EVERYTHING AROUND sl_{2+}^{ϵ} IS DPG. HOORAY!

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Abstract. We construct sl_{2+}^{ϵ}, a certain "lossless approximation" of sl_2, and show that "everything that matters" around its universal enveloping algebra and its quantization, namely the products, the co-products, the R-matrix, and other essential ingredients, can be described in terms of a certain category **DPG** of "**Do**cile **Pe**rturbed **G**aussian differential operators".

Those essential ingredients are what one needs in order to construct powerful knot invariants with good algebraic properties. Also, as we show, **DPG** is "easy" in the sense of computational complexity. Hence we get (and implement and compute) powerful poly-time-computatble knot invariants with favourable algebraic properties. Hooray!

Similar constructions ought to exist for all semi-simple Lie algebras, but we do not pursue this here.

Contents

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1. Plan of the Paper

There is little we want to say by means of an introduction beyond what we said already in the abstract (please read it again). Instead, here's the plan:

In Section sec:DoPeGDU 2 MORE.

In Section sec:U 3 MORE.

Sections sec:DoPeGDU 2 and sec:U 3 completely commute and can be read in either order.

MORE.

1.1. **Acknowledgement.** We wish to thank M. Pugh for Footnote foot:Burger 15.

2. The Category DPG

sec:DoPeGDO

2.1. **Motivation, conventions, generating functions.** This section may seem like an awful way to start a topology paper — it's all about formula-based technicalities. Here are its redeeming features (beyond its usefulness for the later parts of the paper):

- Did you know that quadratic forms (aka "Gaussians") form a category in a natural way? (Theorem 2.3.4).
- Did you know that Feynman diagrams arise in pure algebra in a completely natural way?

mot:PBW

Motivation 2.1.1. The "PBW Principle" says that many algebras U are isomorphic, as vector spaces, to polynomial rings (hence as algebras they are "polynomial rings with funny multiplications"). Many times one needs to understand maps between algebras. Primarily, the algebra's own structure: the multiplication map $m\colon U\otimes U\to U$, perhaps a co-multiplication $\Delta\colon U\to U\otimes U$, and more. Sometimes one may care about specific special elements in U or some tensor power thereof; say, $R\in U\otimes U=\mathrm{Hom}(U^{\otimes\varnothing}\to U^{\otimes 2})$. So we need to understand the category of maps between algebras and their tensor powers, and hence, by PBW, the category of maps between polynomial rings. This category is way too big — one can encode an infinite amount of information into a map between polynomial rings (no matter the base fields) — and so no finite computer can fully store a general such map. Hence we develop a theory of "maps between polynomial rings that can be described using finite formulas (of a certain kind)" and we are lucky that the maps we care about later in this paper can indeed be described by formulas of that kind. Those maps/formulas are "**Do**cile **Pe**rturbed **G**aussian differential operators", and they make a category, **DPG**, which is the main object of study for this section.

Convention 2.1.2. Throughout this paper we will use lower case Latin letters such as z, y, b, a, x, and t to denote the generators of polynomial rings. Each such generator comes with a dual (whose purpose will be explained shortly), and the dual will always be denoted by the corresponding Greek letter: $z^*=\zeta$, $y^*=\eta$, $b^*=\beta$, $a^*=\alpha$, $x^*=\xi$, and $t^*=\tau$. If C is a finite set, we will denote by $z_C=\{z_c\}_{c\in C}$ the set of variables denoted by the letter z with an index $c\in C$; likewise there's y_C, x_C, etc. We will regard z_C sometimes as a set and sometimes as a column vector, as appropriate. We extend duality to indexed variables: $z_C^*=\zeta_C=\{z_c^*=\zeta_c\}_{c\in C}$. We will sometimes treat ζ_C (or η_C, etc) as a row vector.

Next, we establish a bijection

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\to\mathbb{Q}[z_B][\![\zeta_a]\!] \qquad (2.1.3)$$

eq:calG

between linear maps from polynomials in variables z_A to polynomials in variables z_B (A and B are finite sets) and a certain class of power series in the output variables and the duals of the input variables (more precisely, power series in the Greek variables corresponding to the inputs, with coefficients that are polynomials in the Latin variables corresponding to the outputs).

Definition 2.1.4. Let A and B be finite sets and let $L\colon\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]$ be linear. Let $\mathcal{L}=\mathcal{G}(L)$, the generating function of L, be defined as follows:

$$\mathcal{L}=\mathcal{G}(L):=\sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n)\in\mathbb{Q}[z_B][\![\zeta_a]\!]. \qquad (2.1.5)$$

eq:calG1

Here $\mathbb{N}$ denotes the non-negative integers, $n = (n_a)_{a\in A}$ is a multi-index, $\zeta_A^n := \prod_{a\in A} \zeta_a^{n_a}$ and likewise $z_A^n := \prod_{a\in A} z_a^{n_a}$, and $n! := \prod_{a\in A} n_a!$. Extending L without changing its name to an operator $L\colon \mathbb{Q}[z_A][\![\zeta_a]\!] \to \mathbb{Q}[z_B][\![\zeta_a]\!]$ by treating the ζ_A's as scalars, and recalling the definition of the exponential function, we find that (2.1.5) can also be written as

$$\mathcal{L} = \mathcal{G}(L) = L\left(\mathbb{e}^{\zeta_A\cdot z_A}\right),$$

where $\zeta_A \cdot z_A := \sum_{a\in A} \zeta_a z_a$.

Proposition 2.1.6. $\mathcal{G}\colon \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \to \mathbb{Q}[z_B][\![\zeta_a]\!]$ *is a bijection. If* $\mathcal{L} \in \mathbb{Q}[z_B][\![\zeta_a]\!]$ *and* $p \in \mathbb{Q}[z_A]$ *then*

$$\mathcal{G}^{-1}(\mathcal{L})(p) = p(\partial_{\zeta_a})\mathcal{L}(\zeta_a, z_b)\big|_{\zeta_a=0} = \mathcal{L}(\partial_{z_a}, z_b)p(z_a)\big|_{z_a=0}$$

□

Example 2.1.7. Consider $L_i\colon \mathbb{Q}[z] \to \mathbb{Q}[z]$ for $i = 1,2,3,4$, where $L_1(p) = p$ is the identity, $L_2(p) = p(z+1)$ is the shift, $L_3(p) = p'$ is differentiation, and $L_4(p) = \int_0^z p$ is definite integration. Then

$$\mathcal{G}(L_1) = \mathbb{e}^{\zeta z}, \qquad \mathcal{G}(L_2) = \mathbb{e}^{\zeta(z+1)}, \qquad \mathcal{G}(L_3) = \zeta\mathbb{e}^{\zeta z}, \qquad \text{and} \qquad \mathcal{G}(L_4) = (\mathbb{e}^{\zeta z} - 1)/\zeta.$$

A few further examples of generating functions, closer in spirit to the ones we care for the most in this paper, are in Section 2.2, right below.

Linear maps between polynomial rings can be composed, and it is useful to know how their corresponding generating functions compose[1]:

Proposition 2.1.8. *Let* A*,* B*, and* C *be finite sets, and let* $L \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ *and* $M \in \operatorname{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$*. Then, with* b *standing for all elements of* B*,*

$$\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0} = \left(\mathcal{G}(M)|_{\zeta_b\to\partial_{z_b}}\mathcal{G}(L)\right)_{z_b=0}. \tag{2.1.9}$$

□

Said differently, $\mathcal{G}$ is an isomorphism of categories from the category of polynomial rings in finitely many generators to the category $\mathfrak{G}$ whose objects are finite sets with morphisms $\operatorname{mor}_{\mathfrak{G}}(A \to B) = \mathbb{Q}[z_B][\![\zeta_A]\!]$ and compositions

$$\mathcal{L}/\!\!/\mathcal{M} = \left(\mathcal{L}|_{z_b\to\partial_{\zeta_b}}\mathcal{M}\right)_{\zeta_b=0} = \left(\mathcal{M}|_{\zeta_b\to\partial_{z_b}}\mathcal{L}\right)_{z_b=0}, \tag{2.1.10}$$

where $\mathcal{L} \in \operatorname{mor}_{\mathfrak{G}}(A \to B)$ and $\mathcal{M} \in \operatorname{mor}_{\mathfrak{G}}(B \to C)$.

Comment 2.1.11. We call the operation in (2.1.10) "contraction of the variable pairs (ζ_b, z_b) for $b \in B$".

Comment 2.1.12. There is an easily-provable third version for the composition formula (2.1.10), which treats $\mathcal{L}$ and $\mathcal{M}$ and Greek and Latin letters more symmetrically:

$$\mathcal{L}/\!\!/\mathcal{M} = \mathbb{e}^{\sum \partial_{z_b}\partial_{\zeta_b}}(\mathcal{L}\cdot\mathcal{M})\big|_{z_b=\zeta_b=0}, \tag{2.1.13}$$

where the indices b run through the set B. Here $\mathcal{L}\cdot\mathcal{M}$ stands for the ordinary product of power series $\mathbb{Q}[z_B][\![\zeta_A]\!] \otimes \mathbb{Q}[z_C][\![\zeta_B]\!] \to \mathbb{Q}[z_{A\cup B}][\![\zeta_{B\cup C}]\!]$.[2]

[1] Below and throughout we use "$/\!\!/$" for left-to-right composition: $L/\!\!/M = M \circ L$.

[2] Strictly speaking this is valid only if there are no name clashes, namely if $A \cap B = B \cap C = \emptyset$. That's a

Comment 2.1.14. If you are familiar with formal Gaussian integration, especially as it is used in physics and especially in perturbation theory where one allows themselve to pretend that integrals always converge (e.g. [Po]), then there is another easily verified form for the composition formula (see also [Ab]):

$$\mathcal{L}/\!\!/\mathcal{M} = \mathbb{e}^{\sum \partial_{z_b}\partial_{\zeta_b}}(\mathcal{L}\cdot\mathcal{M})\Big|_{z_b=\zeta_b=0} \propto \int \mathbb{e}^{-\sum_b z_b\zeta_b}(\mathcal{L}\cdot\mathcal{M})\prod_{b\in B} dz_b d\zeta_b. \tag{2.1.15}$$

Much of this paper can be re-written in terms of the above formula and Gaussian integration, yet we prefer to use this fact only for inspiration[3]. There is simply nothing to gain: everything one can do with integration we can also do directly with (2.1.13), a bit more simply. Yet there is a lesson to learn from (2.1.15): compositions may have simple formulas (and indeed they do) if $\mathcal{L}$ and $\mathcal{M}$ are themselves Gaussians or perturbed Gaussians, for then the integral in (2.1.15) would be Gaussian or perturbed Gaussian, and these are known to be computable. 2.1.14

Discussion 2.1.16. Later in this paper we will also want to consider power series in the mold of $\mathbb{e}^z \in \mathbb{Q}[\![z]\!]$ or $(1-z)^{-1}$. The generating function formalism does not extend to power series in the most naive way: the space $\mathrm{Hom}\,(\mathbb{Q}[\![z_A]\!] \to \mathbb{Q}[\![z_B]\!])$ is *not* isomorphic to some space of "generating functions" such as $\mathbb{Q}[\![\zeta_A, z_B]\!]$. Indeed, $\mathbb{Q}[\![z_A]\!]$ is of uncountable dimension over $\mathbb{Q}$, and $\mathrm{Hom}\,(\mathbb{Q}[\![z_A]\!] \to \mathbb{Q}[\![z_B]\!])$ is quite wild[4]. One standard way to get around this is to introduce a "small" parameter $\hbar$ and insist that it be present in power series, as in $\mathbb{e}^{\hbar z}$ and $(1-\hbar z)^{-1}$. But first, a discussion and a convention.

In analysis the identity $(1-\hbar z)^{-1} = \sum \hbar^n z^n$ holds true even if $|z|$ isn't small, provided $\hbar$ is small enough[5]. In algebra, if we want to enrich $\mathbb{Q}[z]$ so as to allow such identities[6] we need to do two things:

- Tensor multiply $\mathbb{Q}[z]$ with $\mathbb{Q}[\hbar]$ to get $\mathbb{Q}[z,\hbar]$, so as to allow coefficient depending on $\hbar$.
- Complete relative to the $\hbar$-adic topology so as to get $\mathbb{Q}[z][\![\hbar]\!]$, where series like $\sum \hbar^n z^n$ make sense.

MORE. ??? Add somewhere a comment that exponentials make sense in both $\mathbb{Q}[\![z]\!]$ and $\mathbb{Q}[z][\![\hbar]\!]$, yet $\mathrm{Hom}(\mathbb{Q}[\![z]\!] \to \mathbb{Q}[\![z]\!])$ is of uncountable dimension while $\mathrm{Hom}_{\mathbb{Q}[\![\hbar]\!]}(\mathbb{Q}[z][\![\hbar]\!] \to \mathbb{Q}[z][\![\hbar]\!])$ is countable.

MORE. This whole discussion is still murky. Does God really care about $\hbar$? 2.1.16

Convention 2.1.17 (and subtle point)**.** We slightly abuse notation and use $\mathbb{Q}_\hbar$ as a symbol for both steps:

$$\mathbb{Q}_\hbar[x,y,z] := \mathbb{Q}[x,y,z][\![\hbar]\!].$$

non-issue — if needed the labels in B can be temporarily renamed before the formula is applied.

[3] The constant of proportionality in Equation (2.1.15) has some 2π factors in it. We don't really want dreadful transcendental numbers in an algebra paper.

[4] One may be tempted to restrict attention in $\mathrm{Hom}\,(\mathbb{Q}[\![z_A]\!] \to \mathbb{Q}[\![z_B]\!])$ to *continuous* homomorphisms (relative to the power series topology; see e.g. [Kas, Chapter XVI]). That's wrong in our context — many of the homomorphisms we care about are simply not continuous relative to the power series topology. See an example in Footnote 8.

[5] How small? $|h|$ must be smaller than $|z|^{-1}$, so $\hbar$ must be determined *after* z.

[6] And yet without making z small, that is, without switching to $Q[\![z]\!]$, which our formalism can't handle.

Note that $\mathbb{Q}_\hbar$ is not a ring but a name for an operator: tensor with $\mathbb{Q}[\hbar]$ and complete relative to the $\hbar$-adic topology. In particular, $\mathbb{Q}_\hbar$ isn't $\mathbb{Q}[\![\hbar]\!]$ and $\mathbb{Q}_\hbar[z]$ isn't $\mathbb{Q}[\![\hbar]\!][z]$. Indeed, $\mathbb{e}^{\hbar z}$ and $(1-\hbar z)^{-1}$ are both members of $\mathbb{Q}_\hbar[z]$ but not of $\mathbb{Q}[\![\hbar]\!][z]$.

Yet we further abuse notation, and when $\mathbb{Q}_\hbar$ is on its own, we will regard it as the ring $\mathbb{Q}[\![\hbar]\!]$. So "$\omega \in \mathbb{Q}_\hbar$" means that ω is a power series in $\hbar$ with rational coefficients.

With all this said, in much of this paper one can read $\mathbb{Q}_\hbar$ to simply mean "$\mathbb{Q}$, also with a small parameter $\hbar$", with only a minor disloyalty to precision. conv:Qh 2.1.17

Everything said so far work over $\mathbb{Q}_\hbar$ as well as over $\mathbb{Q}$. The same bijection as in (eq:calG 2.1.3),

$$\mathcal{G}\colon \operatorname{Hom}_{\mathbb{Q}_\hbar}(\mathbb{Q}_\hbar[z_A] \to \mathbb{Q}_\hbar[z_B]) \to \mathbb{Q}_\hbar[z_B][\![\zeta_a]\!],$$

with the same definition (eq:calG1 2.1.5) and the same composition law (eq:LMcomposition 2.1.9).

MORE. A continuity clause is missing.

reekCompletion **Convention 2.1.18.** We also automatically complete spaces relative to the Greek letters α, β, π, τ, η, ξ, and ζ, and also when they come with subscripts. So if a and b are elements of some algebra U then $\mathbb{e}^{\alpha_1 a+\beta_2 b}$ always makes sense, and should be regarded as an element of $U[\![\alpha_1, \beta_2]\!]$.

ssec:realistic

2.2. **Some Real Life Examples.** Let us briefly meet a few generating functions of the type that we will care about the most in this paper. But first,

conv:id **Convention 2.2.1.** Throughout this paper we often put labels on tensor factors in a tensor product instead of ordering them; hence we often write $U^{\otimes A}$, where U is a vector space and A is a finite set, instead of $U^{\otimes n}$, where n is a natural number[7]. If U has a prescribed unit $1 \in U$ and if $z \in U$ and $i \in A$, we write z_i for "z placed in tensor factor i (with 1 in all other tensor factors)". Thus for example, we often write $z_1 + z_2$ for $z \otimes 1 + 1 \otimes z$. If $\psi\colon U^{\otimes A} \to U^{\otimes B}$ is a map, we often emphasize its domain and range by writing "ψ^A_B".

We start with some examples from the realm of commutative polynomials. Here $U = \mathbb{Q}[z]$ denotes the ring of commutative polynomials in a variable z.

exa:m **Example 2.2.2.** Let $m\colon U \otimes U \to U$ be the multiplication of polynomials. With the language of Convention 2.2.1, we choose labels i, j, k and write $m^{ij}_k\colon \mathbb{Q}[z_i, z_j] \simeq U_i \otimes U_j \to U_k \simeq \mathbb{Q}[z_k]$. But now m^{ij}_k is given by $z_i, z_j \mapsto z_k$, and so

$$\mathcal{G}(m^{ij}_k) = m^{ij}_k(\mathbb{e}^{\zeta_i z_i + \zeta_j z_j}) = \mathbb{e}^{(\zeta_i + \zeta_j) z_k}.$$

Note that $\mathcal{G}(m^{ij}_k)$ is a Gaussian — the exponential of a quadratic expression.

Example 2.2.3. Similarly, there is a coproduct $\Delta\colon U \to U \otimes U$, better written as Δ^i_{jk}, given by $z_i \mapsto z_j + z_k$. We have

$$\mathcal{G}(\Delta^i_{jk}) = \Delta(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i(z_j + z_k)}.$$

Again, this is a Gaussian expression.

exa:sigma **Example 2.2.4.** A bit silly but nevertheless useful is the relabelling map $\sigma^i_j\colon U_i \to U_j$, which is merely the identity map $U \to U$, albeit with a change-of-label for the unique tensor factor that appears. We have

$$\mathcal{G}(\sigma^i_j) = \sigma^i_j(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i z_j}. \qquad \text{(A Gaussian!)}$$

Example 2.2.5. There is an inner product $P\colon U \otimes U \to \mathbb{Q}$ given by $\langle z^n, z^m \rangle = \delta_{nm} n!$, and by a quick computation we have

$$\mathcal{G}(P^{ij}) = \mathbb{e}^{\zeta_i \zeta_j}. \qquad \text{(A Gaussian!)}$$

Note that there are no Latin letters in the above expression, because it is the generating function of a morphism whose target space is a polynomial ring on 0 variables.

Example 2.2.6. On a finite dimensional vector space V an inner product P would have an inverse $R \in V \otimes V$ such that $R_{ij} /\!\!/ P^{jk} = \sigma^k_i$ (with σ like in Example 2.2.4). We cannot have that here because U is infinite-dimensional. We come close with $R_{ij} = \mathbb{e}^{\hbar z_i z_j} \in \mathbb{Q}_\hbar U^{\otimes\{i,j\}}$, which satisfies $R_{ij} /\!\!/ P^{jk} = \hbar^{\deg} \sigma^k_i$, where $\hbar^{\deg}$ is the operator defined by $\hbar^{\deg}(z^n) = \hbar^n z^n$. We then have

$$\mathcal{G}(R_{ij}) = \mathbb{e}^{\hbar z_i z_j}. \qquad \text{(A Gaussian!)}$$

Note that there are no Greek letters in the above expression, because it is the generating function of a morphism whose domain space is a polynomial ring on 0 variables.

[7]These conventions only make sense in strict monoidal categories. They are consistent with the "identity" world view as opposed to the "geography" view; see [BN].

Our next few examples are minimally-non-commutative having to do first with the Heisenberg algebra $\mathbb{H}$ and then with the unique non-commutative 2D Lie algebra $\mathfrak{a}$. But first,

Convention 2.2.7. In support of the PBW principle (Motivation 2.1.1) we will often consider both commutative and non-commutative algebras generated by the same set of generators. In such cases we will use ordinary *italics* for the generators regarded within commutative algebras, yet **boldface** letters for the same generators regarded within non-commutative algebras. The following definition is an example.

Definition 2.2.8. Let $\mathbb{H}$ denote the Heisenberg algebra, the free associative algebra with generators $\mathbf{p}$ and $\mathbf{x}$ modulo the "canonical commutation relation" $[\mathbf{p},\mathbf{x}]=1$. The "$\mathbf{p}$ before $\mathbf{x}$" PBW ordering map (or "normal ordering", as physicists would call it) $\mathbb{O}\colon \mathbb{Q}[p,x]\to\mathbb{H}$ defined by $p^mx^n\mapsto \mathbf{p}^m\mathbf{x}^n$ is a vector space isomorphism of a (commutative) polynomial algebra with the (non-commutative) algebra $\mathbb{H}$.

exa:Heis **Example 2.2.9.** Let hm be the multiplication map of $\mathbb{H}$, turned into a map between polynomial rings by using $\mathbb{O}$ to identify $\mathbb{H}$ with $\mathbb{Q}[p,x]$; namely, let hm^{ij}_k be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i\otimes\mathbb{H}_j \xrightarrow{m^{ij}_k} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k,x_k],$$

where $m\colon \mathbb{H}\otimes\mathbb{H}\to\mathbb{H}$ is the (non-commutative) multiplication map of $\mathbb{H}$. Then

$$\mathcal{G}(hm^{ij}_k)=\mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k},\qquad \text{(a Gaussian!)}, \tag{2.2.10}$$ eq:Ghm

for indeed, using the Weyl form of the canonical commutation relation,

$$\mathbb{e}^{\xi\mathbf{x}}\mathbb{e}^{\pi\mathbf{p}}=\mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi\mathbf{p}}\mathbb{e}^{\xi\mathbf{x}}\qquad \text{(in } \mathbb{H}[\![\pi,\xi]\!]\text{; see Convention 2.1.18)}, \tag{2.2.11}$$ eq:Weyl

we have

$$\begin{aligned}\mathcal{G}(hm^{ij}_k)&=\mathbb{e}^{\pi_ip_i+\xi_ix_i+\pi_jp_j+\xi_jx_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m^{ij}_k/\!\!/\mathbb{O}_k^{-1}=\mathbb{e}^{\pi_i\mathbf{p}_i}\mathbb{e}^{\xi_i\mathbf{x}_i}\mathbb{e}^{\pi_j\mathbf{p}_j}\mathbb{e}^{\xi_j\mathbf{x}_j}/\!\!/m^{ij}_k/\!\!/\mathbb{O}_k^{-1}\\ &=\mathbb{e}^{\pi_i\mathbf{p}_k}\mathbb{e}^{\xi_i\mathbf{x}_k}\mathbb{e}^{\pi_j\mathbf{p}_k}\mathbb{e}^{\xi_j\mathbf{x}_k}/\!\!/\mathbb{O}_k^{-1}=\mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)\mathbf{p}_k}\mathbb{e}^{(\xi_i+\xi_j)\mathbf{x}_k}/\!\!/\mathbb{O}_k^{-1}=\mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

Note that as in Example 2.2.2, $\mathfrak{g}(m^{ij}_k)=\mathbb{e}^{(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$, so the only "contribution" of the non-commutativity of hm is the term $-\xi_i\pi_j$ in (2.2.10).

Our last example for this section is split between a definition, a proposition, two proofs, and a discussion.

def:am **Definition 2.2.12.** Let ϵ be a parameter, let $\mathfrak{a}_\epsilon$ be the 2D Lie algebra with generators $\mathbf{a}$ and $\mathbf{x}$ and relation $[\mathbf{a},\mathbf{x}]=\epsilon\mathbf{x}$, and let $\mathbb{A}_\epsilon=\mathcal{U}(\mathfrak{a}_\epsilon)$ be the universal enveloping algebra of $\mathfrak{a}_\epsilon$. Let $\mathbb{O}\colon\mathbb{Q}[a,x]\to\mathbb{A}_\epsilon$ be the "$\mathbf{a}$ before $\mathbf{x}$" ordering map given by $a^mx^n\mapsto\mathbf{a}^m\mathbf{x}^n$ (by PBW, it is a vector space isomorphism). Let $am^{ij}_{\epsilon;k}$ be the composition

$$\mathbb{Q}[a_i,x_i,a_j,x_j]\xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j}\mathbb{A}_{\epsilon;i}\otimes\mathbb{A}_{\epsilon;j}\xrightarrow{m^{ij}_{\epsilon;k}}\mathbb{A}_{\epsilon;k}\xrightarrow{\mathbb{O}_k^{-1}}\mathbb{Q}[a_k,x_k],$$

where $m^{ij}_{\epsilon;k}$ is the multiplication map of $\mathbb{A}_\epsilon$.

prop:am **Proposition 2.2.13.**

$$\mathcal{G}(am^{ij}_{\epsilon;k})=\exp\left((\alpha_i+\alpha_j)a_k+(\mathbb{e}^{-\epsilon\alpha_j}\xi_i+\xi_j)x_k\right)\qquad\left(\begin{array}{c}\text{nearly Gaussian, see}\\ \text{Discussion 2.2.15}\end{array}\right)$$

Proof 1. We first need a Weyl-style exponentiated relation (cf. (2.2.11)). Start with $\mathbf{xa}=(\mathbf{a}-\epsilon)\mathbf{x}$,[8] iterate to get $\mathbf{xa}^n=(\mathbf{a}-\epsilon)^n\mathbf{x}$, sum over n with coefficients $\frac{\alpha^n}{n!}$ to get $\mathbf{x}\mathbb{e}^{\alpha\mathbf{a}}=$

$\mathbb{e}^{\alpha(\mathbf{a}-\epsilon)}\mathbf{x} = \mathbb{e}^{\alpha\mathbf{a}}\mathbb{e}^{-\epsilon\alpha}\mathbf{x}$, iterate again to get $\mathbf{x}^n\mathbb{e}^{\alpha\mathbf{a}} = \mathbb{e}^{\alpha\mathbf{a}}(\mathbb{e}^{-\epsilon\alpha})^n\mathbf{x}^n$, and sum again with coefficients $\frac{\xi^n}{n!}$ to get the exponentiated relation $\mathbb{e}^{\xi\mathbf{x}}\mathbb{e}^{\alpha\mathbf{a}} = \mathbb{e}^{\alpha\mathbf{a}}\mathbb{e}^{\mathbb{e}^{-\epsilon\alpha}\xi\mathbf{x}}$.

Now proceed as in Example 2.2.9:

$$\mathcal{G}(am^{ij}_{\epsilon;k}) = \mathbb{e}^{\alpha_i a_i+\xi_i x_i+\alpha_j a_j+\xi_j x_j} /\!\!/ \mathbb{O}_i\otimes\mathbb{O}_j /\!\!/ m^{ij}_{\epsilon;k} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{\alpha_i\mathbf{a}_i}\mathbb{e}^{\xi_i\mathbf{x}_i}\mathbb{e}^{\alpha_j\mathbf{a}_j}\mathbb{e}^{\xi_j\mathbf{x}_j} /\!\!/ m^{ij}_{\epsilon;k} /\!\!/ \mathbb{O}_k^{-1}$$
$$= \mathbb{e}^{\alpha_i\mathbf{a}_k}\mathbb{e}^{\xi_i\mathbf{x}_k}\mathbb{e}^{\alpha_j\mathbf{a}_k}\mathbb{e}^{\xi_j\mathbf{x}_k} /\!\!/ \mathbb{O}_k^{-1} \stackrel{\text{(key)}}{=} \mathbb{e}^{(\alpha_i+\alpha_j)\mathbf{a}_k}\mathbb{e}^{(\mathbb{e}^{-\epsilon\alpha}\xi_i+\xi_j)\mathbf{x}_k} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{(\alpha_i+\alpha_j)a_k+(\mathbb{e}^{-\epsilon\alpha}\xi_i+\xi_j)x_k}. \tag{2.2.14}$$

eq:GamMain

□

Proof 2. We reprove the key equality of Equation (2.2.14), $\mathbb{e}^{\alpha_i\mathbf{a}}\mathbb{e}^{\xi_i\mathbf{x}}\mathbb{e}^{\alpha_j\mathbf{a}}\mathbb{e}^{\xi_j\mathbf{x}} = \mathbb{e}^{(\alpha_i+\alpha_j)\mathbf{a}}\mathbb{e}^{(\mathbb{e}^{-\epsilon\alpha}\xi_i+\xi_j)\mathbf{x}}$. Let ρ be the 2-dimensional representation of $\mathfrak{a}_\epsilon$ given by $\rho(\mathbf{a}) = \begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix}$ and $\rho(\mathbf{x}) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. The represenattion ρ is faithful on $\mathfrak{a}_\epsilon$ so it extends to a faithful representation of group-like elements in (the Greek-letter completion of) $\mathbb{A}_\epsilon$[9], so it is enough to prove that $\mathbb{e}^{\alpha_i\rho(\mathbf{a})}\mathbb{e}^{\xi_i\rho(\mathbf{x})}\mathbb{e}^{\alpha_j\rho(\mathbf{a})}\mathbb{e}^{\xi_j\rho(\mathbf{x})} = \mathbb{e}^{(\alpha_i+\alpha_j)\rho(\mathbf{a})}\mathbb{e}^{(\mathbb{e}^{-\epsilon\alpha}\xi_i+\xi_j)\rho(\mathbf{x})}$. This we do by brute force matrix exponentiation (see also [BDV, Gam.nb]):

```
ρa = (ϵ 0
      0 0); ρx = (0 1
                  0 0); exp = MatrixExp;
Simplify[ exp[αi ρa].exp[ξi ρx].exp[αj ρa].exp[ξj ρx] ==
  exp[(αi + αj) ρa].exp[(e^(-ϵ αj) ξi + ξj) ρx] ]
```

```
True
```

□

disc:gam **Discussion 2.2.15.** The exponent of $\mathcal{G}(am^{ij}_{\epsilon;k})$ in itself has an exponential term in it ($\mathbb{e}^{-\epsilon\alpha_j}$) hence $\mathcal{G}(am^{ij}_{\epsilon;k})$ is not a Gaussian in $\{\alpha_i,\alpha_j,\xi_i,\xi_j,a_k,x_k\}$, and hence some of the techniques that we introduce in later sections, to compose Gaussian generating functions, appear to break. We have two ways around this, we need both of them below, and in fact, one reason we included Proposition 2.2.13 is to forewarn that these two ways are needed:

(1) If ϵ is considered as "small" we can expand relative to ϵ and find

$$\mathcal{G}(am^{ij}_{\epsilon;k}) = \exp\left((\alpha_i+\alpha_j)a_k + (\xi_i+\xi_j)x_k + \sum_{m\geqslant 1}\frac{(-\epsilon)^m\alpha_j^m}{m!}\xi_i x_k.\right)$$

This is a prime example of a "perturbed Gaussian", and the lesson to take is that we will need to look beyond Gaussians and at perturbation theory.

(2) Even if $\epsilon = 1$, $\mathcal{G}(am^{ij}_k)$ is a Gaussian in the variables $\{\xi_i,\xi_j,x_k\}$ if the variables $\{\alpha_i,\alpha_j,a_k\}$ are held fixed, so contractions involving ξ's and x's create no problems. As for contractions of α'a and a's, $\mathcal{G}(am^{ij}_k)$ is *nearly* Gaussian in $\{\alpha_i,\alpha_j,a_k\}$ for fixed $\{\xi_i,\xi_j,x_k\}$: the offending term is the term $\mathbb{e}^{-\alpha_j}\xi_i x_k$. That term is a manageable perturbation. It is a bit hard to summarize what "manageable" means beyond saying

foot:ncont [8] In continuation of Footnote 4: We have just shown that $am(x\otimes a^n) = (a-\epsilon)^n x = (-\epsilon)^n x + \text{higher powers}$. But $x\otimes a^n \to 0$ while $(-\epsilon)^n x \not\to 0$, so am is not continuous.

[9] That's an algebraic version of the fact that faithful representations of a Lie algebra are also faithful on a neighborhood of the identity element of the corresponding Lie group.

ssec:PDE

"whatever is subject to the techniques of Section 2.5". Yet in short, the manageability here stems from the fact that the quadratic term $(\alpha_i + \alpha_j)a_k$ is "bipartite", involving only αa terms but no $\alpha\alpha$'s or aa's, while the perturbation term $\mathbb{e}^{-\alpha_j}\xi_i x_k$ involves only variables from one side of the partition: only the α's.

The lesson to take is that sometimes we will need to use Gaussian techniques twice, relative to to different sets of variables, while holding the variables from the other set fixed.

2.3. **Gaussian Differential Operators.** In the examples we care about (see Motivation 2.1.1 and Section 2.2) the generating functions turn out to be perturbed Gaussians, whose perturbations are in some sense "docile"[10]. Hence we seek to define a category **DPG** of docile perturbed Gaussian generating functions, with "differential operator" compositions as in Proposition **??**. We start with the unperturbed version, **GDO**:

Definition 2.3.1. **GDO** is the category with objects finite sets and, if A and B are finite, with $\text{mor}(A \to B)$ the set of "Gaussians in $\zeta_A \cup z_B$":

$$\text{mor}(A \to B) = \left\{\omega \mathbb{e}^{Q}\right\},$$

where $\omega \in \mathbb{Q}_\hbar$ is a scalar and where Q is a "small" quadratic expression in $\zeta_A \cup z_B$ with coefficients in $\mathbb{Q}_\hbar$. To define "small" and the composition law, we decompose quadratics in $\zeta_A \cup z_B$ into a Greek-Latin part E, and Greek-Greek part F, and a Latin-Latin part G:

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j.$$

With this, "small" means that G must be a multiple of $\hbar$.[11] Also, we define the composition of $\omega_1 \mathbb{e}^{Q_1} \in \text{mor}(A \to B)$ and $\omega_2 \mathbb{e}^{Q_2}$ to be $\omega \mathbb{e}^{Q}$, with

$$\begin{aligned} E &= E_1(I - F_2G_1)^{-1}E_2, & F &= F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T, \\ G &= G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2, & \omega &= \omega_1\omega_2 \det(I - F_2G_1)^{-1/2}, \end{aligned} \tag{2.3.2}$$

where (E, F, G) and (E_i, F_i, G_i) are the Greco-Roman decompositions of Q and of Q_i as above. Finally, the identity morphism in $\text{mor}(A \to A)$ is declared to be $\mathbb{e}^{\zeta_A \cdot z_A}$. 2.3.1

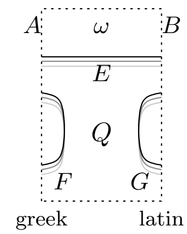
Comment 2.3.3. The formulas in Definition 2.3.1 may appear unfriendly. But appearances are deceiving. Note that the rank of the space of quadratics in a certain set of variables is in itself quadratic in the number of variables, and quadratics grow very slowly relative to exponentials. Hence the storage and time requirements to store and compute with elements of **GDO** are much milder than those for many other computations in quantum algebra, which tend to be exponential.

Theorem 2.3.4. *(i)* **GDO** *is indeed a category (the composition law is associative, the identity morphisms are identity morphisms).*
(ii) The explicit composition law of (2.3.2) *agrees with the "differential operator" one of* (2.1.9).

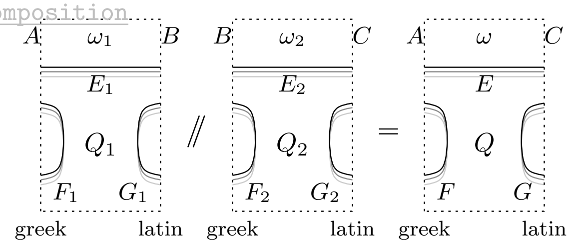
Proof. Part (i) can be verified by explicit matrix computations. It can also be implemented and tested, and seeing that we are committed to computability, we do that in Appendix 6.1. Finally, part (i) follows from part (ii) and the fact that the composition law of (2.1.9) is obviously associative. Hence we concentrate on proving (ii). We do it in two ways: pictorial, right below, for those who are familiar with diagrammatic algebra, and pure algebraic, on page **??**. □

[10] Or perhaps, we care about those examples precisely because their generating functions are docile perturbed Gaussians.

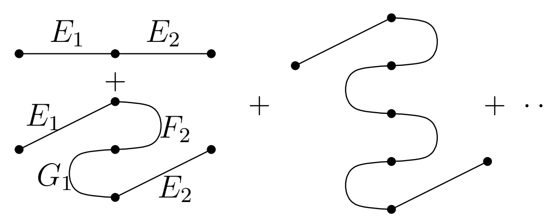
[11] The formulas below make sense either if the G terms are always small or if the F terms are always small. Mostly, in the applications G will be small and so we made the condition "G is small" be a part of the definition of **GDO**. Rarely we will encounter cases where F is small but G isn't. See Discussion **??**.



Pictorial proof of Theorem 2.3.4, (ii). This proof assumes familiarity with the kind of diagrammatics that occurs with Feynman diagrams in quantum field theory and/or with exponentials of connected diagrams as they occur in, say, [BGRT]. Pictorially, we view morphisms in $\text{mor}_{\mathbf{GDO}}(A \to B)$ as in the picture on the right: we put the Greek input variables corresponding to A on the left, the Latin output variable corresponding to B on the right, we indicate the scalar coefficient ω at the top, and we use the bulk of the picture to indicate Q and its Greco-Roman decomposition, with an obvious "Greek facing" placement of F, "Latin facing" placement of G, and "across the divide" placement of E. Note that Q is exponentiated and that exponentials are "reservoirs of multiple copies" $\mathbb{e}^x = 1 + x + xx/2 + xxx/6 + \ldots$. We emphasize this by drawing E, F, and G as having multiple shadows.



With this language, a composition as in (2.1.9) of a pair of morphisms as on the right is interpreted as "sum over all possible contractions of Latin-side ends in $\mathbb{e}^{Q_1}$ with Greek-side ends in $\mathbb{e}^{Q_2}$ (provided their labels, which are elements of B, agree)". Thus to figure out, say, the E part of the output, we need to figure out all the ways to travel from A to C across the composition of $\mathbb{e}^{Q_1}$ and $\mathbb{e}^{Q_2}$ by carrying out such contractions.

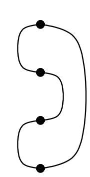


The most obvious way to travel across is the direct route: contract E_1 with E_2. This contributes a term proportional to E_1E_2 to the output E. Another possibility is to travel along E_1, then F_2, then G_1, then E_2, producing a term proportional to $E_1F_2G_1E_2$. Another possibility is to take the F_2G_1 detour twice, producing a term proportional to $E_1(F_2G_1)^2E_2$. In general, and with proper accounting of the combinatorial factors (it turns out that all proportionality factors are 1), we get

$$E = \sum_{r=0}^{\infty} E_1(F_2G_1)^rE_2 = E_1(I - F_2G_1)^{-1}E_2,$$

where the last equality was obtained by summing a geometric series, and where convergence is assured by the "smallness" condition on G in Definition 2.3.1.

Similar reasonings justify the formulas for F and for G.



Yet there is one further contribution to $\mathbb{e}^{Q_1} /\!\!/ \mathbb{e}^{Q_2}$, coming from closed F_2G_1 cycles as on the right (but of an arbitrary length r). This contribution is a scalar that modifies $\omega_1\omega_2$, and it is $\exp\left(\sum_{r=1}^{\infty} \frac{1}{2r} \operatorname{tr}(F_2G_1)^r\right) = \exp(-\frac{1}{2} \operatorname{tr} \log(1 - F_2G_1)) = \det(1 - F_2G_1)^{-1/2}$, justifying the last part of Equation (2.3.2). Note that in the last formula we used the familiar quantum field theory dictum to "divide each diagram by the order of its symmetry group" to get the $1/2r$ factor, and that throughout the proof we regarded only connected diagrams and exponentiated the result, as per the dictum "the logarithm of the partition function is generated by connected diagrams". pictorial

MORE. Add a section about piggyback Gaussians.

ssec:baby

2.4. **A Baby DPG and the Statement of the main DPG Theorem.** In this section we introduce a "baby" version of **DPG**, in which the most interesting features of the "mature" versions are present, yet some inconveniences regarding weights are censored.

def:DoPeGDO **Definition 2.4.1.** Let Ω be some ring of "scalars" and let ϵ be a formal parameter. Like **GDO**, let $\mathbf{DPG}_b$ be the category with objects finite sets and, if A and B are finite, with $\operatorname{mor}(A\to B)$ the set of "docile perturbed Gaussians in $\zeta_A \cup z_B$":

$$\operatorname{mor}(A\to B) = \left\{\omega \mathbb{e}^{Q+P}\right\},$$

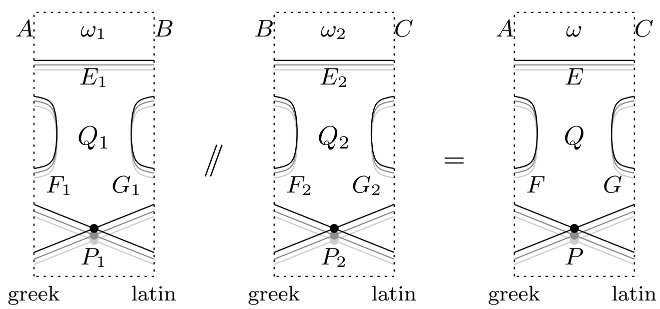
where ω and Q are ϵ-independent and otherwise as in Definition 2.3.1, and where P is a power series in ϵ of the form $P=\sum_{k\geqslant 1} P^{(k)}\epsilon^k$ and where each $P^{(k)}$ is a polynomial in $\zeta_A\cup z_B$ satisfying the "docility condition":

$$\deg P^{(k)} \leqslant 2k+2.$$

The composition law of $\mathbf{DPG}_b$ is "be compatible with (2.1.9)" (so this definition becomes complete only following the discussion of Feynman diagrams below, or in Section 2.5). 2.4.1

Comment 2.4.2. If we mod out by ϵ^{k_0+1} for some $k_0\geqslant 0$, or in other words, restrict our attention to $\mathbf{DPG}_b$ "up to ϵ^{k_0}", then the rank of the space of docile polynomials is polynomial in the number of variables (cf. Comment 2.3.3). Hence storing and manipulating docile polynomials has a chance of being computationally cheap; later we will see that this is indeed the case.

We now seek to understand compositions. With the same diagrammatic language as before, we seek to determine ω, $Q=(E,F,G)$ and P, so that the following would hold, where composition is "all possible contractions":

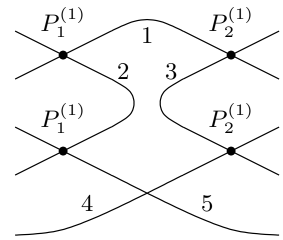


(2.4.3) eq:DoPeGDOComp

Looking only at the ϵ-independent part, it is clear that the composition law for ω and for Q is the same as for **GDO** (2.3.2) (so **DPG** is an "extension" of **GDO**). We just have to find $P=\sum_{k\geqslant 1}P^{(k)}\epsilon^k$ as a function of $Q_{1,2}$ and $P_{1,2}$.

Well, $P^{(k)}$ must get k factors of ϵ and it can only get them from P_1 and P_2. So $P^{(k)}$ is a sum of diagrams that have at most k vertices[12]. These vertices can be connected to each other (including self-connections), or to the outside, either directly, or by travelling along $E_{1,2}$ lines, or by travelling along F_2G_1 or G_1F_2 cycles as before. The latter cycles produce geometric series that sum to either $(I-F_2G_1)^{-1}$ or $(I-G_1F_2)^{-1}$. We arrive at the following theorem, which we state in a slightly informal manner as a more rigorous treatment follows in Section 2.5:

[12]Less than k if a single vertex brings along more than one factor of ϵ. Namely, if it comes from $P_{1,2}^{(l\geqslant 2)}$.



Theorem 2.4.4. *In a composition as in (2.4.3) the term $P^{(k)}$ in P is the sum of all connected Feynman diagrams as on the right, each divided by the order of its automorphism group, and in which the vertices are determined by P_1 and P_2 and in which there are five types of propagators (all sampled on the right):*

(1) *A P_1-to-P_2 propagator which equals $(I - F_2G_1)^{-1}$.*
(2) *A P_1-to-P_1 propagator which equals $(I - F_2G_1)^{-1}F_2$.*
(3) *A P_2-to-P_2 propagator which equals $G_1(I - G_1F_2)^{-1}$.*
(4) *A Greek-to-P_2 propagator which equals $E_1(I - F_2G_1)^{-1}$.*
(5) *A P_1-to-Latin propagator which equals $(I - F_2G_1)^{-1}E_2$.*

The figure here depicts a contribution to $P^{(4)}$. In general the valencies of vertices may be higher and self-contractions of two edges coming out of the same vertex are allowed. □

Proposition 2.4.5. DPG$_b$, *as defined in Definition 2.4.1 and with composition as in the above theorem, is indeed a category. Namely, with notation as in Equation (2.4.3) and with P as in the theorem, if P_1 and P_2 are docile then so is P.*

Proof. Consider a diagram contributing to P that has m vertices $v_1, \ldots, v_m$. Each v_i comes from either P_1 or P_2 and brings along some power k_i of ϵ, so the diagram overall contributes a term T in which the power of ϵ is $k = \sum_{i=1}^m k_i$. We need to show that the degree of T in the Greek and Latin variables satisfies $\deg T \leqslant 2k + 2$. Indeed, by the docility of P_1 and P_2 each v_i contributes at most $2k_i + 2$ to that degree. Also, the diagram is connected[13] so it has at least $m - 1$ edges, and each one contracts to variables, so each one reduces the overall degree by 2. So $\deg T \leqslant (\sum_{i=1}^m 2k_i + 2) - 2(m - 1) = 2k + 2$. □

MORE. Add a "formula" version and a demo.

The full **DPG** category needed in this paper is merely a "garnished" version of **DPG**$_b$, in which every variable has a "weight", and some weight restriction apply. We now turn to its formal definition, which we give in a slightly informal manner.

Context 2.4.6. Let $n > 0$ be a positive integer, and let us work in some universe of Latin and Greek variables in which every variable z (or ζ) has a weight $\mathrm{wt}(z)$ (or $\mathrm{wt}(\zeta)$) with $0 \leqslant \mathrm{wt}(z) \leqslant n$ (and $0 \leqslant \mathrm{wt}(\zeta) \leqslant n$), so that if z and ζ are dual then $\mathrm{wt}(z) + \mathrm{wt}(\zeta) = n$. Every monomial in our universe now has a weight, the sum of the weights of all the variables appearing in it, counted with multiplicity. The variables $\hbar$ and ϵ are special and do not carry a weight.

Example 2.4.7. In the main context of this paper, that of Section 4, we will have $n = 2$ and we will have variables y_i, b_i, a_i, and x_i (where i can run in some sets of labels), and their duals η_i, β_i, α_i, and ξ_i, with weights $\mathrm{wt}(y_i, b_i, a_i, x_i) = (1, 0, 2, 1)$ and $\mathrm{wt}(\eta_i, \beta_i, \alpha_i, \xi_i) = (1, 2, 0, 1)$. In this context, $\mathrm{wt}(\alpha_3^{62} a_1^8 y_{41}^3 \hbar^1 \epsilon^7) = 62 \cdot 0 + 8 \cdot 2 + 3 \cdot 1 + 1 \cdot 0 + 7 \cdot 0 = 19$.

Definition 2.4.8. A power series $D = \sum D^{(k)}\epsilon^k$ is called "docile" if for every k every monomial appearing in $D^{(k)}$ has weight less than $n(k + 1)$ (with a slight imprecision, this is $\mathrm{wt}(D^{(k)}) \leqslant n(k + 1)$). The same D is called "G_n-docile" if it is docile and in addition the following "Condition G_{n0}" holds:

[13] Da liegt der Hund begraben. Had we used $\omega \mathbb{e}^Q P$ instead of $\omega \mathbb{e}^{Q+P}$ for the morphisms of **DPG** we'd have had no connectedness here and the docility bound would have been $\deg P^{(k)} \leqslant 4k$, leading to slower computations.

Condition G_{n0}. For any weight-n variable z, $\partial_z D^{(0)}$ is affine-linear in the weight-0 variables.

Comment 2.4.9. Note that if D is docile then $\mathrm{wt}(D^{(0)}) \leqslant n$ so if also $wt(z) = n$, then $\mathrm{wt}(\partial_z D^{(0)}) = 0$, so $\partial_z D^{(0)}$ depends only on the weight 0 variables. Condition G_{n0} says that this dependence is particularly simple.

Example 2.4.10. The generating function of multiplication in the algebra $\mathbb{A}_\epsilon$,

$$\exp\left((\alpha_i + \alpha_j)a_k + (\mathbb{e}^{-\epsilon\alpha_j}\xi_i + \xi_j)x_k\right)$$

(see Proposition prop:am 2.2.13), is G_2-docile, with variable weights as in Example exa:MainWeights 2.4.7: $\mathrm{wt}(a, x, \alpha, \xi) = (2, 1, 0, 1)$.

Possible improvement: **DPG** are things which are ϵ-weight-docile, $\hbar$-Latin-docile, and have no Greek-only pairs. Can the last condition also be phrased as a docility condition?

MORE: State up front a full EDDO/**DPG** theorem.

The diagrammatic discussion of this section can be continued and extended to the full **DPG**$_n$ category of Section ssec:FullDPG 2.6 but we prefer the more solid grounds of pure algebra as in the next section, Section ssec:PDE 2.5.

2.5. **Algebra by means of Partial Differential Equations.** Much as we love intuitive graphical reasonings such as in the previous sections, we also like the more solid grounds of algebra. Hence we repeat the content of Sections 2.3 and 2.4 in a purely algebraic language (as it turns out, it is the language of partial differential equations, though they are only used with power series and hence we remain in pure algebra).

Recall from Comment 2.1.12 that in order to compute compositions of generating functions we need to evaluate contractions like $\mathbb{e}^{\sum \partial_{z_b}\partial_{\zeta_b}}(\mathcal{L}\cdot\mathcal{M})\big|_{z_b=\zeta_b=0}$. This inspires the following slightly more general definition:

Definition 2.5.1. Let B be a finite set, let F be a $B\times B$ matrix, and let $\mathcal{E}$ be a power series in variables that include the variable z_B. Set the "partial contraction" and the "full contraction" of $\mathcal{E}$ using F to be

$$[F\colon \mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F\colon \mathcal{E}\rangle_B := [F\colon \mathcal{E}]_B\big|_{z_B\to 0}.$$

Note 2.5.2. To ensure convergence one must assume some "smallness" condition on either F or $\mathcal{E}$. We defer this to a later point.

Note 2.5.3. In the above definition,

- $\mathcal{E}$ replaces the product $\mathcal{L}\cdot\mathcal{M}$ of (2.1.13),
- we restrict to a single "type" of variables z_B instead of the $z_B\cup\zeta_B$ of (2.1.13) (so B here is "twice" the B of (2.1.13)),
- instead of a pairing matrix of the form $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ as in (2.1.13), we allow a general matrix F.

This added generality will become beneficial soon.

Note 2.5.4. The computations of $[F\colon\cdot]_B$ and of $\langle F\colon\cdot\rangle_B$ are equivalent by "soft" means: $[F\colon\cdot]_B$ clearly determines $\langle F\colon\cdot\rangle_B$, and we also have $[F\colon\mathcal{E}]_B = \left\langle F\colon\ \mathcal{E}|_{z_b\to z_b+z'_b}\right\rangle\Big|_{z'_b\to z_b}$, where z'_B is a new set of variables indexed by B. The full contraction $\langle F\colon\cdot\rangle_B$ is used in (2.1.13), yet the partial contraction $[F\colon\cdot]_B$ is easier to manipulate as below.

Let λ be a formal variable and let $\mathcal{Z}_\lambda := [\lambda F\colon\mathcal{E}]_B$. Then $\mathcal{Z}_\lambda$ (and hence all we care about in this section) is determined by the following initial value problem, a heat equation:

$$\mathcal{Z}_0 = \mathcal{E} \qquad\text{and}\qquad \partial_\lambda\mathcal{Z}_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_iz_j}\mathcal{Z}_\lambda. \tag{2.5.5}$$

Yet we like to write generating functions as exponentials[14], and hence the following proposition:

Proposition 2.5.6. *With $E=\log\mathcal{E}$ and $Z_\lambda=\log\mathcal{Z}_\lambda$ Equation (2.5.5) becomes*

$$Z_0 = E \qquad\textit{and}\qquad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_iz_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right). \tag{2.5.7}$$

Proof. Simply substitute $\mathcal{Z}_\lambda = \mathbb{e}^{Z_\lambda}$ into (2.5.5) and carry out the differentiations. □

[14]The equations become non-linear, but as we will see later, their solutions lie in smaller spaces, allowing for more efficient manipulations.

Example 2.5.8. With $B = \{1\}$ a singleton, so we have just one variable $z = z_1$, with F the 1×1 matrix (1), with t a small scalar (namely, a commuting extra variable and with all work carried out over $\mathbb{Q}[\![t]\!]$), and with $\mathcal{E} = \mathbb{e}^{\frac{t}{2}z^2}$, let us compute

$$[(1)\colon \mathcal{E}] = \mathbb{e}^{\frac{1}{2}\partial_z^2}\mathbb{e}^{\frac{t}{2}z^2} \qquad \text{and} \qquad \langle (1)\colon \mathcal{E}\rangle = [(1)\colon \mathcal{E}]|_{z=0} = \mathbb{e}^{\frac{1}{2}\partial_z^2}\mathbb{e}^{\frac{t}{2}z^2}|_{z=0}.$$

With Proposition 2.5.6 in mind, we set $Z_\lambda := \log[(\lambda)\colon \mathcal{E}]$ and Equation (2.5.7) becomes

$$Z_0 = \frac{t}{2}z^2 \qquad \text{and} \qquad \partial_\lambda Z_\lambda = \frac{1}{2}\left(\partial_z^2 Z_\lambda + (\partial_z Z_\lambda)^2\right).$$

The differential operator $Z \mapsto \partial_z^2 Z + (\partial_z Z)^2$ maps the space of linear combinations of 1 and of z^2 into itself, and so our solution must be of the form $\frac{f}{2}\cdot 1 + \frac{g}{2}z^2$ where f and g are power series in t and λ. Substituting this back into the equation, we get the system

$$f|_{\lambda=0} = 0, \qquad \partial_\lambda f = g,$$
$$g|_{\lambda=0} = t, \qquad \partial_\lambda g = g^2,$$

whose solution is $g = \frac{t}{1-t\lambda}$ and $f = \log\frac{1}{1-t\lambda}$. Thus $Z_\lambda = \log\frac{1}{\sqrt{1-t\lambda}} + \frac{t}{1-t\lambda}\frac{z^2}{2}$, and so

$$[(1)\colon \mathcal{E}] = \mathbb{e}^{\frac{1}{2}\partial_z^2}\mathbb{e}^{\frac{t}{2}z^2} = \mathbb{e}^{Z_1} = \frac{1}{\sqrt{1-t}}\exp\left(\frac{t}{1-t}\frac{z^2}{2}\right)$$

and

$$\langle (1)\colon \mathcal{E}\rangle = \mathbb{e}^{\frac{1}{2}\partial_z^2}\mathbb{e}^{\frac{t}{2}z^2}|_{z=0} = \frac{1}{\sqrt{1-t}}.$$

2.5.8

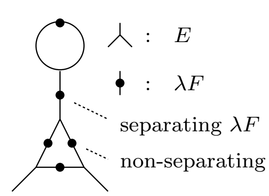
MORE: An exercise about the relationship with integration.

A sometimes-useful alternative to (2.5.7) is to allow F to be implicitly dependent on λ in an arbitrary (differentiable) manner with the condition $F|_{\lambda=0} = 0$ and to suppress the λ subscript in Z_λ. The resulting equation is

$$Z|_{\lambda=0} = E \qquad \text{and} \qquad \partial_\lambda Z = \frac{1}{2}\sum_{i,j\in B}(\partial_\lambda F_{ij})\left(\partial_{z_iz_j}Z + (\partial_{z_i}Z)(\partial_{z_j}Z)\right). \tag{2.5.7'}$$

We call Equation (2.5.7) (and its variant Equation (2.5.7')) "the synthesis equation", as it governs how the "vertices" in E merge and contract to synthesize larger and larger connected diagrams, as in the interpretation below.[15]

Interpretation 2.5.9. For the initiated, we cannot resist including a Feynman-diagram interpretation of Equation (2.5.7). With E "the vertices" and F "the contraction tensor" (roughly, "the propagator"), $\mathcal{Z}_\lambda = \langle \lambda F\colon \mathbb{e}^E\rangle$ is the sum of all Feynman diagrams that can be made with vertices in E and contractions as dictated by F, with each contraction multiplied by an additional factor of λ. Then $Z_\lambda = \log \mathcal{Z}_\lambda$ is the same, except restricting to connected Feynman diagrams. And then $\partial_\lambda Z_\lambda$ picks out one contraction in Z_λ. If it is "separating", it contributes an F-weighted product of two connected diagrams — the term $(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)$. If it not separating, it can be seen to contribute the $\partial_{z_iz_j}Z_\lambda$ term. See the picture on the right. 2.5.9



[15]M. Pugh told us that Equation (2.5.7) is a variant of "Burger's equation", and that it's relationship with the heat equation (2.5.5) is a variant of the "Cole-Hopf transformation".

Lemma 1. $\left\langle F\colon \mathcal{E}\, \mathbb{e}^{\sum_{i\in B} y_i z_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$ and

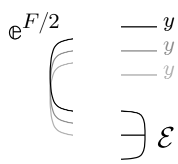
$$\left[F\colon \mathcal{E}\, \mathbb{e}^{\sum_{i\in B} y_i z_i}\right]_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j + \sum_{i\in B} y_i z_i} \left[F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right]_B$$
$$= \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j + \sum_{i\in B} y_i z_i} \left([F\colon \mathcal{E}]_B|_{z_B\to z_B+Fy_B}\right).$$

Lemma 2. With convergences left to the reader,

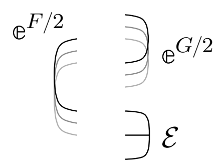
$$\left\langle F\colon \mathcal{E}\, \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon \mathcal{E}\right\rangle_B,$$

and

$$\left[F\colon \mathcal{E}\, \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right]_B = \det(1-GF)^{-1/2} \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} (G(I-FG)^{-1})_{ij} z_i z_j}$$
$$\cdot \left(\left[F(1-GF)^{-1}\colon \mathcal{E}\right]_B\right)_{z_B\to (I-FG)^{-1} z_B}.$$



Lemma 1



Lemma 2

MORE.

ssec:FullDPG

2.6. **Full DPG.** MORE.

3. sl^ϵ_{2+}, CU, AND QU

sec:U

For a minimalistic reading of this paper it is enough to know the definitions and some basic propeties of the Lie algebra sl^ϵ_{2+} and its associated associative algebras CU, and QU. Hence we start this section by declaring these algebras by fiat and listing some of their properties, postponing some of their proofs to Section 3.3. In Section 3.2 we explain the motivation behind sl^ϵ_{2+} and find that it extends to arbitrary semi-simple Lie algebras.

In anticipation of Section **??**, in which we show that everything that matters around sl^ϵ_{2+} is **DPG**, we emphasize the first occurrence of every object in this section that is later shown to be **DPG** with a lollipop symbol 🍭. Within the context of the current section the lollipops are purely motivational.

3.1. Definitions and Basic Properties.

Our ground ring throughout this section is $\mathbb{Q}[\epsilon]$, the ring of polynomials with rational coefficients over a formal parameter ϵ. Quantum algebra people should note that ϵ is distinct from $\hbar$.

Definition 3.1.1. Let sl^ϵ_{2+} be the Lie algebra $L\langle y,b,a,x\rangle$ with generators $\{y,b,a,x\}$ and with commutation relations

$$[a,x]=x,\quad [b,y]=-\epsilon y,\quad [a,b]=0,\quad [a,y]=-y,\quad [b,x]=\epsilon x,\quad [x,y]=b+\epsilon a. \tag{3.1.2}$$

eq:slepsrelati

Remark 3.1.3. It is easy to verify that $t:=b-\epsilon a$ is central in sl^ϵ_{2+}, and that if ϵ is invertible[16] then sl^ϵ_{2+} splits as a direct sum: $sl^\epsilon_{2+}\cong sl_2\oplus\langle t\rangle$, explaining its name. (Though we will mostly care about the vicinity of $\epsilon=0$, and at $\epsilon=0$[17] our algebra is not a direct sum).

Definition 3.1.4. Let $CU:=\mathcal{U}(sl^\epsilon_{2+})$ be the universal enveloping algebra of sl^ϵ_{2+}. Namely, CU is the associative algebra $A\langle y,b,a,x\rangle$ generated by the same $\{y,b,a,x\}$, subject to the same relations as in (3.1.2). We denote the multiplication map of CU with ${}^c m\colon CU\otimes CU\to CU$ (or, with the language of Convention 2.2.1 and Example 2.2.2, with ${}^c m^{ij}_k\colon CU_i\otimes CU_j\to CU_k$) 🍭. CU is a Hopf algebra in the standard way; namely, with its given associative algebra structure and with unit ${}^c\eta\colon\mathbb{Q}\to CU$ 🍭, counit ${}^c\varepsilon\colon CU\to\mathbb{Q}$[18] 🍭, antipode ${}^cS\colon CU\to CU$ 🍭, and coproduct ${}^c\Delta\colon CU\to CU\otimes CU$ 🍭 given as follows:

$$\begin{aligned}
{}^c\eta(\lambda)&=\lambda\cdot 1,\\
{}^c\varepsilon(1,y,b,a,x)&=(1,0,0,0,0),\\
{}^cS(y,b,a,x)&=(-y,-b,-a,-x),\\
{}^c\Delta(y,b,a,x)&=(y\otimes 1+1\otimes y,b\otimes 1+1\otimes b,a\otimes 1+1\otimes a,x\otimes 1+1\otimes x).
\end{aligned}\tag{3.1.5}$$

eq:CUDef

With the language of Convention 2.2.1, Equations (3.1.5) become:

$$\begin{aligned}
&{}^c\eta_i\colon\mathbb{Q}\to CU^{\otimes\{i\}}, & {}^c\eta_i(\lambda)&=\lambda\cdot 1_i,\\
&{}^c\varepsilon^i\colon CU^{\otimes\{i\}}\to\mathbb{Q}, & {}^c\varepsilon^i(1_i,y_i,b_i,a_i,x_i)&=(1,0,0,0,0),\\
&{}^cS_i:={}^cS^i_i\colon CU^{\otimes\{i\}}\to CU^{\otimes\{i\}}, & {}^cS_i(y_i,b_i,a_i,x_i)&=(-y_i,-b_i,-a_i,-x_i),\\
&{}^c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}, & {}^c\Delta^i_{jk}(y_i,b_i,a_i,x_i)&=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k).
\end{aligned}\tag{3.1.6}$$

eq:CUDefId

[16]E.g., if the ring of scalars is extended to $\mathbb{Q}(\epsilon)$ via $sl^\epsilon_{2+}\mapsto\mathbb{Q}(\epsilon)\otimes_{\mathbb{Q}[\epsilon]}sl^\epsilon_{2+}$.

[17]Evaluation at $\epsilon=\epsilon_0\in\mathbb{Q}$ makes sense via $sl^\epsilon_{2+}\mapsto(\mathbb{Q}[\epsilon]/(\epsilon-\epsilon_0))\otimes_{\mathbb{Q}[\epsilon]}sl^\epsilon_{2+}$, a Lie algebra over $\mathbb{Q}$.

[18]We use `\epsilon` (ϵ) for a perturbation parameter and `\varepsilon` (ε) for counits. There's rarely a

def:QU **Definition 3.1.7.** Let QU, a "quantization" of CU, be the associative algebra $A\langle y,b,a,x\rangle[\![\hbar]\!]$ over the ring $\mathbb{Q}[\![\hbar]\!]$ modulo to the relations

$$[a,x]=x,\quad [b,y]=-\epsilon y,\quad [a,b]=0,\quad [a,y]=-y,\quad [b,x]=\epsilon x,\quad xy-qyx=\frac{1-AB}{\hbar},$$

where $q:=\mathbb{e}^{\hbar\epsilon}$, $A:=\mathbb{e}^{-\hbar\epsilon a}$, and $B:=\mathbb{e}^{-\hbar b}$. We denote the multiplication map of QU with ${}^qm\colon QU\otimes QU\to QU$. We also set

$$\begin{aligned} {}^q\eta_i(\lambda) &= \lambda\cdot 1_i, \\ {}^q\varepsilon^i(1_i,y_i,b_i,a_i,x_i) &= (1,0,0,0,0), \\ {}^qS_i(y_i,b_i,a_i,x_i) &= (-B_i^{-1}y_i,-b_i,-a_i,-A_i^{-1}x_i), \\ {}^q\Delta^i_{jk}(y_i,b_i,a_i,x_i) &= (y_j+B_jy_k,b_j+b_k,a_j+a_k,x_j+A_jx_k). \end{aligned} \tag{3.1.8}$$

eq:QUDefId

The following claim can be verified easily by explicit computations:

Claim 3.1.9. *With the above operations and relative to the $\hbar$-adic topology, QU is a complete topological*[19] *Hopf algebra over the ring $\mathbb{Q}[\epsilon][\![\hbar]\!]$.* □

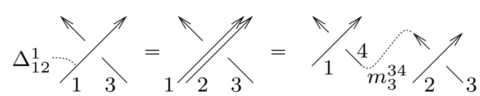
def:R **Definition 3.1.10.** Let R be the element of $QU\otimes QU$[20] given by the following formula:

$$R=\sum_{m,n\geqslant 0}\frac{y^nb^m\otimes(\hbar a)^m(\hbar x)^n}{m![n]_q!},\quad \text{alternatively}\quad R_{ij}=\sum_{m,n\geqslant 0}\frac{y_i^nb_i^m(\hbar a_j)^m(\hbar x_j)^n}{m![n]_q!}\in\mathbb{B}_i\otimes\mathbb{A}_j,$$

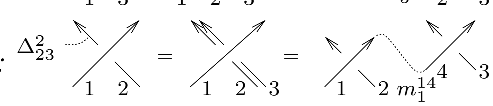
where $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and $[k]_q:=\frac{q^k-1}{q-1}=1+q+q^2+\ldots+q^{k-1}$ (recall that $q=\mathbb{e}^{\hbar\epsilon}$).

prop:R **Proposition 3.1.11** (proof in Section 3.3 ssec:UProofs)**.** *R is an R-matrix. Namely, it has the following properties: (This algebra section can be self contained, yet when we can, we can't resist including knot-theoretic interpretations, prefixed with "KT". Pure algebraists can ignore.)*

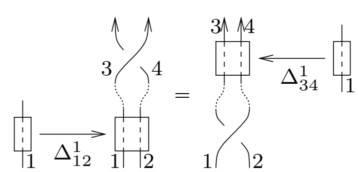
$$R_{13}/\!\!/{}^q\Delta^1_{12}=(R_{14}R_{23})/\!\!/{}^qm^{34}_3$$ *KT:*



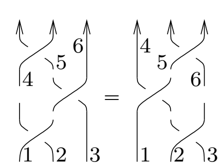
$$R_{12}/\!\!/{}^q\Delta^2_{23}=(R_{12}R_{43})/\!\!/{}^qm^{14}_1$$ *KT:*



$$({}^q\Delta^1_{12}R_{34})/\!\!/({}^qm^{13}_1\,{}^qm^{24}_2)=(R_{12}\ {}^q\Delta^1_{34})/\!\!/({}^qm^{14}_1\,{}^qm^{23}_2)$$ *KT:*



$$(R_{12}R_{63}R_{45}/\!\!/({}^qm^{16}_1\,{}^qm^{24}_2\,{}^qm^{35}_3)=(R_{23}R_{14}R_{56})/\!\!/({}^qm^{15}_1\,{}^qm^{26}_2\,{}^qm^{34}_3)$$ *KT:*



reason for confusion.

[19]Most people can safely ignore the "topological" language: it just means that everything can be a power series in $\hbar$, and only reasonable things are done to such series.

[20]Tensor products are completed relative to the $\hbar$-adic topology with no further mention.

We have finished listing the atomic pieces we need for the purpose of knot theory. Yet these pieces in themselves are assembled from even lower level pieces — perhaps "quarks" — and we need to introduce those as they are necessary for both the proof of Proposition 3.1.11 and for the proofs in Section 4 that all the lollipopped items above are indeed in **DPG**. Here we go:

Definition 3.1.12. Let $\mathfrak{a}$ be the 2-dimensional Lie algebra $L\langle a, x\rangle/[a,x]=x)$ and let $\mathbb{A} := \mathcal{U}(\mathfrak{a})[\![\hbar]\!]$ be the $\hbar$-adic completed universal enveloping algebra of the two dimensional Lie algebra with generators a and x and with the same bracket as in Definition 3.1.7. We turn $\mathbb{A}$ into a complete topological Hopf algebra with the obvious definitions for ${}^am$, ${}^a\varepsilon$, and ${}^a\eta$ (all), and with the definitions for ${}^aS$ and ${}^a\Delta$ (both) induced from (3.1.8). Namely,

$$\begin{aligned} {}^aS_i(a_i, x_i) &= (-a_i, -A_i^{-1}x_i), \\ {}^a\Delta^i_{jk}(a_i, x_i) &= (a_j + a_k, x_j + A_j x_k). \end{aligned} \tag{3.1.13}$$

Let $\mathbb{A}'$ be the subalgebra of $\mathbb{A}$ generated by $\hbar a$ and by $\hbar x$[21]. It is easy to check that $\mathbb{A}'$ is a sub-Hopf-algebra of $\mathbb{A}$.

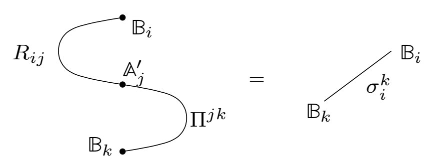
Definition 3.1.14. Similarly let $\mathbb{B} := \mathcal{U}(L\langle y, b\rangle/[b,y] = -\epsilon y)[\![\hbar]\!]$ be the $\hbar$-adic completed universal enveloping algebra of the two dimensional Lie algebra with generators y and b and with the same bracket as in Definition 3.1.7. We turn $\mathbb{B}$ into a complete topological Hopf algebra with the obvious definitions for ${}^bm$, ${}^b\varepsilon$, and ${}^b\eta$ (all), with ${}^bS$ taken to be the inverse of ${}^qS$ (but only on y and b) and with ${}^b\Delta$ taken to be the opposite of ${}^q\Delta$ (but only on y and b). Namely,

$$\begin{aligned} {}^bS_i(y_i, b_i) &= (-y_iB_i^{-1}, -b_i), \\ {}^b\Delta^i_{jk}(y_i, b_i) &= (B_ky_j + y_k, b_j + b_k). \end{aligned} \tag{3.1.15}$$

Clearly, $R \in \mathbb{B} \otimes \mathbb{A}'$. We claim that it has an inverse, a pairing $\Pi \in (\mathbb{A}')^* \otimes \mathbb{B}^*$:

Proposition 3.1.16. *There is a unique pairing* $\Pi \in (\mathbb{A}')^* \otimes \mathbb{B}^*$ *satisfying*

$$R_{ij} /\!\!/ \Pi^{jk} = \sigma_i^k, \qquad \textit{FD:}$$



where $\sigma_i^k \colon \mathbb{B}_k \to \mathbb{B}_i$ *is the identity map (more preciely, the factor renaming map) and where "FD" stands for "Flow Diagram(s)", a rather standard graphical language for representing compositions of tensors (e.g.* [ES, Lecture 12]*) which nevertheless seems not to have a standard name.*

defined on the generators by

$$\Pi\langle\hbar a, b\rangle = \Pi\langle\hbar x, y\rangle = 1, \qquad \Pi\langle\hbar a, y\rangle = \Pi\langle\hbar x, b\rangle = 0,$$

MORE.
MORE.

[21]Elements of $\mathbb{A}$ are infinite series $\sum w_n\hbar^n$ where $w_n \in \mathcal{U}(\mathfrak{a})$. Elements of $\mathbb{A}'$ are such series in which each

ssec:UMotivation

3.2. **Motivation for sl_{2+}^{ϵ}, CU, and QU.** MORE.

ssec:UProofs

3.3. **Proofs.** MORE.

w_n is a (non-commutative) polynomial in a and x of degree at most n. So using language similar to the language of Section 2, $\mathbb{A}'$ is the "docile" subspace of $\mathbb{A}$.

4. Everything Around sl_{2+}^{ϵ} is **DPG**

sec:Everything

MORE.

5. Tangles and Knots and Algebraic Knot Theory

MORE.

6. Computational Appendices

We believe in implementing as much as possible. Actually, we hardly believe ourselves unless we implement.

All code in these appendices is written in *Mathematica* [Wo].

6.1. Computational Verification of Theorem 2.3.4, (i).

We test that the composition law of **GDO** is indeed associative, by defining it general and verifying associativity on random (and hence likely generic) morphisms. First, we define the composition law of two morphisms. The program first determines E_i, F_i, and G_i from Q_i ($i = 1, 2$) by taking partial derivatives, and then outputs the scalar ω and quadratic Q, with equations (2.3.2) converted nearly literally into code (see also [BDV, GDOCompositions.nb]):

```
𝕄_{A_→B_}[ω1_, Q1_] // 𝕄_{B_→C_}[ω2_, Q2_] := Module[{ζA, zC, E1, F1, G1, E2, F2, G2, I},
  ζA = Table[ζ_i, {i, A}]; zC = Table[z_i, {i, C}]; I = IdentityMatrix@Length@B;
  E1 = Table[∂_{ζ_i,z_j} Q1, {i, A}, {j, B}]; E2 = Table[∂_{ζ_i,z_j} Q2, {i, B}, {j, C}];
  F1 = Table[∂_{ζ_i,ζ_j} Q1, {i, A}, {j, A}]; F2 = Table[∂_{ζ_i,ζ_j} Q2, {i, B}, {j, B}];
  G1 = Table[∂_{z_i,z_j} Q1, {i, B}, {j, B}]; G2 = Table[∂_{z_i,z_j} Q2, {i, C}, {j, C}];
  Expand /@ 𝕄_{A→C}[ω1 ω2 Det[I - F2.G1]^(-1/2), ζA.E1.Inverse[I - F2.G1].E2.zC
     + 1/2 ζA.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).ζA +
    1/2 zC.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).zC]]
```

Next we implement "random morphisms" (`RM`) by picking their quadratic parts to have small random integer coefficients. We also set M_1, M_2, and M_3 to be random morphisms in $\mathrm{mor}(\{1,2\} \to \{1,2,3\})$, $\mathrm{mor}(\{1,2,3\} \to \{1,2,3\})$, and $\mathrm{mor}(\{1,2,3\} \to \{1,2\})$, respectively:

```
R𝕄_{A_→B_} := Module[{vs = Table[ζ_i, {i, A}] ⋃ Table[z_i, {i, B}]},
    𝕄_{A→B}[1, Sum[RandomInteger[{-3, 3}] vi vj, {vi, vs}, {vj, vs}]]];
{M1 = R𝕄_{{1,2}→{1,2,3}}, M2 = R𝕄_{{1,2,3}→{1,2,3}}, M3 = R𝕄_{{1,2,3}→{1,2}}} // Column
```

$$\mathbb{M}_{\{1,2\}\to\{1,2,3\}}\big[1,\ -z_1^2 + 4 z_1 z_2 + z_2^2 - z_1 z_3 - 2 z_2 z_3 + 2 z_3^2 + 4 z_1 \zeta_1 + 3 z_2 \zeta_1 + 2 z_3 \zeta_1 + 6 z_1 \zeta_2 + z_2 \zeta_2 + 5 z_3 \zeta_2 - 2 \zeta_1 \zeta_2 - \zeta_2^2\big]$$

$$\mathbb{M}_{\{1,2,3\}\to\{1,2,3\}}\big[1,\ z_1 z_2 + 3 z_2^2 - z_1 z_3 + 5 z_2 z_3 - z_3^2 + 2 z_1 \zeta_1 - 2 z_2 \zeta_1 + \zeta_1^2 - 5 z_1 \zeta_2 + 3 z_2 \zeta_2 + 5 z_3 \zeta_2 - 3 \zeta_1 \zeta_2 + 2 \zeta_2^2 - 5 z_1 \zeta_3 - 2 z_2 \zeta_3 - 4 z_3 \zeta_3 - \zeta_1 \zeta_3 - 2 \zeta_2 \zeta_3 + \zeta_3^2\big]$$

$$\mathbb{M}_{\{1,2,3\}\to\{1,2\}}\big[1,\ -z_1^2 + 4 z_1 z_2 - 3 z_2^2 + 5 z_1 \zeta_1 - z_2 \zeta_1 + 2 \zeta_1^2 + 2 z_1 \zeta_2 - 4 z_2 \zeta_2 + 2 \zeta_1 \zeta_2 + \zeta_2^2 + 4 z_1 \zeta_3 - z_2 \zeta_3 + \zeta_1 \zeta_3 + 2 \zeta_3^2\big]$$

Just to get an appreciation of what compositions look like, we compute $(M_1 /\!/ M_2) /\!/ M_3$:

```
(M1 // M2) // M3
```

$$\mathbb{M}_{\{1,2\}\to\{1,2\}}\Big[-\frac{1}{2\sqrt{655102}},\ -\frac{6526189 z_1^2}{1310204} + \frac{4887535 z_1 z_2}{655102} - \frac{3883913 z_2^2}{1310204} - \frac{258319 z_1 \zeta_1}{327551} - \frac{2762891 z_2 \zeta_1}{327551} - \frac{8260873 \zeta_1^2}{2620408} - \frac{73313 z_1 \zeta_2}{93586} - \frac{867195 z_2 \zeta_2}{93586} - \frac{467207 \zeta_1 \zeta_2}{46793} - \frac{1189699 \zeta_2^2}{187172}\Big]$$

Finally, we verify that composition is associative:

```
((M1 // M2) // M3) == (M1 // (M2 // M3))
```

```
True
```

The last `True` above is an in-practice proof of Theorem thm:GDO 2.3.4, (i).

References

cCombinatorics [Ab] A. Abdesselam, *Feynman Diagrams in Algebraic Combinatorics,* Sém. Lotharingien de Comb. **49** (2003), arXiv:math/0212121. See pp. 7.

k:Toronto-1912 [BN] D. Bar-Natan, *Geography vs. Identity,* talk given at the Canadian Mathematical Society winter 2019 meeting in Toronto, handout and video at `http://drorbn.net/to19`. See pp. 9.

hurston:Aarhus [BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky, and D. P. Thurston, *The Århus integral of rational homology 3-spheres I–III,* Selecta Math., New Series **8** (2002) 315–339, arXiv:q-alg/9706004, **8** (2002) 341–371, arXiv:math.QA/9801049, **10** (2004) 305–324, arXiv:math.QA/9808013. See pp. 14.

Self [BDV] D. Bar-Natan and R. van der Veen, *Everything Around sl_{2+}^ϵ is* **DPG**. *Hooray!,* (self-reference), paper and related files at `http://drorbn.net/DPG`. The arXiv:????.????? edition may be older. See pp. 11, 31.

:QuantumGroups [ES] P. Etingof and O. Schiffman, *Lectures on Quantum Groups,* International Press, Boston, 1998. See pp. 25.

actionOfGroups [IW] E. Inonu and E. P. Wigner, *On the Contraction of Groups and their Representations,* Proc. Nat. Acad. Sci. **39** (1953) 510–524. See pp. .

:QuantumGroups [Kas] C. Kassel, *Quantum Groups,* Springer-Verlag GTM **155**, Heidelberg 1994. See pp. 7.

alVirtualKnots [Kau] L. H. Kauffman, *Rotational Virtual Knots and Quantum Link Invariants,* arXiv:1509.00578. See pp. .

ntumInvariants [Oh] T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything **29**, World Scientific 2002. See pp. .

Overbay:Thesis [Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov. See pp. .

Polyak:Feynman [Po] M. Polyak, *Feynman Diagrams for Mathematicians and Pedestrians,* Proc. Symp. Pure Math. **73** (2005) 15–42, arXiv:math/0406251. See pp. 7.

Rozansky:Flat1 [Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061. See pp. .

Rozansky:Burau [Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005. See pp. .

Rozansky:U1RCC [Ro3] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139. See pp. .

am:Mathematica [Wo] *Wolfram Language & System Documentation Center,* ωεβ/Wolf. See pp. 31.

7. Scratch Work — will be removed before posting

7.1. An ∞-dimensional DD Theorem, take 1. Suppose $\mathbb{A} = \mathcal{U}(\mathfrak{a})$ and $\mathbb{B} = \mathcal{U}(\mathfrak{b})$ are Hopf algebras with their native products and with aS, $a\Delta$, bS, $b\Delta$, etc. Suppose $\langle\cdot,\cdot\rangle\colon \mathfrak{b}\otimes\mathfrak{a}\to\mathbb{Q}$ is a pairing such that:

- Compatibility of $[]_b$ and $a\Delta$ etc.
- Non degeneracy.

Then

(1) $\langle\cdot,\cdot\rangle$ extends uniquely to a non-degenerate pairing $\mathbb{B}\otimes\mathbb{A}\to\mathbb{Q}$ such that m and Δ are compatible.
(2) $\mathbb{D} = \mathbb{B}\otimes\mathbb{A}$ is a Hopf algebra with the DD formulas and $\mathbb{A}\to\mathbb{D}$ and $\mathbb{B}\to\mathbb{D}$ are Hopf morphisms.
(3) If b_i and a_i are dual bases of $\mathbb{B}$ and $\mathbb{A}$ relative to our pairing, then $R = \sum b_{\otimes} a_i$ satisfies the quasi-triangularity axioms.

7.2. An ∞-dimensional DD Theorem, take 2. Step 1. Everybody knows that if H is a finite dimensional Hopf algebra then $D = H^*\otimes$ is a quasi-triangular Hopf algebra, with R, m, Δ, S given by the following formulas…

Step 2. If A and B are Hopf algebras over a ring Ω with a Hopf pairing $P\colon A\otimes B\to\Omega$ and $R\in B\otimes A$ contracts P to the identity, the same conclusion holds for $D = B\otimes A$.

Step 3. Over $\Omega = \mathbb{Q}[\![\hbar]\!]$ let $\mathbb{A} = \mathcal{U}(\mathfrak{a})[\![\hbar]\!]$, $\mathbb{A}' = \langle\hbar\mathfrak{a}\rangle\subset\mathbb{A}$, and $\mathbb{B} = \mathcal{U}(\mathfrak{b})$, with $P\colon \mathbb{A}'\otimes\mathbb{B}\to\Omega$ be given by $\langle\hbar a, b\rangle = \langle\hbar x, y\rangle = 1$, and let $R = \sum_{m,n} y^n b^m\otimes(\hbar a)^m(\hbar x)^n/m![n]_q!$. Then we're in the situation of Step 2, with $A = \mathbb{A}'$ and $B = \mathbb{B}$, and hence $\mathbb{D}' = \mathbb{B}\otimes\mathbb{A}'$ is a quasi-triangular Hopf algebra.

Step 4. All the formulas extend Ω-linearly to $\mathbb{D} = \mathbb{B}\otimes\mathbb{A}$ and hence all identities hold there too.

7.3. **A Naming Question.** The following was posted on Facebook on May 10, 2021:

> **A naming question follows.**
> Physicists (and some mathematicians) know how to integrate Gaussians multiplied by polynomials, and they do it often, especially when they think about "perturbation theory".
> There are two types of Gaussians: the "one type of variables" kind, which looks like $\mathbb{e}^{x^TAx}$, and the "dual variables" kind, which looks like $\mathbb{e}^{x^TBy}$. With the first type, we study $\int_{\mathbb{R}^n} dx\, \mathbb{e}^{x^TAx}p(x)$ where p is a polynomial. With the second type we study $\int_{\mathbb{R}^{2n}} dxdy\, \mathbb{e}^{x^TBy}f(x,y)$, where f is a polynomial. But for the second type, the answer is 0 unless $\deg_x f = \deg_y f$, so in fact we can extend to the case where f is a polynomial in (say) x yet is allowed to be a power series in y.
> **Question.** What is the second type of Gaussians called? "Polarized Gaussians"? "Bipartite Gaussians"? Is there a name for the fact that perturbations in the second case vanish if not balanced? A name or a precedent for the (trivial) fact that f can be a power series in one of its sets of variables? Mathematicians, please don't complain about convergence. Add conditions if you must, or think that I'm really imitating some QFT-like context in which convergence is not an issue.

7.4. **Iterated Gaussian Integration.** We wish to compute the formal $(2m+n)$-dimensional near-Gaussian integral $I_{\alpha\beta\xi} = \int \mathbb{e}^L da\, db\, dx$, where

$$L = \lambda^{ij}a_ib_j + \frac{1}{2}q^{kl}(b_j)x_kx_l + \alpha^i a_i + \beta^i b_i + \xi^k x_k,$$

and where $i,j \in \underline{m}$ and $k,l \in \underline{n}$.

Method 1. First compute the ab-integral.

$$\int \mathbb{e}^L da\, db = \mathbb{e}^{\xi^k x_k} \exp\left(\frac{1}{2}q^{kl}(\partial_{\beta^j})x_kx_l\right) \int \exp\left(\lambda^{ij}a_ib_j + \alpha^i a_i + \beta^i b_i\right) da\, db$$

$$= \mathbb{e}^{\xi^k x_k} \exp\left(\frac{1}{2}q^{kl}(\partial_{\beta^j})x_kx_l\right) \exp\left(-\lambda_{ij}\alpha^i\beta^j\right) \int \exp\left(\lambda^{ij}(a_i + \lambda_{ii'}\alpha^{i'})(b_j + \lambda_{j'j}\beta^{j'})\right) da\, db$$

$$= \det(\lambda)^{-1}\mathbb{e}^{\xi^k x_k} \exp\left(\frac{1}{2}q^{kl}(\partial_{\beta^j})x_kx_l\right) \exp\left(-\lambda_{ij}\alpha^i\beta^j\right)$$

$$= \det(\lambda)^{-1}\mathbb{e}^{\xi^k x_k} \exp\left(\frac{1}{2}q^{kl}(-\lambda_{ij}\alpha^i)x_kx_l\right) \exp\left(-\lambda_{ij}\alpha^i\beta^j\right)$$

$$= \det(\lambda)^{-1} \exp\left(-\lambda_{ij}\alpha^i\beta^j\right) \exp\left(\frac{1}{2}q^{kl}(-\lambda_{ij}\alpha^i)x_kx_l + \xi^k x_k\right)$$

Seeking a Precedent — Two-Stage Gaussian Integration? (Posted at https://mathoverflow.net/questions/395934).
Sometimes, by iteration, linear algebra can be used to solve non-linear equations. For example, consider the system

$$Ax = a \qquad B(x)y = b(x),$$

where a is a vector with scalar entries, A is a matrix with scalar entries, $b(x)$ is a vector whose entries are functions of x, and $B(x)$ is a matrix whose entries are functions of x. This system can be solved by first solving $Ax = b$, then substituting the solution into the second equation $By = b$, and then solving the second equation. The system can also be solved by first solving $By = b$ over the ring of functions of x, and then solving the first equation.
Similarly, formal* Gaussian integration techniques can sometimes be used iteratively to compute the *exact* integrals of non-Gaussian integrands. Here's a 3D example in the variables a, b, x; it is easy to raise this example to higher dimensions by replacing scalars with vectors and matrices.
Let $L = \lambda ab + \frac{1}{2}q(b)x^2 + \alpha a + \beta b + \xi x$, where all the letters represent scalars except for $q(b)$ which is a function of b. We wish to compute $I := \int \mathbb{e}^L da\, db\, dx$. This is not a Gaussian integral because the $q(b)x^2$ term is not quadratic in the integration variables.
Yet first computing the ab integral we get

$$I(x) := \int \mathbb{e}^L da\, db = \mathbb{e}^{\xi x}\mathbb{e}^{q(\partial_\beta)x^2/2} \int \mathbb{e}^{\lambda ab + \alpha a + \beta b} da\, db$$
$$= \frac{2\pi}{\lambda}\mathbb{e}^{\xi x}\mathbb{e}^{q(\partial_\beta)x^2/2}\mathbb{e}^{-\alpha\beta/\lambda} = \frac{2\pi}{\lambda}\mathbb{e}^{-\alpha\beta/\lambda + \xi x + q(-\alpha/\lambda)x^2/2}.$$

Thus $I(x)$ is a Gaussian with respect to x, so we can (formally) compute

$$I = \int I(x)dx = \frac{(2\pi)^{3/2}}{\lambda\sqrt{q(-\alpha/\lambda)}}\mathbb{e}^{-\alpha\beta/\lambda - q(-\alpha/\lambda)^{-1}\xi^2/2}.$$

We could have arrived at the same result by first computing the x integral as a formal Gaussian over the ring of functions of b and then computing the ab integral.
Question. Is there a precedent for this procedure? A name? Is there a place where people routinely iterate Gaussian integration to integrate non-Gaussians?

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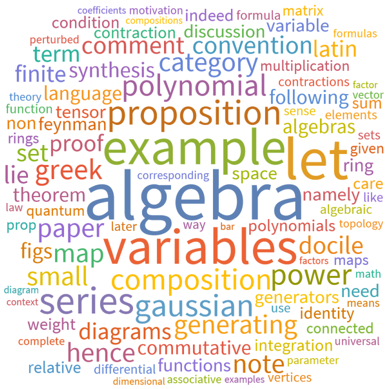
*Meaning, applying standard formulas without worrying about convergence. Add conditions if you must, or think that I'm really imitating some QFT-like context in which convergence is not an issue.



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