Notice of Intent Summary

July 8, 2017 11:09 AM

Application Title: Poly-Time Knot Theory and Quantum Algebra

Summary of Proposal (5000 characters max, the PPSA abstract is about 5%)

Totally by definition, once in a lifetime, a researcher is working on his personal best project. For me this is now, and I'm very excited about it. Let me explain.

Here and there math has immense philosophical value or beauty to justify the effort. Yet everyday math is mostly about, or should be about, "doing useful things". Deciding if *A* has property *B*, counting how many *C*'s satisfy *D*, computing *E*.
When *A* and *B* and *C* and *D* and *E* are small, we do the computations on the back of an envelope and write them as "Example 3.14" in some paper. But these are merely the demos, and sooner or later we worry (or ought to worry) about bigger inputs. I'm more aware then most mathematicians (though perhaps less than many computer scientists), how much the complexity of obtaining the solution as a function of the size of the inputs matters. Hence I firmly believe that incomputable mathematics is intrinsically less valuable than computable mathematics (allowing some exceptions for philosophical value and/or beauty), and that within computable mathematics, what can be computed in linear time is generally more valuable than what can be computed in polynomial (poly-) time, which in itself is more valuable than what can be computed in exponential (exp-) time, which in itself is more valuable than what can be computed just in theory.

I've always been an exp-time mathematician. Almost everything I've worked on, finite-type invariants and invariants of certain 3-manifolds, categorification, matters related to associators and to free Lie algebras, etc., boils down to computable things, though they are computable in exp-time.

My current project (joint with Roland van der Veen and continuing Lev Rozansky and Andrea Overbay) is poly-time, which puts it ahead of everything else I have done. IMHO it is also philosophically interesting and beautiful, but I'm biased.

On to content:

There is a standard construction that produces a knot invariant given a certain special element *R* ("the R-matrix") in the second tensor power of some algebra *U*. Roughly speaking, one independent copy of *R* is placed next to each crossing of a knot *K*, yielding an element in some high tensor power of *U*. Then the edges of *K* provide ordering instructions for how to multiply together these tensor factors using the algebra structure of *U* so as to get a *U*-valued knot invariant *Z*. Typically *U* is the universal enveloping algebra of some semisimple Lie algebra *L* (or some completed or quantized variant thereof). These algebras are infinite dimensional, and so *Z* is not immediately computable. The standard resolution is to also choose a finite-dimensional representation *V* of *L* and to carry out all computations within *V* and its tensor powers.

This works incredibly well. In fact, almost all the "knot polynomials" that arose following the work of Jones and Witten, the Jones and coloured Jones polynomials, the HOMFLY-PT polynomial, the Kauffman polynomial, and more, arise in this way, and much if not all of "quantum topology" is derived from these seeds. Yet these polynomials take an exponential time to compute: within the computation high tensor powers of *V* must be considered, the dimensions of such powers grow exponentially with the size of the input knot, and computations within exp-sized spaces take at least exp-time. The same criticism applies to almost everything else in quantum topology: whenever there is a "braided monoidal category" or a "topological quantum field theory" within some chain of reasoning, at some point high tensor powers of some vector spaces must be considered and the results become (at least) exp-time.

(An alternative mean to extract computable information from *Z* is to reduce modulo various "powers of *h*" filtrations on *U*. This yields the theory of finite type invariants. Individual finite type invariants are poly-time, but each single one is rather weak, and only when infinite sequences of finite type invariants are considered together, they become strong. Such sequences reproduce the aforementioned knot polynomials, but they are hard to compute).

Our approach is different. We explain how one can "fade out" roughly a half of a given semisimple Lie algebra *L* (namely its lower Borel subalgebra) by appropriately multiplying the structure constants that pertain to that half by some new coupling constant *b*. When $b=0$, the original *L* collapses to a solvable Lie algebra inside which the computation of *Z* is easy (as the name suggests, solvable algebras are easy to "solve"). Alas at $b=0$ the result is always the same - the classical (yet poly-time and very useful) Alexander polynomial. We find that in a formal neighborhood of $b=0$, namely in a ring in which $b^{k+1}=0$ for some natural number *k*, the invariant *Z* remains poly-time to compute.

By explicit experimentation with knots in the standard tables, the resulting poly-time invariants are very strong: with just $L=sl_2$ and $k=1$, the resulting invariant separates more knots than the exp-time HOMFLY-PT and Khovanov taken together. By both theory and experimentation, we know that our invariants give genus bounds for knots (hence they "see" some topology), and we have reasons to suspect that they may give a way to show that certain knots are not-ribbon, potentially assisting with the long-standing *slice=ribbon* conjecture.

None of the above is written yet, though I have given many talks on the subject, and most are online with videos and handouts and running code. See http://www.math.toronto.edu/drorbn/Talks/.

Within the time of the requested grant, I plan to complete my work on these poly-time invariants. Much remains to be done: writing from several perspectives, implementation for cases beyond sl_2 at $k=1$, a complete analysis of the relationship with genus and with the ribbon property, an analysis of the relationship with the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial, and more.

Proposed Research Topics

Enter up to five Research Topics in order of relevance. The first Research Topic must be selected within the Suggested Evaluation Group. (required)

1.	1508 Mathematics and Statistics
	MS07 Topology
2.	1508 Mathematics and Statistics
	MS01 Algebra
3.	1508 Mathematics and Statistics
	MS22 Computational Methods
4.	1508 Mathematics and Statistics
	MS10 Mathematical Physics
5.	Select

Keywords (up to 10):

1. Knot Theory.
2. Knot Polynomials.
3. Poly-Time Complexity.
4. Lie Algebras.
5. Lie Bialgebras.
6. Solvable Approximation.
7. Quantization.
8. 2-Parameter Quantum Groups.
9. Drinfel'd double.
10. Virtual Knots.

Five external reviewers: Rozansky, Jones, Etingof, Andersen, Reshetikhin.

Reviewer exclusion: None.

CCV: Your CV has been submitted. The confirmation number is : 679886