Knot Theory as an Excuse

Discovery Grant Proposal

Recent Progress.

For the purpose of this proposal, I would like to concentrate on topic #4 in my *Notice of Intent*. Recently, as a direct outcome of the research supported by my previous NSERC grant, Roland van der Veen and I commenced a study of a knot invariant Θ, which is the **strongest genuinely computable knot invariant presently known**. Let me start by discussing the words in this bold statement.

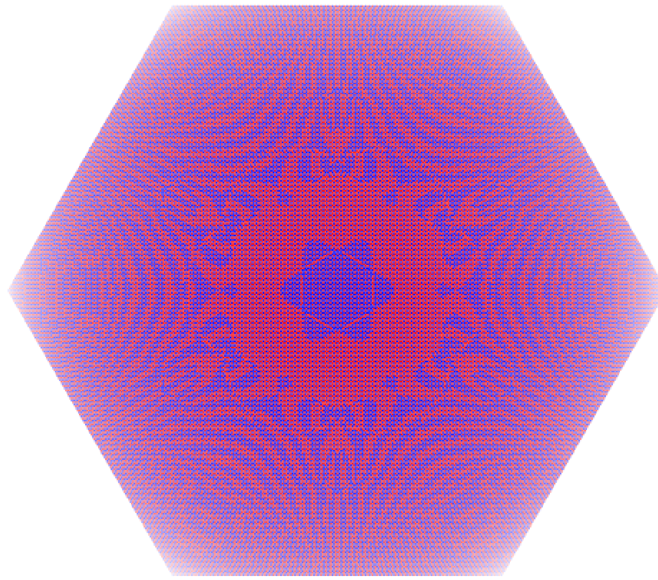
A **knot** is a piece of string tangled up in 3D space; some examples appear on the right. We consider two such tanglings to be equivalent if you can get from one to the other by continuously deforming the strings from one shape to the other, without cutting them at any point. Knots may appear esoteric, yet they are key to the understanding of all 3-dimensional and 4-dimensional spaces. For a light introduction, see [Mu].

It is in general very difficult to decide if two knots are equivalent; the best algorithms to do so take an exponential amount of time and hence they are impractical. So we seek what's called "knot **invariants**" – computable functions that assign to a knot some simpler quantities, such as polynomials or matrices, in a way so that equivalent knots are assigned equal invariants. It is even better if one can read topological properties of the knot from the values of its invariants.

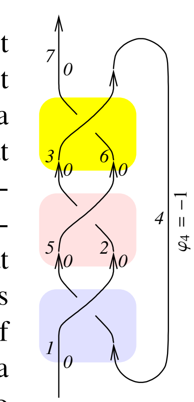
By **genuinely computable** we mean that we can compute Θ on arbitrary knots with up to about 300 crossings. For almost any other invariant presently known, that would be science fiction.

By **strongest** we mean that Θ appears to be quite good at separating knots. For example, on the 313,230 prime knots with up to 15 crossings, Θ attains 306,472 distinct values – a deficit of 6,758. The better known yet less computable HOMFLY-PT polynomial and Khovanov homology, taken together, have a deficit of 70,245, more than 10 times the worse.

Aside. The main part of the value of Θ is a two-variable polynomial. Such polynomials can be regraded as 2D arrays of coefficients, and these can be considered as coding the colour values of pixels. Hence the values of Θ can be displayed as pictures. The picture corresponding to a random 317 crossing knot (from [DHOEBL]) appears below. There are patterns in these pictures, and one of my less-major goals within the grant period will be to understand them.

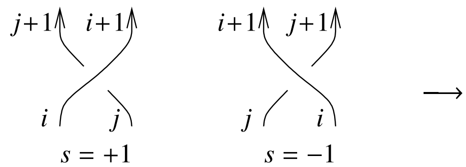


While perhaps not strictly necessary, I'd like to give here a complete definition of Θ.



Preparation. Given an oriented knot K, we draw it in the plane as a long knot diagram D with n crossings in such a way that the two strands intersecting at each crossing are pointing up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 except for φ_4, which is -1.

Making a matrix. We let A be the $(2n+1)\times(2n+1)$ matrix with entries in the ring $\mathbb{Z}[T^{\pm 1}]$ of Laurent polynomials in a formal variable T obtained by starting with the identity matrix I_{2n+1} and adding to it one contribution per crossing as follows (s is the sign of the crossing):



$$\begin{array}{c|cc} \text{add at} & \text{column } i{+}1 & \text{column } j{+}1 \\ \hline \text{row } i & -T^s & T^s-1 \\ \text{row } j & 0 & -1 \end{array} \qquad (1)$$

For our example, A comes out to be:

$$A=\begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Please count everything so far as "trivial". The matrix A is a presentation matrix for the Alexander module of K, obtained by using Fox calculus on the lower Wirtinger presentation. Up to a unit $\pm T^\bullet$, its determinant is the normalized Alexander polynomial Δ and there's nothing new about it. In fact,

$$\Delta = T^{(-\varphi-w)/2}\det(A), \quad \text{with} \quad \varphi=\sum_k \varphi_k,\ w=\sum_c s.$$

Note that in our example $\Delta = T-1+T^{-1}$.

Doing something new. Let $G=(g_{\alpha\beta})=A^{-1}$ be the inverse matrix of A, so in our example, G is

$$\begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

There is little precedence for inverting a presentation matrix, so already here we are in little-explored territory.

Let T_1 and T_2 be indeterminates and let $T_3=T_1T_2$. For $\nu=1,2,3$ let Δ_ν and $G_\nu=(g_{\nu\alpha\beta})$ be Δ and G subject to the substitution $T\to T_\nu$. Now define

$$\theta(K) := \Delta_1\Delta_2\Delta_3\left(\sum_c R_{11}(c)+\sum_{c_0,c_1} R_{12}(c_0,c_1) + \sum_k \Gamma_1(\varphi_k,k)\right), \qquad (2)$$

where the first summation is over crossings $c=(s,i,j)$ (with s, i, j as in (1)), the second is over pairs of crossings ($c_0=(s_0,i_0,j_0), c_1=(s_1,i_1,j_1)$), and the third is over edges k, and where

```
R11[s_, i_, j_] =
  s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
      (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji
          (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
             (T2^s - 2) g2jj + g2ij)) / (T2^s - 1));
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R12[{s0_, i0_, j0_}, {s1_, i1_, j1_}] =
  s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0));
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and

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Γ1[φ_, k_] = -φ/2 + φ g3kk;
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(The formulas above were computer-generated from the source code of a program that computes Θ and that was verified throughly. This guarantees the absence of typos.)

Now let $\Theta(K)=(\Delta(K),\theta(K))$ (the computation of Δ is a part of the computation of θ, so including it here is not artificial). This completes the definition of $\Theta(K)$. Yet, to emphasize that the definition above is actually quite simple, here is a complete implementation of Θ, written in Matematica [Wo]:

```
T3 = T1 T2;
Θ[K_] :=
  Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α,
     β, gEval, c, z},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
      (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                         0      -1   ))];
   Δ = T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   gEval[ℰ_] :=
    Factor[ℰ /. g_ν_,α_,β_ :> (G[[α, β]] /. T → T_ν)];
   z = gEval[Sum_{k=1}^{n} R11 @@ Cs[[k]]];
   z += gEval[Sum_{k1=1}^{n} Sum_{k2=1}^{n} R12[Cs[[k1]], Cs[[k2]]]];
   z += gEval[Sum_{k=1}^{2 n} Γ1[φ[[k]], k]];
   {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) z} //
    Factor];
```

Note 1. We note the similarity between the formulas written here for Θ with evaluations of perturbed Gaussian integrals via Feynman diagrams. In both cases the end result is a sum of polynomials in the entries of the inverse of a matrix. This similarity can be made precise, and indeed, in [BN5] I write a perturbed Gaussian integral formula for Θ, in which the Lagrangian is a sum over the crossings of K of quadratic terms that correspond to the matrix A (and whose inverse becomes G) and of higher order perturbation terms. The integration is carried out over a space whose dimension is 6 times the number of edges in a diagram D of K — a space that has some combinatorial significance (as it pertains to a knot diagram) but no immediate topological significance.

Objectives.

I have two long-term career goals. They are radical and it is time to out them.

Goal 1. *It's time to depreciate Witten-Reshetikhin-Turaev invariants (WRT) within knot theory and low dimensional topology.* This may sound like the words of a lunatic, seeing that so so much has been written about the Chern-Simons-Witten quantum field theory and about Reshetikhin-Turaev invariants (including by myself). Yet please, bear with me and keep an open mind:

- WRT invariants came to topology from outside, from representation theory and from quantum algebra and quantum field theory, and we still can't quite motivate them in the language of topology. Seen in the eyes of a topologist who studies objects (and not a dual-topologist, who studies invariants for their own sake), WRT invariants seem like artificial constructs.

- Other than their separation power WRT invariants simply don't do much in topology.

These two bullets are not unrelated, of course. What isn't natural in topology is unlikely to do much for topology.

Yet topologists can't fairly pretend that WRT invariants don't exist. They exist for sure. We simply don't understand them.

One of my long-term career goals is to find the proper topology home for WRT invariants. I used to think that this entailed understanding them and the processes leading to them better, but it may be that by now I understand those well enough to suspect that that's not where the keys are hiding. I now believe in the following instead:

> There is a natural home in topology for the invariant Θ discussed earlier, and in that home live many other invariants with formulas similar to Θ's. The collection of all such invariants is equivalent to the collection of WRT invariants; except that the WRT invariants make "a wrong basis" to that collection, within which it is hard to see their naturality and their utility.

The matrix $g_{\alpha\beta}$, a key to the construction of Θ, is the inverse matrix of a presentation matrix of the Alexander module M of a knot K, the first homology of the universal Abelian cover of the knot complement. There are other presentation matrices for M and I expect that many of them can be used to write alternative formulas for Θ. In particular, I expect that there should be a "Seifert formula" for Θ, presenting it as a perturbed Gaussian integral of the exponential of a Lagrangian L which is a naturally-defined function on the homology H of a Seifert surface Σ for K (or of a finite number of copies of that homology). The quadratic part of the Lagrangian should be the Seifert linking form, repeated over several copies of H (and taken with different parameters $T_1, T_2, \ldots$).

Where would the Lagrangian L be coming from? To compute the topologically most interesting knot polynomial, the Alexander polynomial, one only studies the linking and intersection numbers of curves on Σ. We are clearly missing a lot of topology here. I expect that other finite type invariants of curves on Σ will be used to produce the "perturbation" terms of L.

In my dreams, given a knot K we will pick a Seifert surface Σ for K with homology vector space H. We will then be asking ourselves, "which Lagrangians L on nH (n copies of H, for various fixed values of n), when integrated on nH using the rules of perturbative Gaussian integration (namely, Feynman diagrams), will yield knot invariants"? I expect that the answer to that question is non-empty, for the currently strongest genuinely-computable knot invariant Θ is most likely an example (and a few more are at [BN5]). We ought to be able to classify in simple terms the set of such Lagrangians. There should be plenty, and it should be possible to describe them in topologically more natural terms than "semi-simple Lie algebras and their representations". The corresponding invariants will be strong and easy to compute (as Θ is) and they should be topologically meaningful, as they will be intrinsically aware of the topology of the Seifert surface Σ.

I expect that ultimately these Seifert-type invariants will replace WRT invariants as a center of attention in algebraic knot theory and low dimensional topology.

Goal 2. *Disprove the ribbon-slice conjecture*. It is well known that every ribbon knot is slice, and one of the greatest open problems in knot theory is whether the converse holds true; namely, whether every slice knot is ribbon. This possible equivalence of ribbon knots with slice knots is known as the "ribbon-slice conjecture", and many people believe it to be false. There are even proposed counterexample (e.g., in [GST], and note that these proposed counterexamples are rather large knots). What is missing is a proof that these proposed counter examples aren't ribbon knots.

What is needed is a knot invariant Ψ whose values on ribbon knots are especially constrained. If such an invariant can be computed on those rather large proposed counterexamples, and if its values on the proposed counterexamples do not satisfy the ribbon constraints, we will have disproved the ribbon-slice conjecture.

So we need an invariant Ψ that can be computed efficiently on large knots, and that "sees" the Seifert surface of a knot (as the Seifert surfaces of ribbon knots can be taken to be of a special form, possibly leading to the restrictions on the values of Ψ; see "Seifert for Ribbon" below).

If there ever was a good candidate for Ψ, it is our Θ (in fact, a lot of my motivation for the development of Θ was precisely that it would serve as Ψ). Yet a lot of work remains to be done. I am sure there are Seifert formulas for Θ, but I don't know them yet. And once these formulas are written, it would still be necessary to find what constraints on their values can be obtained from the existence of Seifert surfaces of the form that arises from ribbons. This is a hard problem and I expect it will take several years to find the answer.

Seifert for Ribbon. A ribbon knot with g ribbon singularities always has a Seifert surface Σ of genus g, in which g of the $2g$ homology cycles can be jointly represented by a g-component unlink. See Figure 1.

If as I believe Θ has a Seifert formula as discussed in Goal 1, in which all the ingredients of the Lagrangian L are finite type invariants of curves representing homology classes on Σ, the half-triviality of these curves as indicated above will lead to strong restrictions on the form of L which in themselves may lead to strong restrictions on the values of Θ.

For the Alexander polynomial Δ, the same reasoning leads to the Fox-Milnor condition. See e.g. [Kau, pp. 212-213].

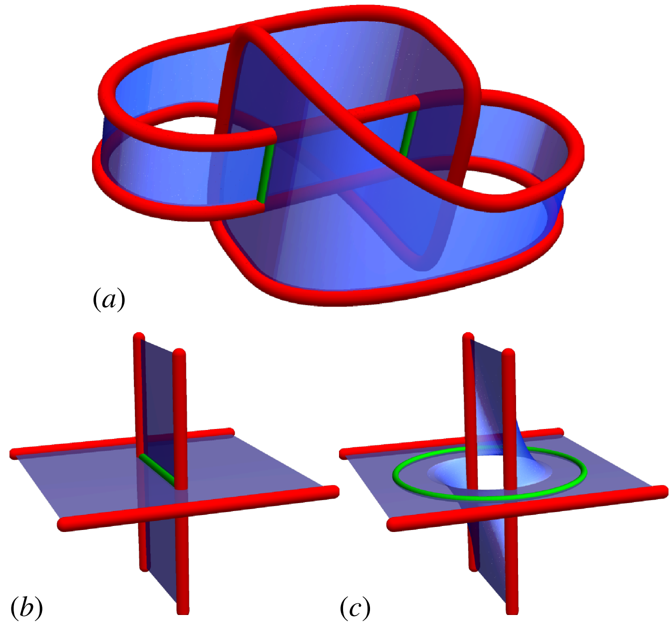

Figure 1. A ribbon knot (a) and a ribbon singularity (b) (singularities in green), and a piece of a Seifert surface for a ribbon knot near a ribbon singularity (c), with an unknotted homology cycle in green. A ribbon knot with g ribbon singularities will have g of those, unlinked with each other.

As for the other topics within my Notice of Intent (ωεβ/NOI): Topic #4 is the core of this proposal as above. Topic #2 was mentioned in passing within the above, and will not be mentioned further. Topic #3 is mostly subsumed within the discussion of topic #4 above. To a large extent, ρ_1 is very much like θ except with somewhat different specific formulas. For topics #1, #5, and #6, I will simply repeat ωεβ/NOI with some modifications:

Topic 1. I plan to continue to study, along with Roland van der Veen and others, how "solvable approximation" of semisimple Lie algebras (Inonu-Wigner contractions of their lower Borel subalgebras) leads via perturbed Gaussian formulas (in spirit, QFT) to poly-time computable knot invariants that "behave well" under useful knot theoretic operations. I hope this sounds powerful; it certainly sounds highly technical. Can we make it less technical? Can we rely less on Lie algebra and quantum algebra techniques and instead make the topic intrinsic to knot theory? See [BN1, BV1, BV2, BN2].

Topic 5. Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich [BN3], and also along with Yusuke Kuno [BK], I plan to continue to study knots and tangles in a "Pole Dancing Studio" (PDS, a cylinder with a few vertical lines removed) and their relationship with the Goldman-Turaev Lie bialgebra and Kashiwara-

Vergne (KV) equations [AKKN1, AKKN2]. Are solutions of the KV equations sufficient to construct a homomorphic expansion of tangles in a PDS up to strand-strand degree 1? How is this related to my earlier work with Dancso [BD1, BD2] on welded knots? The subject is beautiful, yet it is a hard-to-penetrate patchwork of results and techniques and papers by different authors. In the past, this feeling that a subject's beauty is incongruous with its complexity had been a great motivator for me, often leading to deeper understanding. I have high hopes for this topic too.

Topic 6. Recently [BN4], along with my Ph.D. student Jessica Liu, we've found a truly elegant "signatures for tangles" invariant (sorry for complimenting ourselves, yet hey, it really is elegant). There is more to do before we can claim to fully understand these signatures. Is there an Alexander invariant for tangles obtained using the same "pushforward" techniques? Are its roots related to the jumping points of the signature? Does it generalize to the multi-variable case? This topic was originally conceived within an attempt to prove the Kashaev Signature Conjecture [Kas], but that conjecture is by now my student's Jessica Liu's theorem [Li].

Literature review.

Θ is most likely equal to "the two loop contribution to the Kontsevich integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh], continuing an older study by myself and Garoufalidis [BG] (we haven't proven that, yet I expect we will soon). But the definitions used by these authors are a lot more complicated than ours, and do not lend themselves to efficient computations.

Θ is probably related to the invariant considered by Garoufalidis, Kashaev, and Li in [GK, GL], as they share many properties. Yet they are different, and our definitions are simpler and lead to vastly faster computations.

Other than that, there isn't a lot written yet about Θ.

Methodology.

To some extent my methodology is as dull as it gets. I sit in my office (with my feet up if nobody's looking), or in a coffee shop (never with my feet up), or I ride the bus or I lie in bed, and I think. Sometimes a pencil and a piece of paper are involved too.

Yet in one way my methodology differs from that of most mathematicians. Almost everything I do I implement on the computer almost immediately. It isn't just that I implement what I conceived with pencil and paper once the latter matures. Rather, the pencil and paper and the implementation are fully interleaved and integrated. My implementations are mathematically-informed, and my thoughts and scribbles never diverge much from what can be implemented. I believe that Θ, the strongest genuinely-computable knot invariant we presently know, is a great success. Its strength is the result of a bit of informed luck. Its computability isn't luck. Its computability is because I think about computability at nearly all times. Computability was a part of the development process of Θ throughout. I expect that my future work will follow the same lines.

I am asked to comment here on issues of EDI (Equity, Diversity, and Inclusion). On the surface, such issues do not arise in mathematical research or within my methodology of research. Math is gender- and race-neutral, and computers don't know the races and genders of the people punching their keyboards. This said, I am aware, and over the years I became more and more aware, of how differing backgrounds may lead to differing levels of initial preparedness, of how differing societal and cultural expectations lead to different ways in which we present ourselves, and of how inconsiderate feedback, or worse, the wrong kind of attention, can greatly harm the motivation and success of young researchers.

With this (ever growing) awareness I'm doing my best to make the atmosphere in my research group supportive to all members of all under-represented groups. The alternative, of losing great minds because perhaps they dress or look differently, would be offensively stupid.

I think I've had some success. Of my 17 PhD students so far, 5 are women: one is a current student, and the four that have already graduated all continued within academia, two with tenured or tenure-track positions (at the University of Sydney and at Northeastern University). My most recent graduated PhD student, who defended his thesis last August, came from Ghana to Canada back in 2017 specifically in order to work with me, following links I have established when I volunteered to give a course on algebraic topology at the University of Ghana in 2010. I've had (and I have) a number of other BIPOC students, but the definitions here are sometimes ambiguous and do not belong in this document, so I will refrain from including statistics. Of my 5 post-doctoral fellows, 3 were women (one is current).

Impact.

I expect the work proposed here to revolutionize what we know about knot invariants. The invariant Θ is already the strongest genuinely-computable invariant we have, and it stands to get better by acquiring a solid topological foundation. In addition, I think there is a fair chance that the work that I propose will lead to disproving the ribbon-slice conjecture, one of the most major outstanding problems in knot theory.