}\,t_{13}t_{13}t_{13}t_{13}t_{23},\ \frac{31\,t_{13}t_{23}t_{23}t_{23}t_{23}t_{23}}{967680}-\frac{139339\,t_{13}t_{13}t_{23}t_{23}t_{13}t_{23}}{13547520}-\frac{31\,t_{13}t_{23}t_{13}t_{23}t_{23}t_{23}}{387072}-$$
$$\frac{31\,t_{13}t_{13}t_{23}t_{23}t_{23}t_{23}}{483840}+\frac{11\,t_{13}t_{13}t_{13}t_{23}t_{13}t_{23}}{290304}-\frac{51623\,t_{13}t_{13}t_{23}t_{13}t_{23}t_{23}}{5080320}-$$
$$\frac{68539\,t_{13}t_{13}t_{13}t_{23}t_{23}t_{23}}{6773760}+\frac{17189\,t_{13}t_{13}t_{13}t_{13}t_{23}t_{23}}{1693440}+\frac{t_{13}t_{13}t_{13}t_{13}t_{13}t_{23}}{60480},\ \ldots\Big]$$

In[∘]:=
```
R = DKS[t[1, 2] / 2];
Z_B = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

Out[∘]=
$$\mathrm{DKS}\Big[\frac{\overline{t_{23}}}{2},\ -\frac{1}{12}\overline{t_{13}\,t_{23}}-\frac{1}{24}\overline{t_{14}\,t_{24}}+\frac{1}{24}\overline{t_{14}\,t_{34}}+\frac{1}{12}\overline{t_{24}\,t_{34}},\ -\frac{1}{7}\overline{t_{13}\,\overline{t_{23}\,t_{23}}}-\frac{1}{7}\overline{t_{24}\,\overline{t_{34}\,t_{34}}}-\frac{1}{7}\overline{t_{14}\,\overline{t_{14}\,t_{24}}}+\frac{1}{7}\overline{t_{14}\,\overline{t_{14}\,t_{34}}}+\frac{3}{7}\overline{t_{14}\,\overline{t_{24}\,t_{34}}}+\frac{1}{7}\overline{t_{24}\,\overline{t_{24}\,t_{34}}},\ \ldots\Big]$$

In[∘]:=
```
V_B = Z_B // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
  tσ[{2, 4} → {x, y}]
```

Out[∘]=
$$\mathrm{Es}\Big[\Big\langle x\to \mathrm{LS}\Big[0,\ -\frac{\overline{x\,y}}{24},\ \frac{1}{7}\overline{\overline{x\,y}\,y},\ \ldots\Big],\ y\to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{x\,y}}{12},\ \frac{1}{7}\overline{x\,\overline{x\,y}},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}[0,0,0,\ldots]\Big]$$

In[∘]:=
```
γ = CWS[{x, y}, γs];
V0 = Es[V_B[[1]], γ];
```

In[∘]:=
```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

In[∘]:=
```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn =
  (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡ (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

In[∘]:=
```
SeriesSolve[{γ, κ}, (ℏ^-1 R4Eqn) ∧ UnitarityEqn ∧ CapEqn];
{V0@{4}, κ@{6}}
```

SeriesSolve: In degree 1 arbitrarily setting {κs[x] → 0}.

SeriesSolve: In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

Out[∘]=
$$\Big\{\mathrm{Es}\Big[\Big\langle x\to \mathrm{LS}\Big[0,\ -\frac{\overline{x\,y}}{24},\ \frac{1}{7}\overline{\overline{x\,y}\,y},\ \frac{7\,\overline{x\,\overline{x\,\overline{x\,y}}}}{5760}-\frac{7\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5760}+\frac{\overline{\overline{\overline{x\,y}\,y}\,y}}{1440},\ \ldots\Big],$$
$$y\to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{x\,y}}{12},\ \frac{1}{7}\overline{x\,\overline{x\,y}},\ \frac{1447\,\overline{x\,\overline{x\,\overline{x\,y}}}}{40\,320}-\frac{367\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5040}+\frac{1}{720}\overline{\overline{\overline{x\,y}\,y}\,y},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xy}}{48},\ \frac{\widehat{xxy}}{14}+\frac{\widehat{xyy}}{14},\ \frac{\widehat{xxxy}}{2880}+\frac{\widehat{xxyy}}{2880}+\frac{\widehat{xyxy}}{5760}+\frac{\widehat{xyyy}}{2880},\ \ldots\Big]\Big],$$
$$\mathrm{CWS}\Big[0,\ -\frac{\widehat{xx}}{96},\ \frac{\widehat{xxx}}{42},\ \frac{\widehat{xxxx}}{11\,520},\ \frac{\widehat{xxxxx}}{130},\ -\frac{\widehat{xxxxxx}}{725\,760},\ \ldots\Big]\Big\}$$

In[∘]:=
```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Out[∘]=
$$\Big\{\mathrm{El}\Big[\Big\langle x\to \mathrm{LS}[\overline{y},\ 0,\ 0,\ \ldots],\ y\to \mathrm{LS}[\overline{x},\ 0,\ 0,\ \ldots]\Big\rangle,\ \mathrm{CWS}[0,0,0,\ldots]\Big],$$
$$\mathrm{Es}\Big[\Big\langle x\to \mathrm{LS}\Big[\overline{y},\ \frac{\overline{x\,y}}{2},\ \frac{1}{6}\overline{x\,\overline{x\,y}}-\frac{1}{12}\overline{\overline{x\,y}\,y},\ \ldots\Big],\ y\to \mathrm{LS}\Big[\overline{x},\ -\frac{\overline{x\,y}}{2},\ -\frac{1}{12}\overline{x\,\overline{x\,y}}+\frac{1}{6}\overline{\overline{x\,y}\,y},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}[0,0,0,\ldots]\Big]\Big\}$$

```
In[]:= τV = Rs[x, y] ** (V0 // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
       (V0 ≡ τV) @ {6}
Out[]= BS[7 True, ...]
```