Knot Theory as an Excuse

Discovery Grant Proposal

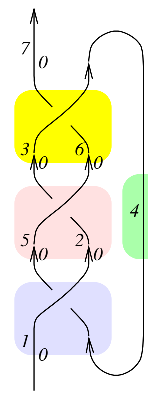
Grant proposals are often written as if they describe a victory parade. “The principal investigator will march from A to B to C collecting trophies along the way”. I hope this one is written differently. There is a single “!” in it, and it is in quotes. There are plenty of “?” in it. Each one represents a dream. A question that I plan to study and that I hope I have the tools to address and to elucidate, if not resolve. Will the NSERC help?

I have chosen to concentrate in this proposal on what was topic #2 in my Notice of Intent (ωεβ/NOI) — the invariant ρ_1: a well-connected, strong, homomorphic, and ridiculously easy to define and to compute knot invariant, which is nevertheless far from being understood and utilized. Let me tell you some more about it.

ρ_1 is not new. It traces back to work by Rozansky [Ro3, Ro4] and Overbay [Ov] and to work by Ohtsuki [Oh], which in itself traces back to work by Garoufalidis and myself [BNG] proving the Melvin-Morton-Rozansky Conjecture [MM, Ro1, Ro2] which relates the Coloured Jones polynomial [Jo] with the Alexander polynomial [Al]. Yet my recent work with Roland van der Veen [BV3] makes it ridiculously easy to define and to compute and shows it to be “homomorphic” (see below) and hence suggests that ρ_1 may have far-reaching topological implications and applications. Does it?

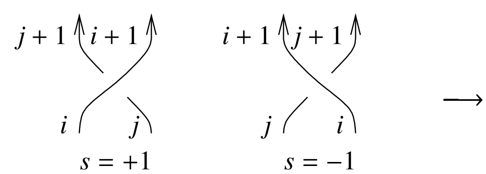
ρ_1 is thus dominated by the coloured Jones polynomial. That can be seen as a handicap, for supposedly we already “understand” the coloured Jones polynomial. But no, we don’t really. The coloured Jones polynomial is complicated to define and nearly impossible to compute for knots with more than just a few crossings. A section of the coloured Jones that is simple and easy and which is more than the “classical” Alexander polynomial may well be the golden key that many have been looking for, that will finally bring the power of quantum invariants to use within classical topology. Is it? (It is a bit of an absurd, and a bit of a sore point, that quantum invariants that are so much stronger than the Alexander polynomial say so little, beyond what Alexander already knows, on classical properties of knots such as their genus, unknotting numbers, and whether or not they are slice or ribbon or fibred).

ρ_1 really is easy to define, so here’s a definition, in full and with a worked-out example, following [BV3].



Preparation. Given an oriented knot K, we draw it in the plane as a long knot diagram D with n crossings in such a way that the two strands intersecting at each crossing are pointed up (that’s always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a “rotation number” φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

Making a matrix. We let A be the $(2n+1)\times(2n+1)$ matrix with entries in the ring $\mathbb{Z}[T^{\pm 1}]$ of Laurent polynomials in a formal variable T obtained by starting with the identity matrix I_{2n+1} and adding to it one contribution per crossing as follows (s is the sign of the crossing):



add at	column $i+1$	column $j+1$
row i	$-T^s$	T^s-1
row j	0	-1

(1)

For our example, A comes out to be:

$$A=\begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -T & 0 & 0 & T-1\\ 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & T-1 & 0 & 1 & -T & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Please count everything so far as “trivial”. The matrix A is a presentation matrix for the Alexander module of K, obtained by using Fox calculus on the lower Wirtinger presentation. Up to a unit $\pm T^\bullet$, it’s determinant is the normalized Alexander polynomial Δ and there’s nothing new about it. Note that in our example $\Delta = T-1+T^{-1}$.

Doing something new. Let $G = (g_{\alpha\beta}) = A^{-1}$ be the inverse matrix of A, so in our example, G is

$$\begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now define

$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\,(g_{kk} - 1/2)\right), \qquad (2)$$

where the first summation is over crossings c and the second is over edges k, and where

$$R_1(c) = R_1(s,i,j) :=$$
$$s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right) \qquad (3)$$

where for a crossing c, the parameters s, i, and j are as in (1). This completes the definition of ρ_1. It's invariance is proven by elementary means[1] in [BV3].

For our trefoil example, using the values of Δ and of $g_{\alpha\beta}$ established before,

$$\begin{aligned} \rho_1 &= \Delta^2(R_1(1,3,6) + R_1(1,5,2) + R_1(1,1,4) \\ &\qquad - (-1)(g_{44} - 1/2)) \\ &= \Delta^2(g_{63}(g_{76} + g_{67} - g_{36}) - g_{33}(g_{67} - 1) - 1/2 \\ &\qquad g_{25}(g_{32} + g_{23} - g_{52}) - g_{55}(g_{23} - 1) - 1/2 \\ &\qquad g_{41}(g_{54} + g_{45} - g_{14}) - g_{11}(g_{45} - 1) - 1/2 \\ &\qquad + g_{44} - 1/2) \\ &= -T^2 + 2T - 2 + 2T^{-1} - T^{-2}. \end{aligned}$$

But wait, what? Inverting a presentation matrix? What does it mean? Who does that? Forming a quadratic expression out of the entries of said inverse? Who does that? What does it mean?

ρ_1 really is easy to implement. As evidence for that, here is a complete implementation, written in Mathematica [Wo]. The only reason it is included to make a point: It is ridiculously short.

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R₁[s_, i_, j_] :=
  s (g_{j i} (g_{j⁺,j} + g_{j,j⁺} - g_{i j}) - g_{i i} (g_{j,j⁺} - 1) - 1/2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
      (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                       0     -1    ))];
   Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   ρ1 = Σ_{k=1}^{n} R₁ @@ Cs⟦k⟧ - Σ_{k=1}^{2 n} φ⟦k⟧ (g_{kk} - 1/2);
   Factor@
    {Δ, Δ² ρ1 /. α_⁺ :> α + 1 /. g_{α_,β_} :> G⟦α, β⟧}];
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ρ_1 is easy to compute, also in a technical sense. Except for the computation of A^{-1}, the computation of ρ_1 takes only a linear number of additions and multiplications in the ring $\mathbb{Z}[T^{\pm 1}]$, as a function of the number of crossings n (and the degrees and the digit-lengths of the coefficients of all the polynomials that appear are easily linearly bounded by n, so ring operations are cheap). The hardest part of the computation of ρ_1, inverting a matrix with entries that are affine linear in $T^{\pm 1}$, is standard and efficient and takes polynomial time (in n), though it's better not to commit to a specific bound because the bounds on the complexity of matrix operations are still improving.

ρ_1 is strong. Direct computations show that, at least on knots with up to 12 crossings, it has more separation power then the HOMFLY-PT polynomial and Khovanov homology taken together. Both are considered rather strong, and both are much harder to compute: the programs are longer, and they run in non-polynomial time. To the best of my knowledge, presently ρ_1 is the strongest knot invariant we know, both per line of code and per CPU cycle.

ρ_1 has a home in quantum algebra. Indeed, in [BV1, BV2, BV3] and in future publications, Roland van der Veen and I explain how the formulas (2) and (3) arise in a natural way from the quantization of a natural contraction of the lie algebra sl_2. Very roughly, up to a central factor, sl_2 is the double of its half, its upper Borel subalgebra $\mathfrak{b}$. If one takes $\mathfrak{b}$, scales its cobracket by ϵ and then doubles, one gets a new algebra sl_{2+}^{ϵ}, which is isomorphic to sl_2 (plus a central factor) if ϵ is invertible, yet is solvable when $\epsilon = 0$. The algebra sl_{2+}^{ϵ} can be quantized using the Drinfel'd double procedure, and its uni-

[1]Elementary is better than fancy and complicated. If you do something new that an undergraduate student can understand you contribute more than if you do something new that only graduate students can understand.

versal quantum knot invariant Z_ϵ may be considered. It turns out that all the tensors that appear within the study of sl^ϵ_{2+} are "perturbed Gaussians", and can be effectively computed using techniques reminiscent of the techniques used in perturbative quantum field theory, with ϵ as the perturbation parameter. (Can we make this lofty quantum algebra / QFT discussion a lot easier?)

Thus if we expand $Z_\epsilon = \sum_{d\geq 0} Z^{(d)}\epsilon^d$, then $Z^{(0)}$ is computable using pure Gaussian techniques[2]. It turns out that $Z^{(0)}$ reduces to the Alexander polynomial. The computation of $Z^{(1)}$ then involves minimal perturbation theory, and when the dust settles, ρ_1 emerges from it.

So can we say we understand ρ_1? Oh no. Something so simple as formulas (2) and (3), and so close to the Alexander polynomial with its purely topological definitions, ought to have a much simpler home, hopefully within the hamlet of topology down the street from Alexander's. What does this home look like?

ρ_1 has neighbors. From quantum algebra it follows that there are also ρ_d for $d > 1$ that arise in a similar manner from $Z^{(d)}$. Quantum algebra gives as recipes for computing ρ_d, and Roland van der Veen and myself have implemented them and computed them. They are stronger than ρ_1, but get progressively harder to compute. For ρ_2, the bottleneck remains inverting A (so it is still "easy"). For ρ_3 and beyond the bottleneck moves to perturbation theory. Each ρ_d remains polynomial time, but the exponents get bigger and bigger. It also follows from quantum algebra that there should be similar poly-time computable $\rho_{\mathfrak{g},d}$ for other semisimple Lie algebras $\mathfrak{g}$. How can we compute them? Do they have homes in topology?

I think it is unreasonable to believe that looking from topology into perturbations of the Alexander polynomial the theory of semisimple Lie algebras will naturally emerge. I believe the Lie algebras appear in the collection $\{\rho_{\mathfrak{g},d}\}$ because we are searching under the existing lamppost of quantum algebra. Once we find the right vintage point, it will become a different collection, $\rho_{\clubsuit,d}$, parameterized by some unknown moduli $\clubsuit$ which will be meaningful in topology. What is it? Will it merely be a change of basis, or will it be stronger then $\{\mathfrak{g}\}$, the collection of all semisimple Lie algebras?

ρ_1 is "homomorphic", meaning that it extends to tangles, and that its value on a given tangle $\mathcal{T}$ determines its value on any tangle obtained from $\mathcal{T}$ by strand doubling or by composition with other tangles[3]. I've been shouting for a long time now [BN], that being homomorphic may be a very valuable property. For indeed, certain classes of knots that carry great interest, such as ribbon knots and slice knots and knots of a given genus, are definable in terms of tangles and tangle compositions and strand doublings. Invariants that respect tangle operations, namely which are homomorphic, thus have a greater a priori chance of "saying something" about these classes of knots: giving genus bounds, or slice or ribbon obstructions. I believe the Alexander polynomial has this kind of topological applications precisely because it is homomorphic [BNS], and I believe the homomorphic properties of ρ_1 mean that it is much more likely to be of interest in classical low dimensional topology than almost anything else quantum algebra ever produced. Am I right?

ρ_1 is far from understood. There are the heavy questions as above, but even the light requires further work. There are plenty of other matrices like A, whose determinant computes the Alexander polynomial (arising from the Dehn presentation, from Seifert surfaces and forms, from braid closures or plat closures of braids and the Burau representation, from arc presentations, from w-knots, from strange formulas by Kashaev and Liu [Ka, Li], and more). Are there formulas for ρ_1 in terms of the inverses of each of these matrices? In particular, will the formulas coming from Seifert surfaces produce genus bounds and ribbon obstructions, as they do for the Alexander polynomial? Will the arc presentation formula speak with knot Floer homology, as its Alexander counterpart does?

The formulas we've presented here for ρ_1 are directly related to the lower Wirtinger presentation, and there are similar formulas coming from the upper Wirtinger presentation. Is there an elementary proof that these formulas agree? Is there an elementary proof that ρ_1 is palindromic (satisfies $\rho_1(T) = \rho_1(T^{-1})$)? If we're having difficulty already with that, we are clearly missing something. What is it?

ρ_1 extends to tangles without closed components. Is there a natural extension of ρ_1 to links and to tangles that are allowed to have closed components?

[2] Stricktly speaking, "two-step Gaussians", but that need not concern us here.

[3] Reshetikhin-Turaev invariants with fixed representations (namely, those that are computable, even if in exponential time), do not have the "strand doubling" part of this property. In particular, the Jones polynomial and the HOMFLY-PT polynomial do not have the "strand doubling" part of this property.

As for the other topics within my Notice of Intent (ωεβ/NOI), topic #1 was mentioned in passing within the above, and will not be mentioned further. For topics #3 and #4, I will simply repeat ωεβ/NOI with some modifications:

#3 Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich (ωεβ/PDS), I plan to continue to study knots and tangles in a "Pole Dancing Studio" (PDS, a cylinder with a few vertical lines removed) and their relationship with the Goldman-Turaev Lie bialgebra and Kashiwara-Vergne (KV) equations [AKKN1, AKKN2]. Are solutions of the KV equations sufficient to construct a homomorphic expansion of tangles in a PDS up to strand-strand degree 1? How is this related to my earlier work with Dancso [BD1, BD2] on welded knots? The subject is beautiful, yet it is a hard-to-penetrate patchwork of results and techniques and papers by different authors. In the past, this feeling that a subject's beauty is incongruous with its complexity had been a great motivator for me, often leading to deeper understanding. I have high hopes for this topic too.

#4 Recently (ωεβ/PQ), along with Jessica Liu, we've found a truly elegant "signatures for tangles" invariant (sorry for complimenting ourselves, yet hey, it really is elegant). There is more to do before we can claim to fully understand these signatures. Is there an Alexander invariant for tangles obtained using the same "pushforward" techniques? Are its roots related to the jumping points of the signature? Does it generalize to the multivariable case? Within the Notice of Intent I also had a question about proving the Kashaev signatures conjecture [Ka], but that conjecture is by now my student's Jessica Liu's theorem [Li].