Pensieve header: A fresh implementation of baby DoPeGDO. Continues pensieve://2020-09/, pensieve://2020-03/Testing123.nb, and pensieve://People/VanDerVeen/TimidHeisenbergRGeneralForm@.nb.

$\mathbb{E}[\omega, Q, P_\epsilon\text{Series}]$ represents ωe^{Q+P}, where ω is a scalar, Q is an ϵ-free quadratic, and $P = \sum_{k=0}^{\$k} P[\![k]\!]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}$[_] // $\mathbb{E}$[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

Initialization, minor utilities, and "Define" Code

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\BabyDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

In[•]:=
```
$k=1;
```

In[•]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ ℰ;
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | x | η | ξ)_, ∞] ⋃ {y | x | η | ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
   ];
(*CF[ℰ_] := PP_CF@CCF[ℰ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[•]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[0] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: S1_ϵSeries + S2_ϵSeries :=
  ϵSeries @@ Table[S1[[k]] + S2[[k]], {k, Min[Length@S1, Length@S2]}];
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1[[j + 1]] * S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂vs___ S_ϵSeries := (s ↦ ∂vs s) /@ S;
```

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
In[•]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_nisp,$k_Integer, Block[{i, j, k}, op_isp,$k = ℰ; op_nis,$k]];
      SD[op_isp, op_{is},$k]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Basic Tensors

```
In[•]:= Define[m_{i,j→k} = 𝔼_{{i,j}→{k}}[1, -ξ_i η_j + (η_i + η_j) y_k + (ξ_i + ξ_j) x_k, ϵSeries[0]]]
```

```
In[•]:= AllMonomials[{}, 0] = {1};
AllMonomials[{}, d_Integer] /; d > 0 := {};
AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

```
In[•]:= Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[y_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}]
```

```
In[•]:= Basis[{i, j}, {2}]
```

```
Out[•]= {1, x_i y_i, x_j y_i, x_i y_j, x_j y_j, x_i^2 y_i^2, x_i x_j y_i^2, x_j^2 y_i^2, x_i^2 y_i y_j, x_i x_j y_i y_j, x_j^2 y_i y_j, x_i^2 y_j^2, x_i x_j y_j^2, x_j^2 y_j^2}
```

```
In[•]:= GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
GenericCombination[bas_, c_k_] := bas.Table[c_k,j, {j, Length@bas}];
```

```
In[•]:= GenericCombination[Basis[{i, j}, {2}], c_1]
```

```
Out[•]= c_1,1 + x_i y_i c_1,2 + x_j y_i c_1,3 + x_i y_j c_1,4 + x_j y_j c_1,5 + x_i^2 y_i^2 c_1,6 + x_i x_j y_i^2 c_1,7 + x_j^2 y_i^2 c_1,8 +
  x_i^2 y_i y_j c_1,9 + x_i x_j y_i y_j c_1,10 + x_j^2 y_i y_j c_1,11 + x_i^2 y_j^2 c_1,12 + x_i x_j y_j^2 c_1,13 + x_j^2 y_j^2 c_1,14
```

```
In[•]:= R_{i_,j_} := 𝔼_{}→{i,j}[1, (-1+T) x_j (y_i - y_j),
           ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i, j}, {k+1}], c_k], {k, $k}]];
         R̄_{i_,j_} := 𝔼_{}→{i,j}[1, (-1+1/T) x_j (y_i - y_j),
           ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i, j}, {k+1}], d_k], {k, $k}]];
         CC_{i_} := 𝔼_{}→{i}[√T, 0,
           ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i}, {k+1}], e_k], {k, $k}]];
         C̄C̄_{i_} := 𝔼_{}→{i}[1/√T, 0,
           ϵSeries @@ Prepend[0] @ Table[GenericCombination[Basis[{i}, {k+1}], f_k], {k, $k}]];
```

```
In[•]:= {R_{1,2}, R̄_{1,2}, CC_1, C̄C̄_1}
```

```
Out[•]=
{𝔼_{}→{1,2}[1, (-1+T) x_2 (y_1 - y_2),
  ϵSeries[0, c_{1,1} + x_1 y_1 c_{1,2} + x_2 y_1 c_{1,3} + x_1 y_2 c_{1,4} + x_2 y_2 c_{1,5} + x_1^2 y_1^2 c_{1,6} + x_1 x_2 y_1^2 c_{1,7} + x_2^2 y_1^2 c_{1,8} +
    x_1^2 y_1 y_2 c_{1,9} + x_1 x_2 y_1 y_2 c_{1,10} + x_2^2 y_1 y_2 c_{1,11} + x_1^2 y_2^2 c_{1,12} + x_1 x_2 y_2^2 c_{1,13} + x_2^2 y_2^2 c_{1,14}]],
 𝔼_{}→{1,2}[1, (-1+1/T) x_2 (y_1 - y_2), ϵSeries[0, d_{1,1} + x_1 y_1 d_{1,2} + x_2 y_1 d_{1,3} +
    x_1 y_2 d_{1,4} + x_2 y_2 d_{1,5} + x_1^2 y_1^2 d_{1,6} + x_1 x_2 y_1^2 d_{1,7} + x_2^2 y_1^2 d_{1,8} + x_1^2 y_1 y_2 d_{1,9} +
    x_1 x_2 y_1 y_2 d_{1,10} + x_2^2 y_1 y_2 d_{1,11} + x_1^2 y_2^2 d_{1,12} + x_1 x_2 y_2^2 d_{1,13} + x_2^2 y_2^2 d_{1,14}]],
 𝔼_{}→{1}[√T, 0, ϵSeries[0, e_{1,1} + x_1 y_1 e_{1,2} + x_1^2 y_1^2 e_{1,3}]],
 𝔼_{}→{1}[1/√T, 0, ϵSeries[0, f_{1,1} + x_1 y_1 f_{1,2} + x_1^2 y_1^2 f_{1,3}]]}
```

The Main Program

Variables and their duals:

```
In[•]:= {y*, x*, η*, ξ*} = {η, ξ, y, x};
        (vs_List)* := (v ↦ v*) /@ vs;
        (u_i_)* := (u*)_i;
```

𝔼 operations:

```
In[•]:= 𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
        𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
        𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
        𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1∪d2)→(r1∪r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
        𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

```
In[•]:= 𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{x$@i, y$@i}, {i, is}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(v : x | y)_i → v$@i, {i, is}],
      𝔼[ℰ2s] /. Table[(v : ξ | η)_i → v$@i, {i, is}]
    ]])
 ]
```

$$[F:\mathcal{E}]_B := e^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

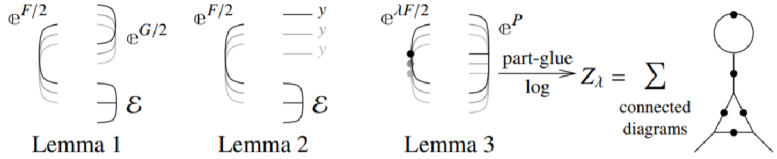
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\, e^{\sum_{i\in B} y_iz_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\, e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$




```
In[•]:= Zip_{vs_}[ℱ_, ℰ_] := ⟨ℱ, ℰ⟩ // Zip1_{vs} // Zip2_{vs} // Zip3_{vs}
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

```
In[•]:= Zip1_{{}} = Identity;
Zip1_{vs_}@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PP_{Zip1}@Module[{ℐ, F, G, u, v},
   ℐ = IdentityMatrix@Length@vs;
   F = Table[∂_{u,v}ℱ, {u, vs*}, {v, vs*}];
   G = Table[∂_{u,v}Q, {u, vs}, {v, vs}];
   CF /@ ⟨vs*.F.Inverse[ℐ - G.F].vs* / 2,
     𝔼[PowerExpand@Factor[ω Det[ℐ - G.F]^{-1/2}], Q - vs.G.vs / 2, P]⟩
  ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F:\mathcal{E}\, e^{\sum_{i\in B} y_iz_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

```
In[•]:= Zip2{} = Identity;
Zip2vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PPZip2@Module[{F, Y, u, v},
   F = Table[∂u,vℱ, {u, vs*}, {v, vs*}];
   Y = Table[∂vQ, {v, vs}];
   CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
  ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathfrak{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

```
In[•]:= Zip3vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := PPZip3@Module[{Z, u, v, m, j},
   Z[0] = P;
   For[m = 0, m < 2 $k, ++m,
    Z[m + 1] = CF[1/(2 (m + 1))
       Sum[∂u*,v*ℱ (∂u,vZ[m] + Sum[(∂uZ[j]) (∂vZ[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
   ];
   𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k}] /. Table[v → 0, {v, vs}]]]
  ]
```

Solving for *R*, CC, $k = 1

```
In[•]:= $k = 1;
{R1,2, CC1}
unknowns = Cases[{R1,2, R̄1,2, CC1, C̄C̄1}, (c | d | e | f)$k,_, ∞] // Union
```

```
Out[•]= {𝔼{}→{1,2}[1, (-1 + T) x2 (y1 - y2),
   ∈Series[0, c1,1 + x1 y1 c1,2 + x2 y1 c1,3 + x1 y2 c1,4 + x2 y2 c1,5 + x1^2 y1^2 c1,6 + x1 x2 y1^2 c1,7 + x2^2 y1^2 c1,8 +
     x1^2 y1 y2 c1,9 + x1 x2 y1 y2 c1,10 + x2^2 y1 y2 c1,11 + x1^2 y2^2 c1,12 + x1 x2 y2^2 c1,13 + x2^2 y2^2 c1,14]],
  𝔼{}→{1}[√T, 0, ∈Series[0, e1,1 + x1 y1 e1,2 + x1^2 y1^2 e1,3]]}

Out[•]= {c1,1, c1,2, c1,3, c1,4, c1,5, c1,6, c1,7, c1,8, c1,9, c1,10, c1,11, c1,12, c1,13, c1,14, d1,1, d1,2, d1,3,
 d1,4, d1,5, d1,6, d1,7, d1,8, d1,9, d1,10, d1,11, d1,12, d1,13, d1,14, e1,1, e1,2, e1,3, f1,1, f1,2, f1,3}
```

In[•]:=
```
Short[errors = {(R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]] -
    (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)[[3, -1]],
   (R1,2 R̄3,4 // m1,3→1 // m2,4→2)[[3, -1]],
   (CC1 C̄C̄2 // m1,2→1)[[3, -1]],
   (CC3 R1,2 // m2,3→2 // m2,1→1)[[3, -1]] - (C̄C̄3 R1,2 // m1,3→1 // m1,2→1)[[3, -1]]},
  10]
```

Out[•]//Short=

$$\Big\{-x_3\,y_1\,c_{1,3}-x_2\,y_1\,(c_{1,2}-T\,c_{1,2}+c_{1,3})+x_1\,y_2\,c_{1,4}+x_1\,y_3\,c_{1,4}-$$
$$T\,x_1\,y_3\,c_{1,4}+T\,x_2\,y_3\,c_{1,4}+\text{<<99>>}+x_3^2\,y_1\,y_3\,\left(T\,c_{1,11}+2\,T\,c_{1,14}-2\,T^2\,c_{1,14}\right)-$$
$$x_3^2\,y_2\,y_3\,\left(T^2\,c_{1,11}+2\,T\,c_{1,14}-2\,T^2\,c_{1,14}\right)-x_3^2\,y_2^2\,\left(T^2\,c_{1,8}+c_{1,14}-2\,T\,c_{1,14}+T^2\,c_{1,14}\right)+$$
$$x_3^2\,y_1^2\,\left(T^2\,c_{1,6}-2\,T^3\,c_{1,6}+T^4\,c_{1,6}+T\,c_{1,7}-2\,T^2\,c_{1,7}+T^3\,c_{1,7}+2\,c_{1,8}-\right.$$
$$\left.4\,T\,c_{1,8}+3\,T^2\,c_{1,8}+c_{1,11}-2\,T\,c_{1,11}+T^2\,c_{1,11}+c_{1,14}-2\,T\,c_{1,14}+T^2\,c_{1,14}\right)+$$
$$x_3^2\,y_2^2\,\left(T^2\,c_{1,8}+T^2\,c_{1,12}-2\,T^3\,c_{1,12}+T^4\,c_{1,12}+T\,c_{1,13}-2\,T^2\,c_{1,13}+T^3\,c_{1,13}+c_{1,14}-2\,T\,c_{1,14}+T^2\,c_{1,14}\right),$$
$$c_{1,1}+\text{<<13>>}+\frac{\text{<<1>>}}{\text{<<1>>}},\ \text{<<1>>},\ -c_{1,1}+\text{<<12>>}\Big\}$$

In[•]:=
```
eqns = Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, y1, y2, y3}][[;; , 2]] & /@ errors)]
```

Out[•]=

$$\Big\{0==c_{1,4}-T\,c_{1,4},\ 0==-c_{1,4}+T\,c_{1,4},\ 0==T\,c_{1,4}-T^2\,c_{1,4},\ 0==-c_{1,4}+2\,T\,c_{1,4}-T^2\,c_{1,4},$$
$$0==-T\,c_{1,4}+T^2\,c_{1,4},\ 0==T\,c_{1,2}-T^2\,c_{1,2}+c_{1,3}-T\,c_{1,3}+c_{1,5}-T\,c_{1,5},$$
$$0==-2\,c_{1,6}+2\,T\,c_{1,6},\ 0==2\,T\,c_{1,6}-2\,T^2\,c_{1,6},\ 0==c_{1,9}-T\,c_{1,9},$$
$$0==-c_{1,9}+T\,c_{1,9},\ 0==2\,T\,c_{1,9}-2\,T^2\,c_{1,9},\ 0==-2\,c_{1,9}+4\,T\,c_{1,9}-2\,T^2\,c_{1,9},$$
$$0==-2\,T\,c_{1,9}+2\,T^2\,c_{1,9},\ 0==2\,T\,c_{1,6}-2\,T^2\,c_{1,6}-c_{1,9}+4\,T\,c_{1,9}-4\,T^2\,c_{1,9}+T^3\,c_{1,9},$$
$$0==2\,T\,c_{1,8}-2\,T^2\,c_{1,8}+T^2\,c_{1,9}-2\,T^3\,c_{1,9}+T^4\,c_{1,9}+T\,c_{1,10}-2\,T^2\,c_{1,10}+T^3\,c_{1,10},$$
$$0==2\,T\,c_{1,7}-2\,T^2\,c_{1,7}-c_{1,10}+4\,T\,c_{1,10}-3\,T^2\,c_{1,10}+2\,c_{1,11}-2\,T\,c_{1,11},$$
$$0==T^2\,c_{1,9}-T^3\,c_{1,9}+2\,T\,c_{1,12}-2\,T^2\,c_{1,12},\ 0==c_{1,12}-T^2\,c_{1,12},\ 0==-c_{1,12}+2\,T\,c_{1,12}-T^2\,c_{1,12},$$
$$0==c_{1,9}-2\,T\,c_{1,9}+T^2\,c_{1,9}+c_{1,12}-2\,T\,c_{1,12}+T^2\,c_{1,12},\ 0==-2\,T\,c_{1,12}+2\,T^2\,c_{1,12},$$
$$0==-4\,T\,c_{1,12}+8\,T^2\,c_{1,12}-4\,T^3\,c_{1,12},\ 0==-2\,c_{1,12}+6\,T\,c_{1,12}-6\,T^2\,c_{1,12}+2\,T^3\,c_{1,12},$$
$$0==-2\,T^2\,c_{1,12}+2\,T^3\,c_{1,12},\ 0==-T^2\,c_{1,12}+2\,T^3\,c_{1,12}-T^4\,c_{1,12},$$
$$0==-c_{1,12}+4\,T\,c_{1,12}-6\,T^2\,c_{1,12}+4\,T^3\,c_{1,12}-T^4\,c_{1,12},\ 0==-2\,T\,c_{1,12}+6\,T^2\,c_{1,12}-6\,T^3\,c_{1,12}+2\,T^4\,c_{1,12},$$
$$0==2\,T\,c_{1,13}-2\,T^2\,c_{1,13},\ 0==T\,c_{1,13}-T^2\,c_{1,13},\ 0==2\,T\,c_{1,12}-2\,T^2\,c_{1,12}+T\,c_{1,13}-T^2\,c_{1,13},$$
$$0==2\,c_{1,8}-2\,T\,c_{1,8}+c_{1,10}-2\,T\,c_{1,10}+T^2\,c_{1,10}+c_{1,13}-2\,T\,c_{1,13}+T^2\,c_{1,13},$$
$$0==-2\,T\,c_{1,13}+2\,T^2\,c_{1,13},\ 0==-2\,T\,c_{1,13}+4\,T^2\,c_{1,13}-2\,T^3\,c_{1,13},$$
$$0==T^2\,c_{1,12}-2\,T^3\,c_{1,12}+T^4\,c_{1,12}+T\,c_{1,13}-2\,T^2\,c_{1,13}+T^3\,c_{1,13},\ 0==-T^2\,c_{1,13}+T^3\,c_{1,13},$$
$$0==-c_{1,13}+4\,T\,c_{1,13}-4\,T^2\,c_{1,13}+T^3\,c_{1,13}+2\,c_{1,14}-2\,T\,c_{1,14},\ 0==2\,T\,c_{1,14}-2\,T^2\,c_{1,14},$$
$$0==T^2\,c_{1,6}-2\,T^3\,c_{1,6}+T^4\,c_{1,6}+T\,c_{1,7}-2\,T^2\,c_{1,7}+T^3\,c_{1,7}+c_{1,8}-4\,T\,c_{1,8}+3\,T^2\,c_{1,8}+c_{1,11}-$$
$$2\,T\,c_{1,11}+T^2\,c_{1,11}+c_{1,14}-2\,T\,c_{1,14}+T^2\,c_{1,14},\ 0==-2\,T\,c_{1,14}+2\,T^2\,c_{1,14},\ 0==c_{1,1}+d_{1,1},$$
$$0==c_{1,2}+d_{1,2}+d_{1,4}-T\,d_{1,4},\ 0==c_{1,4}+T\,d_{1,4},\ 0==c_{1,2}-\frac{c_{1,2}}{T}+\frac{c_{1,3}}{T}+d_{1,3}+d_{1,5}-T\,d_{1,5},$$
$$0==c_{1,4}-\frac{c_{1,4}}{T}+\frac{c_{1,5}}{T}+T\,d_{1,5},\ 0==c_{1,9}+T\,d_{1,9}+2\,T\,d_{1,12}-2\,T^2\,d_{1,12},$$
$$0==c_{1,12}+T^2\,d_{1,12},\ 0==c_{1,6}+d_{1,6}+d_{1,9}-T\,d_{1,9}+d_{1,12}-2\,T\,d_{1,12}+T^2\,d_{1,12},$$
$$0==2\,c_{1,9}-\frac{2\,c_{1,9}}{T}+\frac{c_{1,10}}{T}+T\,d_{1,10}+2\,T\,d_{1,13}-2\,T^2\,d_{1,13},\ 0==2\,c_{1,12}-\frac{2\,c_{1,12}}{T}+\frac{c_{1,13}}{T}+T^2\,d_{1,13},$$

$$0 == 2\,c_{1,6} - \frac{2\,c_{1,6}}{T} + \frac{c_{1,7}}{T} + d_{1,7} + d_{1,10} - T\,d_{1,10} + d_{1,13} - 2\,T\,d_{1,13} + T^2\,d_{1,13},$$

$$0 == c_{1,9} + \frac{c_{1,9}}{T^2} - \frac{2\,c_{1,9}}{T} - \frac{c_{1,10}}{T^2} + \frac{c_{1,10}}{T} + \frac{c_{1,11}}{T^2} + T\,d_{1,11} + 2\,T\,d_{1,14} - 2\,T^2\,d_{1,14},$$

$$0 == c_{1,12} + \frac{c_{1,12}}{T^2} - \frac{2\,c_{1,12}}{T} - \frac{c_{1,13}}{T^2} + \frac{c_{1,13}}{T} + \frac{c_{1,14}}{T^2} + T^2\,d_{1,14},$$

$$0 == c_{1,6} + \frac{c_{1,6}}{T^2} - \frac{2\,c_{1,6}}{T} - \frac{c_{1,7}}{T^2} + \frac{c_{1,7}}{T} + \frac{c_{1,8}}{T^2} + d_{1,8} + d_{1,11} - T\,d_{1,11} + d_{1,14} - 2\,T\,d_{1,14} + T^2\,d_{1,14},$$

$$0 == -\frac{c_{1,3}}{T} + c_{1,4} + \frac{2\,c_{1,8}}{T^2} - 2\,c_{1,12} + e_{1,1} + e_{1,2} - \frac{e_{1,2}}{T} + 2\,e_{1,3} + \frac{2\,e_{1,3}}{T^2} - \frac{4\,e_{1,3}}{T} - f_{1,1},$$

$$0 == e_{1,1} + f_{1,1},\ 0 == e_{1,2} + f_{1,2},\ 0 == c_{1,2} - T\,c_{1,2} - c_{1,3} + \frac{c_{1,3}}{T} + c_{1,4} - T\,c_{1,4} - c_{1,5} + \frac{c_{1,5}}{T} - \frac{2\,c_{1,7}}{T} -$$
$$\frac{4\,c_{1,8}}{T^2} + 2\,T\,c_{1,9} + c_{1,10} - \frac{c_{1,10}}{T} - \frac{2\,c_{1,11}}{T^2} + 4\,T\,c_{1,12} + 2\,c_{1,13} + \frac{e_{1,2}}{T} - \frac{4\,e_{1,3}}{T^2} + \frac{4\,e_{1,3}}{T} - T\,f_{1,2},$$

$$0 == e_{1,3} + f_{1,3},\ 0 == c_{1,6} - T^2\,c_{1,6} + \frac{c_{1,7}}{T} - T\,c_{1,7} - c_{1,8} + \frac{c_{1,8}}{T^2} + c_{1,9} - T^2\,c_{1,9} + \frac{c_{1,10}}{T} -$$
$$T\,c_{1,10} - c_{1,11} + \frac{c_{1,11}}{T^2} + c_{1,12} - T^2\,c_{1,12} + \frac{c_{1,13}}{T} - T\,c_{1,13} - c_{1,14} + \frac{c_{1,14}}{T^2} + \frac{e_{1,3}}{T^2} - T^2\,f_{1,3}\Big\}$$

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\Big\{\Big\{c_{1,4} \to 0,\ c_{1,5} \to -T\,c_{1,2} - c_{1,3},\ c_{1,6} \to 0,\ c_{1,8} \to -\frac{1}{2}\,(1-T)\,c_{1,10},\ c_{1,9} \to 0,$$
$$c_{1,11} \to -T\,c_{1,7} - \frac{1}{2}\,(-1+3\,T)\,c_{1,10},\ c_{1,12} \to 0,\ c_{1,13} \to 0,\ c_{1,14} \to 0,\ d_{1,1} \to -c_{1,1},\ d_{1,2} \to -c_{1,2},$$
$$d_{1,3} \to -\frac{c_{1,3}}{T^2},\ d_{1,4} \to 0,\ d_{1,5} \to \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T^2},\ d_{1,6} \to 0,\ d_{1,7} \to -\frac{c_{1,7}}{T} - \frac{(-1+T)\,c_{1,10}}{T^2},$$
$$d_{1,8} \to -\frac{(1-T)\,c_{1,10}}{2\,T^3},\ d_{1,9} \to 0,\ d_{1,10} \to -\frac{c_{1,10}}{T^2},\ d_{1,11} \to \frac{c_{1,7}}{T^2} - \frac{(-1-T)\,c_{1,10}}{2\,T^3},\ d_{1,12} \to 0,$$
$$d_{1,13} \to 0,\ d_{1,14} \to 0,\ e_{1,1} \to \frac{c_{1,3}}{2\,T},\ e_{1,2} \to -\frac{c_{1,10}}{T},\ e_{1,3} \to 0,\ f_{1,1} \to -\frac{c_{1,3}}{2\,T},\ f_{1,2} \to \frac{c_{1,10}}{T},\ f_{1,3} \to 0\Big\}\Big\}$$

In[•]:=
```
sol /. (a_ → b_) :> (a = b)
```

Out[•]=

$$\Big\{0,\ -T\,c_{1,2} - c_{1,3},\ 0,\ -\frac{1}{2}\,(1-T)\,c_{1,10},\ 0,\ -T\,c_{1,7} - \frac{1}{2}\,(-1+3\,T)\,c_{1,10},\ 0,\ 0,$$
$$0,\ -c_{1,1},\ -c_{1,2},\ -\frac{c_{1,3}}{T^2},\ 0,\ \frac{c_{1,2}}{T} + \frac{c_{1,3}}{T^2},\ 0,\ -\frac{c_{1,7}}{T} - \frac{(-1+T)\,c_{1,10}}{T^2},\ -\frac{(1-T)\,c_{1,10}}{2\,T^3},$$
$$0,\ -\frac{c_{1,10}}{T^2},\ \frac{c_{1,7}}{T^2} - \frac{(-1-T)\,c_{1,10}}{2\,T^3},\ 0,\ 0,\ 0,\ \frac{c_{1,3}}{2\,T},\ -\frac{c_{1,10}}{T},\ 0,\ -\frac{c_{1,3}}{2\,T},\ \frac{c_{1,10}}{T},\ 0\Big\}$$

In[•]:=
$$\left\{R_{1,2},\ \overline{R}_{1,2},\ CC_1,\ \overline{CC}_1\right\}$$

Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ (-1+T)\ x_2\ (y_1-y_2),\right.\right.$$
$$\in\text{Series}\left[0,\ c_{1,1}+x_1\ y_1\ c_{1,2}+x_2\ y_2\ (-T\ c_{1,2}-c_{1,3})+x_2\ y_1\ c_{1,3}+x_1\ x_2\ y_1^2\ c_{1,7}-\frac{1}{2}\ (1-T)\ x_2^2\ y_1^2\ c_{1,10}+\right.$$
$$\left.\left.x_1\ x_2\ y_1\ y_2\ c_{1,10}+x_2^2\ y_1\ y_2\ \left(-T\ c_{1,7}-\frac{1}{2}\ (-1+3\ T)\ c_{1,10}\right)\right]\right],\ \mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ \left(-1+\frac{1}{T}\right)\ x_2\ (y_1-y_2),\right.$$
$$\in\text{Series}\left[0,\ -c_{1,1}-x_1\ y_1\ c_{1,2}-\frac{x_2\ y_1\ c_{1,3}}{T^2}+x_2\ y_2\ \left(\frac{c_{1,2}}{T}+\frac{c_{1,3}}{T^2}\right)-\frac{(1-T)\ x_2^2\ y_1^2\ c_{1,10}}{2\ T^3}-\right.$$
$$\left.\left.\frac{x_1\ x_2\ y_1\ y_2\ c_{1,10}}{T^2}+x_2^2\ y_1\ y_2\ \left(\frac{c_{1,7}}{T^2}-\frac{(-1-T)\ c_{1,10}}{2\ T^3}\right)+x_1\ x_2\ y_1^2\ \left(-\frac{c_{1,7}}{T}-\frac{(-1+T)\ c_{1,10}}{T^2}\right)\right]\right],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\sqrt{T},\ 0,\ \in\text{Series}\left[0,\ \frac{c_{1,3}}{2\ T}-\frac{x_1\ y_1\ c_{1,10}}{T}\right]\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\left[0,\ -\frac{c_{1,3}}{2\ T}+\frac{x_1\ y_1\ c_{1,10}}{T}\right]\right]\right\}$$

In[•]:=
$$c_{1,1}=c_{1,2}=c_{1,3}=c_{1,7}=0;\ \ c_{1,10}=1;$$
$$\left\{R_{1,2},\ \overline{R}_{1,2},\ CC_1,\ \overline{CC}_1\right\}$$

Out[•]=
$$\left\{\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ (-1+T)\ x_2\ (y_1-y_2),\ \in\text{Series}\left[0,\ \frac{1}{2}\ (-1+T)\ x_2^2\ y_1^2+x_1\ x_2\ y_1\ y_2+\frac{1}{2}\ (1-3\ T)\ x_2^2\ y_1\ y_2\right]\right],\right.$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ \left(-1+\frac{1}{T}\right)\ x_2\ (y_1-y_2),\right.$$
$$\left.\left.\in\text{Series}\left[0,\ -\frac{(-1+T)\ x_1\ x_2\ y_1^2}{T^2}-\frac{(1-T)\ x_2^2\ y_1^2}{2\ T^3}-\frac{x_1\ x_2\ y_1\ y_2}{T^2}-\frac{(-1-T)\ x_2^2\ y_1\ y_2}{2\ T^3}\right]\right],\right.$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\sqrt{T},\ 0,\ \in\text{Series}\left[0,\ -\frac{x_1\ y_1}{T}\right]\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\left[0,\ \frac{x_1\ y_1}{T}\right]\right]\right\}$$

In[•]:=
$$\left\{(R_{1,2}\ R_{4,3}\ R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3})\equiv(R_{2,3}\ R_{4,5}\ R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}),\right.$$
$$(R_{1,2}\ \overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2})\equiv\mathbb{E}_{\{\}\to\{1,2\}}[1,0,\in\text{Series}[0]],$$
$$(CC_1\ \overline{CC}_2\ //\ m_{1,2\to1})\equiv\mathbb{E}_{\{\}\to\{1\}}[1,0,\in\text{Series}[0]],$$
$$\left.(CC_3\ R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1})\equiv(\overline{CC}_3\ R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1})\right\}$$

Out[•]=
$$\{\text{True},\ \text{True},\ \text{True},\ \text{True}\}$$

Solving for *R*, CC, $k = 2

In[•]:=
```
$k = 2;
Short[#, 10] & [
 {(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}),
  (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ 𝔼_{{}→{1,2}}[1, 0, ϵSeries[0]],
  (CC_1 C̄C̄_2 // m_{1,2→1}) ≡ 𝔼_{{}→{1}}[1, 0, ϵSeries[0]],
  (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})}]
```

Out[•]//Short=

$$\Big\{(-1+T)\,x_1 x_2 x_3 y_1^2 y_3 - 3\,T\,x_1 x_2 x_3 y_1 y_2 y_3 + \left(-2\,T+4\,T^2\right) x_1 x_3^2 y_1 y_2 y_3 + \text{<<112>>} + x_3^3 y_2^3 \left(T^3 c_{2,18} + T^3 c_{2,27} - 3\,T^4 c_{2,27} + 3\,T^5 c_{2,27} - T^6 c_{2,27} + T^2 c_{2,28} - 3\,T^3 c_{2,28} + 3\,T^4 c_{2,28} - T^5 c_{2,28} + T\,c_{2,29} - 3\,T^2 c_{2,29} + 3\,T^3 c_{2,29} - T^4 c_{2,29} + c_{2,30} - 3\,T\,c_{2,30} + 3\,T^2 c_{2,30} - T^3 c_{2,30}\right) + x_3^3 y_1^2 y_3 \left(T^2 - 4\,T^3 + 3\,T^4 + T\,c_{2,22} - 2\,T^2 c_{2,22} + 2\,T^3 c_{2,22} + 2\,T\,c_{2,26} - 4\,T^2 c_{2,26} + 2\,T^3 c_{2,26} + 3\,T\,c_{2,30} - 6\,T^2 c_{2,30} + 3\,T^3 c_{2,30}\right) == 3\,c_{2,1} + 2\,x_1 y_1 c_{2,2} + \text{<<143>>} + x_3^3 y_2^2 y_3 \left(T^3 c_{2,22} + 3\,T\,c_{2,30} - 6\,T^2 c_{2,30} + 3\,T^3 c_{2,30}\right),\ \text{<<2>>},\ \frac{\text{<<1>>}}{2\,T^3} + \frac{\text{<<1>>}}{\text{<<1>>}} + \text{<<1>>} + \frac{\text{<<1>>}}{T^4} == \text{<<1>>}\Big\}$$

In[•]:=
```
unknowns = Cases[{R_{1,2}, R̄_{1,2}, CC_1, C̄C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=
```
{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{2,7}, c_{2,8}, c_{2,9}, c_{2,10}, c_{2,11}, c_{2,12}, c_{2,13}, c_{2,14},
 c_{2,15}, c_{2,16}, c_{2,17}, c_{2,18}, c_{2,19}, c_{2,20}, c_{2,21}, c_{2,22}, c_{2,23}, c_{2,24}, c_{2,25}, c_{2,26}, c_{2,27},
 c_{2,28}, c_{2,29}, c_{2,30}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{2,11},
 d_{2,12}, d_{2,13}, d_{2,14}, d_{2,15}, d_{2,16}, d_{2,17}, d_{2,18}, d_{2,19}, d_{2,20}, d_{2,21}, d_{2,22}, d_{2,23}, d_{2,24},
 d_{2,25}, d_{2,26}, d_{2,27}, d_{2,28}, d_{2,29}, d_{2,30}, e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, f_{2,1}, f_{2,2}, f_{2,3}, f_{2,4}}
```

In[•]:=
```
Short[errors = CF@{(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]] -
     (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]],
   (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2})[[3, -1]],
   (CC_1 C̄C̄_2 // m_{1,2→1})[[3, -1]],
   (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1})[[3, -1]] - (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})[[3, -1]]},
  10]
```

Out[•]//Short=
```
<<1>>
```

In[•]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x_1, x_2, x_3, y_1, y_2, y_3}][[;; , 2]] & /@ errors)]]
```

Out[•]//Short=

$$\Big\{0 == c_{2,4} - T\,c_{2,4},\ 0 == -c_{2,4} + T\,c_{2,4},\ 0 == T\,c_{2,4} - T^2 c_{2,4},\ \text{<<168>>},\ 0 == e_{2,4} + f_{2,4},\ 0 == \frac{1}{2} - \frac{1}{2\,T^3} + \frac{1}{2\,T^2} - \frac{T}{2} + c_{2,15} - T^3 c_{2,15} + \frac{c_{2,16}}{T} - T^2 c_{2,16} + \frac{c_{2,17}}{T^2} - T\,c_{2,17} - c_{2,18} + \frac{c_{2,18}}{T^3} + c_{2,19} - T^3 c_{2,19} + \frac{c_{2,20}}{T} - T^2 c_{2,20} + \frac{c_{2,21}}{T^2} - T\,c_{2,21} - c_{2,22} + \frac{c_{2,22}}{T^3} + c_{2,23} - T^3 c_{2,23} + \frac{c_{2,24}}{T} - T^2 c_{2,24} + \frac{c_{2,25}}{T^2} - T\,c_{2,25} - c_{2,26} + \frac{c_{2,26}}{T^3} + c_{2,27} - T^3 c_{2,27} + \frac{c_{2,28}}{T} - T^2 c_{2,28} + \frac{c_{2,29}}{T^2} - T\,c_{2,29} - c_{2,30} + \frac{c_{2,30}}{T^3} + \frac{e_{2,4}}{T^3} - T^3 f_{2,4}\Big\}$$

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\left\{\left\{c_{2,4}\to 0,\ c_{2,5}\to -\mathrm{T}\,c_{2,2}-c_{2,3},\ c_{2,6}\to 0,\ c_{2,8}\to -\frac{1}{2}\,(1-\mathrm{T})\,c_{2,10},\right.\right.$$
$$c_{2,9}\to 0,\ c_{2,11}\to -\frac{1}{2}-\mathrm{T}\,c_{2,7}-\frac{1}{2}\,(-1+3\,\mathrm{T})\,c_{2,10},\ c_{2,12}\to 0,\ c_{2,13}\to 0,\ c_{2,14}\to 0,$$
$$c_{2,15}\to 0,\ c_{2,17}\to -((-1+\mathrm{T})\,c_{2,16}),\ c_{2,18}\to -\frac{-1+4\,\mathrm{T}-3\,\mathrm{T}^2}{6\,\mathrm{T}},\ c_{2,19}\to 0,\ c_{2,20}\to -\frac{1}{2\,\mathrm{T}},$$
$$c_{2,21}\to -\frac{1-3\,\mathrm{T}}{2\,\mathrm{T}},\ c_{2,22}\to -\frac{1-11\,\mathrm{T}+16\,\mathrm{T}^2}{6\,\mathrm{T}}-\left(\mathrm{T}-\mathrm{T}^2\right)c_{2,16},\ c_{2,23}\to 0,\ c_{2,24}\to 0,$$
$$c_{2,25}\to -\frac{1}{2},\ c_{2,26}\to \frac{1}{6}\,(-1+7\,\mathrm{T})-\mathrm{T}^2\,c_{2,16},\ c_{2,27}\to 0,\ c_{2,28}\to 0,\ c_{2,29}\to 0,\ c_{2,30}\to 0,$$
$$d_{2,1}\to -c_{2,1},\ d_{2,2}\to -c_{2,2},\ d_{2,3}\to -\frac{c_{2,3}}{\mathrm{T}^2},\ d_{2,4}\to 0,\ d_{2,5}\to \frac{c_{2,2}}{\mathrm{T}}+\frac{c_{2,3}}{\mathrm{T}^2},\ d_{2,6}\to 0,$$
$$d_{2,7}\to -\frac{1-\mathrm{T}}{\mathrm{T}^3}-\frac{c_{2,7}}{\mathrm{T}}-\frac{(-1+\mathrm{T})\,c_{2,10}}{\mathrm{T}^2},\ d_{2,8}\to -\frac{-1+\mathrm{T}}{2\,\mathrm{T}^4}-\frac{(1-\mathrm{T})\,c_{2,10}}{2\,\mathrm{T}^3},\ d_{2,9}\to 0,$$
$$d_{2,10}\to \frac{1}{\mathrm{T}^3}-\frac{c_{2,10}}{\mathrm{T}^2},\ d_{2,11}\to -\frac{1}{2\,\mathrm{T}^4}+\frac{c_{2,7}}{\mathrm{T}^2}-\frac{(-1-\mathrm{T})\,c_{2,10}}{2\,\mathrm{T}^3},\ d_{2,12}\to 0,\ d_{2,13}\to 0,\ d_{2,14}\to 0,$$
$$d_{2,15}\to 0,\ d_{2,16}\to -\frac{-1+\mathrm{T}}{2\,\mathrm{T}^3}-\frac{c_{2,16}}{\mathrm{T}},\ d_{2,17}\to -\frac{3-4\,\mathrm{T}+\mathrm{T}^2}{2\,\mathrm{T}^4}-\frac{(-1+\mathrm{T})\,c_{2,16}}{\mathrm{T}^2},\ d_{2,18}\to -\frac{-3+4\,\mathrm{T}-\mathrm{T}^2}{6\,\mathrm{T}^5},$$
$$d_{2,19}\to 0,\ d_{2,20}\to -\frac{1}{2\,\mathrm{T}^3},\ d_{2,21}\to \frac{2}{\mathrm{T}^4},\ d_{2,22}\to -\frac{4+\mathrm{T}+\mathrm{T}^2}{6\,\mathrm{T}^5}-\frac{(1-\mathrm{T})\,c_{2,16}}{\mathrm{T}^3},\ d_{2,23}\to 0,\ d_{2,24}\to 0,$$
$$d_{2,25}\to -\frac{1}{2\,\mathrm{T}^4},\ d_{2,26}\to -\frac{-1+\mathrm{T}}{6\,\mathrm{T}^5}+\frac{c_{2,16}}{\mathrm{T}^3},\ d_{2,27}\to 0,\ d_{2,28}\to 0,\ d_{2,29}\to 0,\ d_{2,30}\to 0,\ e_{2,1}\to \frac{c_{2,3}}{2\,\mathrm{T}},$$
$$\left.e_{2,2}\to -\frac{c_{2,10}}{\mathrm{T}},\ e_{2,3}\to 0,\ e_{2,4}\to 0,\ f_{2,1}\to -\frac{c_{2,3}}{2\,\mathrm{T}},\ f_{2,2}\to -\frac{1}{\mathrm{T}^2}+\frac{c_{2,10}}{\mathrm{T}},\ f_{2,3}\to 0,\ f_{2,4}\to 0\right\}\right\}$$

In[•]:=
```
sol /. (a_ → b_) :↦ (a = b)
```

Out[•]=

$$\left\{0,\ -\mathrm{T}\,c_{2,2}-c_{2,3},\ 0,\ -\frac{1}{2}\,(1-\mathrm{T})\,c_{2,10},\ 0,\ -\frac{1}{2}-\mathrm{T}\,c_{2,7}-\frac{1}{2}\,(-1+3\,\mathrm{T})\,c_{2,10},\ 0,\ 0,\ 0,\ 0,\right.$$
$$-((-1+\mathrm{T})\,c_{2,16}),\ -\frac{-1+4\,\mathrm{T}-3\,\mathrm{T}^2}{6\,\mathrm{T}},\ 0,\ -\frac{1}{2\,\mathrm{T}},\ -\frac{1-3\,\mathrm{T}}{2\,\mathrm{T}},\ -\frac{1-11\,\mathrm{T}+16\,\mathrm{T}^2}{6\,\mathrm{T}}-\left(\mathrm{T}-\mathrm{T}^2\right)c_{2,16},$$
$$0,\ 0,\ -\frac{1}{2},\ \frac{1}{6}\,(-1+7\,\mathrm{T})-\mathrm{T}^2\,c_{2,16},\ 0,\ 0,\ 0,\ 0,\ -c_{2,1},\ -c_{2,2},\ -\frac{c_{2,3}}{\mathrm{T}^2},\ 0,\ \frac{c_{2,2}}{\mathrm{T}}+\frac{c_{2,3}}{\mathrm{T}^2},$$
$$0,\ -\frac{1-\mathrm{T}}{\mathrm{T}^3}-\frac{c_{2,7}}{\mathrm{T}}-\frac{(-1+\mathrm{T})\,c_{2,10}}{\mathrm{T}^2},\ -\frac{-1+\mathrm{T}}{2\,\mathrm{T}^4}-\frac{(1-\mathrm{T})\,c_{2,10}}{2\,\mathrm{T}^3},\ 0,\ \frac{1}{\mathrm{T}^3}-\frac{c_{2,10}}{\mathrm{T}^2},$$
$$-\frac{1}{2\,\mathrm{T}^4}+\frac{c_{2,7}}{\mathrm{T}^2}-\frac{(-1-\mathrm{T})\,c_{2,10}}{2\,\mathrm{T}^3},\ 0,\ 0,\ 0,\ 0,\ -\frac{-1+\mathrm{T}}{2\,\mathrm{T}^3}-\frac{c_{2,16}}{\mathrm{T}},\ -\frac{3-4\,\mathrm{T}+\mathrm{T}^2}{2\,\mathrm{T}^4}-\frac{(-1+\mathrm{T})\,c_{2,16}}{\mathrm{T}^2},$$
$$-\frac{-3+4\,\mathrm{T}-\mathrm{T}^2}{6\,\mathrm{T}^5},\ 0,\ -\frac{1}{2\,\mathrm{T}^3},\ \frac{2}{\mathrm{T}^4},\ -\frac{4+\mathrm{T}+\mathrm{T}^2}{6\,\mathrm{T}^5}-\frac{(1-\mathrm{T})\,c_{2,16}}{\mathrm{T}^3},\ 0,\ 0,\ -\frac{1}{2\,\mathrm{T}^4},$$
$$\left.-\frac{-1+\mathrm{T}}{6\,\mathrm{T}^5}+\frac{c_{2,16}}{\mathrm{T}^3},\ 0,\ 0,\ 0,\ 0,\ \frac{c_{2,3}}{2\,\mathrm{T}},\ -\frac{c_{2,10}}{\mathrm{T}},\ 0,\ 0,\ -\frac{c_{2,3}}{2\,\mathrm{T}},\ -\frac{1}{\mathrm{T}^2}+\frac{c_{2,10}}{\mathrm{T}},\ 0,\ 0\right\}$$

In[•]:=
```
c[2,1] = c[2,2] = c[2,3] = c[2,7] = c[2,10] = c[2,16] = 0;
{R[1,2], Rb[1,2], CC[1], CCb[1]}
```

Out[•]=

$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\,x_2\,(y_1-y_2),\ \in\text{Series}\Big[0,\ \frac{1}{2}\,(-1+T)\,x_2^2\,y_1^2+x_1\,x_2\,y_1\,y_2+\frac{1}{2}\,(1-3\,T)\,x_2^2\,y_1\,y_2,\ -\frac{\left(-1+4\,T-3\,T^2\right)x_2^3\,y_1^3}{6\,T}-\frac{1}{2}\,x_2^2\,y_1\,y_2-\frac{x_1^2\,x_2\,y_1^2\,y_2}{2\,T}-\frac{(1-3\,T)\,x_1\,x_2^2\,y_1^2\,y_2}{2\,T}-\frac{\left(1-11\,T+16\,T^2\right)x_2^3\,y_1^2\,y_2}{6\,T}-\frac{1}{2}\,x_1\,x_2^2\,y_1\,y_2^2+\frac{1}{6}\,(-1+7\,T)\,x_2^3\,y_1\,y_2^2\Big]\Big],\ \mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \Big(-1+\frac{1}{T}\Big)\,x_2\,(y_1-y_2),\ \in\text{Series}\Big[0,\ -\frac{(-1+T)\,x_1\,x_2\,y_1^2}{T^2}-\frac{(1-T)\,x_2^2\,y_1^2}{2\,T^3}-\frac{x_1\,x_2\,y_1\,y_2}{T^2}-\frac{(-1-T)\,x_2^2\,y_1\,y_2}{2\,T^3},\ -\frac{(1-T)\,x_1\,x_2\,y_1^2}{T^3}-\frac{(-1+T)\,x_2^2\,y_1^2}{2\,T^4}-\frac{(-1+T)\,x_1^2\,x_2\,y_1^3}{2\,T^3}-\frac{\left(3-4\,T+T^2\right)x_1\,x_2^2\,y_1^3}{2\,T^4}-\frac{\left(-3+4\,T-T^2\right)x_2^3\,y_1^3}{6\,T^5}+\frac{x_1\,x_2\,y_1\,y_2}{T^3}-\frac{x_2^2\,y_1\,y_2}{2\,T^4}-\frac{x_1^2\,x_2\,y_1^2\,y_2}{2\,T^3}+\frac{2\,x_1\,x_2^2\,y_1^2\,y_2}{T^4}-\frac{\left(4+T+T^2\right)x_2^3\,y_1^2\,y_2}{6\,T^5}-\frac{x_1\,x_2^2\,y_1\,y_2^2}{2\,T^4}-\frac{(-1+T)\,x_2^3\,y_1\,y_2^2}{6\,T^5}\Big]\Big],$$
$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \in\text{Series}\Big[0,\ -\frac{x_1\,y_1}{T},\ 0\Big]\Big],\ \mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\Big[0,\ \frac{x_1\,y_1}{T},\ -\frac{x_1\,y_1}{T^2}\Big]\Big]\Big\}$$

In[•]:=
```
{(R[1,2] R[4,3] R[5,6] // m[1,4→1] // m[2,5→2] // m[3,6→3]) ≡ (R[2,3] R[4,5] R[1,6] // m[1,4→1] // m[2,5→2] // m[3,6→3]),
 (R[1,2] Rb[3,4] // m[1,3→1] // m[2,4→2]) ≡ 𝔼[{}→{1,2}][1, 0, ∈Series[0]],
 (CC[1] CCb[2] // m[1,2→1]) ≡ 𝔼[{}→{1}][1, 0, ∈Series[0]],
 (CC[3] R[1,2] // m[2,3→2] // m[1,1→1]) ≡ (CCb[3] R[1,2] // m[1,3→1] // m[1,2→1])}
```

Out[•]=

{True, True, True, True}

Solving for *R*, CC, $k = 3

In[•]:=
```
$k = 3;
Short[#, 10] & [
 {(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}),
  (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ 𝔼_{{}→{1,2}}[1, 0, ϵSeries[0]],
  (CC_1 C̄C̄_2 // m_{1,2→1}) ≡ 𝔼_{{}→{1}}[1, 0, ϵSeries[0]],
  (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})}]
```

Out[•]//Short=

$$\left\{\frac{(1-T)\,x_1^2\,x_2\,x_3\,y_1^3\,y_3}{2\,T}+\frac{\left(1-4\,T+3\,T^2\right)\,x_1\,x_2^2\,x_3\,y_1^3\,y_3}{2\,T}+\text{<<267>>}+x_3^4\ \text{<<1>>}\ \left(T^4\,c_{\text{<<1>>}}+\text{<<35>>}\right)+\right.$$
$$\frac{1}{24}\,x_3^4\,y_1^2\,y_3^2\left(7\,T^2-97\,T^3+329\,T^4-239\,T^5+24\,T^2\,c_{3,45}-48\,T^3\,c_{3,45}+48\,T^4\,c_{3,45}+\right.$$
$$72\,T^2\,c_{3,50}-144\,T^3\,c_{3,50}+72\,T^4\,c_{3,50}+144\,T^2\,c_{3,55}-288\,T^3\,c_{3,55}+144\,T^4\,c_{3,55}\Big)+$$
$$\frac{1}{12\,T}\,x_3^4\,y_1^4\left(5-19\,T+13\,T^2+38\,T^3-89\,T^4+77\,T^5-25\,T^6+12\,T^5\,c_{3,31}-48\,T^6\,c_{3,31}+\right.$$
$$\text{<<46>>}+12\,T\,c_{3,50}-48\,T^2\,c_{3,50}+72\,T^3\,c_{3,50}-48\,T^4\,c_{3,50}+12\,T^5\,c_{3,50}+$$
$$\left.12\,T\,c_{3,55}-48\,T^2\,c_{3,55}+72\,T^3\,c_{3,55}-48\,T^4\,c_{3,55}+12\,T^5\,c_{3,55}\right)==$$
$$\left.3\,c_{3,1}+2\,x_1\,y_1\,c_{3,2}+\text{<<367>>}+\text{<<1>>}+x_3^4\ \text{<<3>>},\ \text{<<3>>}\right\}$$

In[•]:=
```
unknowns = Cases[{R_{1,2}, R̄_{1,2}, CC_1, C̄C̄_1}, (c | d | e | f)_{$k,_}, ∞] // Union
```

Out[•]=

$$\{c_{3,1}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{3,7}, c_{3,8}, c_{3,9}, c_{3,10}, c_{3,11}, c_{3,12}, c_{3,13}, c_{3,14}, c_{3,15}, c_{3,16},$$
$$c_{3,17}, c_{3,18}, c_{3,19}, c_{3,20}, c_{3,21}, c_{3,22}, c_{3,23}, c_{3,24}, c_{3,25}, c_{3,26}, c_{3,27}, c_{3,28}, c_{3,29}, c_{3,30}, c_{3,31},$$
$$c_{3,32}, c_{3,33}, c_{3,34}, c_{3,35}, c_{3,36}, c_{3,37}, c_{3,38}, c_{3,39}, c_{3,40}, c_{3,41}, c_{3,42}, c_{3,43}, c_{3,44}, c_{3,45},$$
$$c_{3,46}, c_{3,47}, c_{3,48}, c_{3,49}, c_{3,50}, c_{3,51}, c_{3,52}, c_{3,53}, c_{3,54}, c_{3,55}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5},$$
$$d_{3,6}, d_{3,7}, d_{3,8}, d_{3,9}, d_{3,10}, d_{3,11}, d_{3,12}, d_{3,13}, d_{3,14}, d_{3,15}, d_{3,16}, d_{3,17}, d_{3,18}, d_{3,19}, d_{3,20},$$
$$d_{3,21}, d_{3,22}, d_{3,23}, d_{3,24}, d_{3,25}, d_{3,26}, d_{3,27}, d_{3,28}, d_{3,29}, d_{3,30}, d_{3,31}, d_{3,32}, d_{3,33}, d_{3,34}, d_{3,35},$$
$$d_{3,36}, d_{3,37}, d_{3,38}, d_{3,39}, d_{3,40}, d_{3,41}, d_{3,42}, d_{3,43}, d_{3,44}, d_{3,45}, d_{3,46}, d_{3,47}, d_{3,48}, d_{3,49},$$
$$d_{3,50}, d_{3,51}, d_{3,52}, d_{3,53}, d_{3,54}, d_{3,55}, e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,5}, f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, f_{3,5}\}$$

In[•]:=
```
Short[errors = CF@{(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]] -
     (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})[[3, -1]],
   (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2})[[3, -1]],
   (CC_1 C̄C̄_2 // m_{1,2→1})[[3, -1]],
   (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1})[[3, -1]] - (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})[[3, -1]]},
 10]
```

Out[•]//Short=

{ <<1>> }

In[•]:=
```
Short[#, 10] &[eqns =
  Thread[0 == Union @@ (CoefficientRules[#, {x1, x2, x3, y1, y2, y3}][[ ;; , 2]] & /@ errors)]]
```

Out[•]//Short=

$$\left\{0 == c_{3,4} - T\, c_{3,4},\ \text{<<418>>},\right.$$

$$0 == \frac{3}{4} + \frac{5}{12\,T^5} - \frac{3}{4\,T^4} - \frac{1}{6\,T^3} - \frac{5}{12\,T} + \frac{T}{6} + c_{3,31} - T^4\, c_{3,31} + \frac{c_{3,32}}{T} - T^3\, c_{3,32} + \frac{c_{3,33}}{T^2} - T^2\, c_{3,33} + \frac{c_{3,34}}{T^3} -$$

$$T\, c_{3,34} - c_{3,35} + \frac{c_{3,35}}{T^4} + c_{3,36} - T^4\, c_{3,36} + \frac{c_{3,37}}{T} - T^3\, c_{3,37} + \frac{c_{3,38}}{T^2} - T^2\, c_{3,38} + \frac{c_{3,39}}{T^3} - T\, c_{3,39} -$$

$$c_{3,40} + \frac{c_{3,40}}{T^4} + c_{3,41} - T^4\, c_{3,41} + \frac{c_{3,42}}{T} - T^3\, c_{3,42} + \frac{c_{3,43}}{T^2} - T^2\, c_{3,43} + \frac{c_{3,44}}{T^3} - T\, c_{3,44} - c_{3,45} + \frac{c_{3,45}}{T^4} +$$

$$c_{3,46} - T^4\, c_{3,46} + \frac{c_{3,47}}{T} - T^3\, c_{3,47} + \frac{c_{3,48}}{T^2} - T^2\, c_{3,48} + \frac{c_{3,49}}{T^3} - T\, c_{3,49} - c_{3,50} + \frac{c_{3,50}}{T^4} + c_{3,51} -$$

$$\left.T^4\, c_{3,51} + \frac{c_{3,52}}{T} - T^3\, c_{3,52} + \frac{c_{3,53}}{T^2} - T^2\, c_{3,53} + \frac{c_{3,54}}{T^3} - T\, c_{3,54} - c_{3,55} + \frac{c_{3,55}}{T^4} + \frac{e_{3,5}}{T^4} - T^4\, f_{3,5}\right\}$$

In[•]:=
```
{sol} = Solve[eqns, unknowns]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\left\{\left\{c_{3,4} \to 0,\ c_{3,5} \to -T\, c_{3,2} - c_{3,3},\ c_{3,6} \to 0,\ c_{3,8} \to -\frac{1}{2}\,(1-T)\, c_{3,10},\ c_{3,9} \to 0,\right.\right.$$

$$c_{3,11} \to -T\, c_{3,7} - \frac{1}{2}\,(-1+3\,T)\, c_{3,10},\ c_{3,12} \to 0,\ c_{3,13} \to 0,\ c_{3,14} \to 0,\ c_{3,15} \to 0,\ c_{3,17} \to -((-1+T)\, c_{3,16}),$$

$$c_{3,18} \to -\frac{1-T}{6\,T},\ c_{3,19} \to 0,\ c_{3,20} \to 0,\ c_{3,21} \to \frac{1}{2\,T},\ c_{3,22} \to -\frac{-2+5\,T}{2\,T} - \left(T - T^2\right) c_{3,16},$$

$$c_{3,23} \to 0,\ c_{3,24} \to 0,\ c_{3,25} \to 0,\ c_{3,26} \to \frac{5}{6} - T^2\, c_{3,16},\ c_{3,27} \to 0,\ c_{3,28} \to 0,\ c_{3,29} \to 0,$$

$$c_{3,30} \to 0,\ c_{3,31} \to 0,\ c_{3,33} \to -\frac{3}{2}\,(-1+T)\, c_{3,32},\ c_{3,34} \to -\left(\left(-1+2\,T-T^2\right) c_{3,32}\right),$$

$$c_{3,35} \to -\frac{1-12\,T+27\,T^2-16\,T^3}{24\,T^2},\ c_{3,36} \to 0,\ c_{3,37} \to \frac{1}{6\,T^2},\ c_{3,38} \to -\frac{-1+3\,T}{4\,T^2},\ c_{3,39} \to -\frac{-1+11\,T-16\,T^2}{6\,T^2},$$

$$c_{3,40} \to -\frac{-1+31\,T-131\,T^2+125\,T^3}{24\,T^2} - \left(T - 2\,T^2 + T^3\right) c_{3,32},\ c_{3,41} \to 0,\ c_{3,42} \to 0,\ c_{3,43} \to \frac{1}{T},$$

$$c_{3,44} \to -\frac{-5+23\,T}{6\,T},\ c_{3,45} \to -\frac{-5+69\,T-142\,T^2}{24\,T} + \frac{3}{2}\,(-1+T)\,T^2\, c_{3,32},\ c_{3,46} \to 0,\ c_{3,47} \to 0,$$

$$c_{3,48} \to 0,\ c_{3,49} \to \frac{1}{6},\ c_{3,50} \to \frac{1}{24}\,(1-15\,T) - T^3\, c_{3,32},\ c_{3,51} \to 0,\ c_{3,52} \to 0,\ c_{3,53} \to 0,$$

$$c_{3,54} \to 0,\ c_{3,55} \to 0,\ d_{3,1} \to -c_{3,1},\ d_{3,2} \to -c_{3,2},\ d_{3,3} \to -\frac{c_{3,3}}{T^2},\ d_{3,4} \to 0,\ d_{3,5} \to \frac{c_{3,2}}{T} + \frac{c_{3,3}}{T^2},$$

$$d_{3,6} \to 0,\ d_{3,7} \to -\frac{-1+T}{T^4} - \frac{c_{3,7}}{T} - \frac{(-1+T)\, c_{3,10}}{T^2},\ d_{3,8} \to -\frac{1-T}{2\,T^5} - \frac{(1-T)\, c_{3,10}}{2\,T^3},$$

$$d_{3,9} \to 0,\ d_{3,10} \to -\frac{1}{T^4} - \frac{c_{3,10}}{T^2},\ d_{3,11} \to \frac{1}{2\,T^5} + \frac{c_{3,7}}{T^2} - \frac{(-1-T)\, c_{3,10}}{2\,T^3},\ d_{3,12} \to 0,\ d_{3,13} \to 0,$$

$$d_{3,14} \to 0,\ d_{3,15} \to 0,\ d_{3,16} \to -\frac{1-T}{T^4} - \frac{c_{3,16}}{T},\ d_{3,17} \to -\frac{-7+9\,T-2\,T^2}{2\,T^5} - \frac{(-1+T)\, c_{3,16}}{T^2},$$

$$d_{3,18}\to -\frac{7-9\,T+2\,T^2}{6\,T^6},\ d_{3,19}\to 0,\ d_{3,20}\to \frac{1}{T^4},\ d_{3,21}\to -\frac{9-T}{2\,T^5},\ d_{3,22}\to \frac{3}{2\,T^6}-\frac{(1-T)\,c_{3,16}}{T^3},$$

$$d_{3,23}\to 0,\ d_{3,24}\to 0,\ d_{3,25}\to \frac{1}{T^5},\ d_{3,26}\to -\frac{1}{3\,T^6}+\frac{c_{3,16}}{T^3},\ d_{3,27}\to 0,\ d_{3,28}\to 0,\ d_{3,29}\to 0,$$

$$d_{3,30}\to 0,\ d_{3,31}\to 0,\ d_{3,32}\to -\frac{-1+T}{6\,T^4}-\frac{c_{3,32}}{T},\ d_{3,33}\to -\frac{2-3\,T+T^2}{T^5}-\frac{3\,(-1+T)\,c_{3,32}}{2\,T^2},$$

$$d_{3,34}\to -\frac{-16+27\,T-12\,T^2+T^3}{6\,T^6}-\frac{\left(1-2\,T+T^2\right)c_{3,32}}{T^3},\ d_{3,35}\to -\frac{16-27\,T+12\,T^2-T^3}{24\,T^7},\ d_{3,36}\to 0,$$

$$d_{3,37}\to -\frac{1}{6\,T^4},\ d_{3,38}\to -\frac{-3+T}{T^5},\ d_{3,39}\to \frac{3\,(-3+T)}{2\,T^6},\ d_{3,40}\to -\frac{-27+5\,T-T^2-T^3}{24\,T^7}-\frac{\left(-1+2\,T-T^2\right)c_{3,32}}{T^4},$$

$$d_{3,41}\to 0,\ d_{3,42}\to 0,\ d_{3,43}\to -\frac{1}{T^5},\ d_{3,44}\to \frac{2}{T^6},\ d_{3,45}\to -\frac{12-T-5\,T^2}{24\,T^7}+\frac{3\,(-1+T)\,c_{3,32}}{2\,T^4},$$

$$d_{3,46}\to 0,\ d_{3,47}\to 0,\ d_{3,48}\to 0,\ d_{3,49}\to -\frac{1}{6\,T^6},\ d_{3,50}\to -\frac{-1-T}{24\,T^7}+\frac{c_{3,32}}{T^4},\ d_{3,51}\to 0,$$

$$d_{3,52}\to 0,\ d_{3,53}\to 0,\ d_{3,54}\to 0,\ d_{3,55}\to 0,\ e_{3,1}\to \frac{c_{3,3}}{2\,T},\ e_{3,2}\to -\frac{c_{3,10}}{T},\ e_{3,3}\to 0,$$

$$e_{3,4}\to 0,\ e_{3,5}\to 0,\ f_{3,1}\to -\frac{c_{3,3}}{2\,T},\ f_{3,2}\to \frac{1}{T^3}+\frac{c_{3,10}}{T},\ f_{3,3}\to 0,\ f_{3,4}\to 0,\ f_{3,5}\to 0\Big\}\Big\}$$

In[•]:=
```
sol /. (a_ → b_) :> (a = b)
```

Out[•]=

$$\left\{0,\ -T\,c_{3,2}-c_{3,3},\ 0,\ -\frac{1}{2}\,(1-T)\,c_{3,10},\ 0,\ -T\,c_{3,7}-\frac{1}{2}\,(-1+3T)\,c_{3,10},\ 0,\ 0,\ 0,\ 0,\right.$$

$$-((-1+T)\,c_{3,16}),\ -\frac{1-T}{6T},\ 0,\ 0,\ \frac{1}{2T},\ -\frac{-2+5T}{2T}-\left(T-T^2\right)c_{3,16},\ 0,\ 0,\ 0,\ \frac{5}{6}-T^2\,c_{3,16},\ 0,$$

$$0,\ 0,\ 0,\ 0,\ -\frac{3}{2}\,(-1+T)\,c_{3,32},\ -\left(\left(-1+2T-T^2\right)c_{3,32}\right),\ -\frac{1-12T+27T^2-16T^3}{24T^2},\ 0,\ \frac{1}{6T^2},$$

$$-\frac{-1+3T}{4T^2},\ -\frac{-1+11T-16T^2}{6T^2},\ -\frac{-1+31T-131T^2+125T^3}{24T^2}-\left(T-2T^2+T^3\right)c_{3,32},\ 0,\ 0,\ \frac{1}{T},$$

$$-\frac{-5+23T}{6T},\ -\frac{-5+69T-142T^2}{24T}+\frac{3}{2}\,(-1+T)\,T^2\,c_{3,32},\ 0,\ 0,\ 0,\ \frac{1}{6},\ \frac{1}{24}\,(1-15T)-T^3\,c_{3,32},$$

$$0,\ 0,\ 0,\ 0,\ 0,\ -c_{3,1},\ -c_{3,2},\ -\frac{c_{3,3}}{T^2},\ 0,\ \frac{c_{3,2}}{T}+\frac{c_{3,3}}{T^2},\ 0,\ -\frac{-1+T}{T^4}-\frac{c_{3,7}}{T}-\frac{(-1+T)\,c_{3,10}}{T^2},$$

$$-\frac{1-T}{2T^5}-\frac{(1-T)\,c_{3,10}}{2T^3},\ 0,\ -\frac{1}{T^4}-\frac{c_{3,10}}{T^2},\ \frac{1}{2T^5}+\frac{c_{3,7}}{T^2}-\frac{(-1-T)\,c_{3,10}}{2T^3},\ 0,\ 0,\ 0,\ 0,\ -\frac{1-T}{T^4}-\frac{c_{3,16}}{T},$$

$$-\frac{-7+9T-2T^2}{2T^5}-\frac{(-1+T)\,c_{3,16}}{T^2},\ -\frac{7-9T+2T^2}{6T^6},\ 0,\ \frac{1}{T^4},\ -\frac{9-T}{2T^5},\ \frac{3}{2T^6}-\frac{(1-T)\,c_{3,16}}{T^3},\ 0,$$

$$0,\ \frac{1}{T^5},\ -\frac{1}{3T^6}+\frac{c_{3,16}}{T^3},\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{-1+T}{6T^4}-\frac{c_{3,32}}{T},\ -\frac{2-3T+T^2}{T^5}-\frac{3\,(-1+T)\,c_{3,32}}{2T^2},$$

$$-\frac{-16+27T-12T^2+T^3}{6T^6}-\frac{\left(1-2T+T^2\right)c_{3,32}}{T^3},\ -\frac{16-27T+12T^2-T^3}{24T^7},\ 0,\ -\frac{1}{6T^4},\ -\frac{-3+T}{T^5},\ \frac{3\,(-3+T)}{2T^6},$$

$$-\frac{-27+5T-T^2-T^3}{24T^7}-\frac{\left(-1+2T-T^2\right)c_{3,32}}{T^4},\ 0,\ 0,\ -\frac{1}{T^5},\ \frac{2}{T^6},\ -\frac{12-T-5T^2}{24T^7}+\frac{3\,(-1+T)\,c_{3,32}}{2T^4},\ 0,$$

$$\left.0,\ 0,\ -\frac{1}{6T^6},\ -\frac{-1-T}{24T^7}+\frac{c_{3,32}}{T^4},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{c_{3,3}}{2T},\ -\frac{c_{3,10}}{T},\ 0,\ 0,\ 0,\ -\frac{c_{3,3}}{2T},\ \frac{1}{T^3}+\frac{c_{3,10}}{T},\ 0,\ 0,\ 0\right\}$$

In[•]:=
```
c3,1 = c3,2 = c3,3 = c3,7 = c3,10 = c3,16 = c3,32 = 0;
{R1,2, R̄1,2, CC1, C̄C̄1}
```

Out[•]=

$$\Big\{\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ (-1+T)\,x_2\,(y_1-y_2),$$

$$\in\text{Series}\Big[0,\ \frac{1}{2}\,(-1+T)\,x_2^2\,y_1^2+x_1\,x_2\,y_1\,y_2+\frac{1}{2}\,(1-3\,T)\,x_2^2\,y_1\,y_2,\ -\frac{\left(-1+4\,T-3\,T^2\right)x_2^3\,y_1^3}{6\,T}-\frac{1}{2}\,x_2^2\,y_1\,y_2-$$

$$\frac{x_1^2\,x_2\,y_1^2\,y_2}{2\,T}-\frac{(1-3\,T)\,x_1\,x_2^2\,y_1^2\,y_2}{2\,T}-\frac{\left(1-11\,T+16\,T^2\right)x_2^3\,y_1^2\,y_2}{6\,T}-\frac{1}{2}\,x_1\,x_2^2\,y_1\,y_2^2+\frac{1}{6}\,(-1+7\,T)\,x_2^3\,y_1\,y_2^2,$$

$$-\frac{(1-T)\,x_2^3\,y_1^3}{6\,T}-\frac{\left(1-12\,T+27\,T^2-16\,T^3\right)x_2^4\,y_1^4}{24\,T^2}+\frac{x_1\,x_2^2\,y_1^2\,y_2}{2\,T}-\frac{(-2+5\,T)\,x_2^3\,y_1^2\,y_2}{2\,T}+\frac{x_1^3\,x_2\,y_1^3\,y_2}{6\,T^2}-$$

$$\frac{(-1+3\,T)\,x_1^2\,x_2^2\,y_1^3\,y_2}{4\,T^2}-\frac{\left(-1+11\,T-16\,T^2\right)x_1\,x_2^3\,y_1^3\,y_2}{6\,T^2}-\frac{\left(-1+31\,T-131\,T^2+125\,T^3\right)x_2^4\,y_1^3\,y_2}{24\,T^2}+$$

$$\frac{5}{6}\,x_2^3\,y_1\,y_2^2+\frac{x_1^2\,x_2^2\,y_1^2\,y_2^2}{T}-\frac{(-5+23\,T)\,x_1\,x_2^3\,y_1^2\,y_2^2}{6\,T}-\frac{\left(-5+69\,T-142\,T^2\right)x_2^4\,y_1^2\,y_2^2}{24\,T}+$$

$$\frac{1}{6}\,x_1\,x_2^3\,y_1\,y_2^3+\frac{1}{24}\,(1-15\,T)\,x_2^4\,y_1\,y_2^3\Big]\Big],\ \mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ \left(-1+\frac{1}{T}\right)x_2\,(y_1-y_2),$$

$$\in\text{Series}\Big[0,\ -\frac{(-1+T)\,x_1\,x_2\,y_1^2}{T^2}-\frac{(1-T)\,x_2^2\,y_1^2}{2\,T^3}-\frac{x_1\,x_2\,y_1\,y_2}{T^2}-\frac{(-1-T)\,x_2^2\,y_1\,y_2}{2\,T^3},$$

$$-\frac{(1-T)\,x_1\,x_2\,y_1^2}{T^3}-\frac{(-1+T)\,x_2^2\,y_1^2}{2\,T^4}-\frac{(-1+T)\,x_1^2\,x_2\,y_1^3}{2\,T^3}-\frac{\left(3-4\,T+T^2\right)x_1\,x_2^2\,y_1^3}{2\,T^4}-$$

$$\frac{\left(-3+4\,T-T^2\right)x_2^3\,y_1^3}{6\,T^5}+\frac{x_1\,x_2\,y_1\,y_2}{T^3}-\frac{x_2^2\,y_1\,y_2}{2\,T^4}-\frac{x_1^2\,x_2\,y_1^2\,y_2}{2\,T^3}+\frac{2\,x_1\,x_2^2\,y_1^2\,y_2}{T^4}-\frac{\left(4+T+T^2\right)x_2^3\,y_1^2\,y_2}{6\,T^5}-$$

$$\frac{x_1\,x_2^2\,y_1\,y_2^2}{2\,T^4}-\frac{(-1+T)\,x_2^3\,y_1\,y_2^2}{6\,T^5},\ -\frac{(-1+T)\,x_1\,x_2\,y_1^2}{T^4}-\frac{(1-T)\,x_2^2\,y_1^2}{2\,T^5}-\frac{(1-T)\,x_1^2\,x_2\,y_1^3}{T^4}-$$

$$\frac{\left(-7+9\,T-2\,T^2\right)x_1\,x_2^2\,y_1^3}{2\,T^5}-\frac{\left(7-9\,T+2\,T^2\right)x_2^3\,y_1^3}{6\,T^6}-\frac{(-1+T)\,x_1^3\,x_2\,y_1^4}{6\,T^4}-\frac{\left(2-3\,T+T^2\right)x_1^2\,x_2^2\,y_1^4}{T^5}-$$

$$\frac{\left(-16+27\,T-12\,T^2+T^3\right)x_1\,x_2^3\,y_1^4}{6\,T^6}-\frac{\left(16-27\,T+12\,T^2-T^3\right)x_2^4\,y_1^4}{24\,T^7}-\frac{x_1\,x_2\,y_1\,y_2}{T^4}+\frac{x_2^2\,y_1\,y_2}{2\,T^5}+$$

$$\frac{x_1^2\,x_2\,y_1^2\,y_2}{T^4}-\frac{(9-T)\,x_1\,x_2^2\,y_1^2\,y_2}{2\,T^5}+\frac{3\,x_2^3\,y_1^2\,y_2}{2\,T^6}-\frac{x_1^3\,x_2\,y_1^3\,y_2}{6\,T^4}-\frac{(-3+T)\,x_1^2\,x_2^2\,y_1^3\,y_2}{T^5}+$$

$$\frac{3\,(-3+T)\,x_1\,x_2^3\,y_1^3\,y_2}{2\,T^6}-\frac{\left(-27+5\,T-T^2-T^3\right)x_2^4\,y_1^3\,y_2}{24\,T^7}+\frac{x_1\,x_2^2\,y_1\,y_2^2}{T^5}-\frac{x_2^3\,y_1\,y_2^2}{3\,T^6}-$$

$$\frac{x_1^2\,x_2^2\,y_1^2\,y_2^2}{T^5}+\frac{2\,x_1\,x_2^3\,y_1^2\,y_2^2}{T^6}-\frac{\left(12-T-5\,T^2\right)x_2^4\,y_1^2\,y_2^2}{24\,T^7}-\frac{x_1\,x_2^3\,y_1\,y_2^3}{6\,T^6}-\frac{(-1-T)\,x_2^4\,y_1\,y_2^3}{24\,T^7}\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\sqrt{T},\ 0,\ \in\text{Series}\Big[0,\ -\frac{x_1\,y_1}{T},\ 0,\ 0\Big]\Big],$$

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\frac{1}{\sqrt{T}},\ 0,\ \in\text{Series}\Big[0,\ \frac{x_1\,y_1}{T},\ -\frac{x_1\,y_1}{T^2},\ \frac{x_1\,y_1}{T^3}\Big]\Big]\Big\}$$

```
In[•]:= {(R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}),
         (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ 𝔼_{{}→{1,2}}[1, 0, ϵSeries[0]],
         (CC_1 C̄C̄_2 // m_{1,2→1}) ≡ 𝔼_{{}→{1}}[1, 0, ϵSeries[0]],
         (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (C̄C̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})}
Out[•]=
{True, True, True, True}
```

Some Knot Theory

```
In[•]:= Define[Kink_i = CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→i}, ‾Kink‾_i = CC_3 R̄_{1,2} // m_{1,3→1} // m_{1,2→i}]
```

```
In[•]:= RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
          n = Length@pd; rots = Table[0, {2 n}];
          xs = Cases[pd, x_X :> {Xp[x[[4]], x[[1]]] PositiveQ@x
                                 Xm[x[[2]], x[[1]]]      True   ];
          For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
            front = Flatten[front /. k → (xs /. {
                    Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
                    Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
                   })],
            Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
           ]];
          RVK[xs, rots] ];
        RVK[K_] := RVK[PD[K]];
```

```
In[•]:= rot[i_, 0] := 𝔼_{{}→{i}}[1, 0, ϵSeries@0];
        rot[i_, n_] := Module[{j},
          rot[i, n] = If[n > 0, rot[i, n - 1] CC_j, rot[i, n + 1] C̄C̄_j] // m_{i,j→i}];
```

```
In[•]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ℰ, done, st, cx, ℰ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ℰ = 𝔼{}→{0}[1, 0, ϵSeries@0];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ℰ1 = Switch[Head[cx],
      Xp, (R_{i,j} \[OverBar]Kink_k) // m_{j,k→j},
      Xm, (\[OverBar]R_{i,j} Kink_k) // m_{j,k→j}
    ];
   ℰ1 = (rot[k, rots[[i]]] ℰ1) // m_{k,i→i}; rots[[i]] = 0;
   ℰ1 = (ℰ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ℰ1 = (rot[k, rots[[j]]] ℰ1) // m_{k,j→j}; rots[[j]] = 0;
   ℰ1 = (ℰ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ℰ *= ℰ1;
   If[MemberQ[done, i], ℰ = ℰ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ℰ = ℰ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ℰ = ℰ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ℰ = ℰ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ℰ (*/. {x0→x,y0→y,a0→a}*))
 ]
```

```
In[•]:= BeginProfile[];
PopupWindow[Button["Show Profile Monitor"],
 Dynamic[PrintProfile[], UpdateInterval → 3, TrackedSymbols → {}]]
```

```
Out[•]=
```

Show Profile Monitor

```
In[•]:= $k = 1
```

```
Out[•]=
```

1

```
In[•]:= NewBit[K_] := Module[{Alex = Alexander[K][T]},
  T^3 Alex^2/(T - 1) Z[K][[3, 2]] // Factor]
```

In[•]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$\left\{2 - T + T^2,\ (1 + T)\left(1 - 3T + T^2\right),\ \frac{4 - 3T + 5T^2 - 3T^3 + 3T^4 - T^5 + T^6}{T^2},\ 9 - 11T + 7T^2 - T^3\right\}$$

In[•]:=
```
(*Two knots with equal Alexander, new bit does not agree*)
Alexander[Knot[6, 1]] == Alexander[Knot[9, 46]]
Timing[NewBit[Knot[6, 1]] == NewBit[Knot[9, 46]]]
```

Out[•]=

True

Out[•]=

$$\left\{53.1563,\ 5 - 11T - T^2 + 3T^3 == 7 - 21T + 9T^2 + T^3\right\}$$

In[•]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[Z /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[•]=

1

In[•]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[Z /@ equiv]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

Out[•]=

1

In[•]:=
```
$k = 2
```

Out[•]=

2

In[•]:=
```
equiv = {Knot[10, 106], Knot[12, NonAlternating, 369]};
Length@Union[Z /@ equiv]
```

Out[•]=

2

In[•]:=
```
equiv =
  {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]};
Length@Union[Z /@ equiv]
```

In[•]:=
```
PrintProfile[]
```

Out[•]=
```
ProfileRoot is root. Profiled time: 79.031
  (  24)   0.032/  0.032 above CF
  ( 237)   1.581/  6.183 above Zip1
  ( 237)   0.799/ 38.897 above Zip2
  ( 237)  28.773/ 33.919 above Zip3
CF: called 3816 times, time in 47.878/47.878
  (  24)   0.032/  0.032 under ProfileRoot
  ( 1185)   4.602/  4.602 under Zip1
  ( 1185)  38.098/ 38.098 under Zip2
  ( 1422)   5.146/  5.146 under Zip3
Zip3: called 237 times, time in 28.773/33.919
  ( 237)  28.773/ 33.919 under ProfileRoot
  ( 1422)   5.146/  5.146 above CF
Zip1: called 237 times, time in 1.581/6.183
  ( 237)   1.581/  6.183 under ProfileRoot
  ( 1185)   4.602/  4.602 above CF
Zip2: called 237 times, time in 0.799/38.897
  ( 237)   0.799/ 38.897 under ProfileRoot
  ( 1185)  38.098/ 38.098 above CF
```