

Tingley: [Elementary Construction of Lusztig's Canonical Basis](#).

Quesne@arXiv:math-ph/0305003: $e_q^x = \exp\left(\sum_{k \geq 1} \frac{(1-q)^k x^k}{k(1-q^k)}\right)$.

[Ma1, Example 1.3.2]. A Hopf algebra over $\mathbb{Q}[q^{\pm 1}]$:

$$\langle 1, X, g^{\pm 1} \rangle / \left((Xg^{\pm 1} = q^{\pm 1} g^{\pm 1} X) \right),$$

with $\Delta: (X, g^{\pm 1}) \mapsto (X \otimes 1 + g \otimes X, g^{\pm 1} \otimes g^{\pm 1})$, $\epsilon: (X, g^{\pm 1}) \mapsto (0, 1)$, and $S: (X, g^{\pm 1}) \mapsto (-g^{-1} X, g^{\mp 1})$. Has $S^2 \neq 1$.

Self-duality: Seed $\langle g, g \rangle = q^{-1}$, $\langle X, X \rangle = 1$, $\langle g, X \rangle = \langle X, g \rangle = 0$ then $\langle g^a, g \rangle = q^{-a}$; $\langle g^a, g^b \rangle = q^{-ab}$; $\langle g^a X g^b, X \rangle = q^{-a}$; $\langle X^c, X^d \rangle = \delta_{cd} [c]_{q^{-1}}!$; $\langle X^a g^b, X^c g^d \rangle = \delta_{ac} [a]_{q^{-1}}! [d]_{q^{-b}}$.

[CP, Definition-Proposition 6.4.3], $U_h(sl_2)$ in detail:

DEFINITION-PROPOSITION 6.4.3 Quantum $sl_2(\mathbb{C})$ is the topological Hopf algebra $U_h(sl_2(\mathbb{C}))$ over $\mathbb{C}[[h]]$ defined as follows.

Let $P = \mathbb{C}\{H, X^+, X^-\}$ be the algebra of non-commutative polynomials in three generators H, X^+ and X^- , let I be the two-sided ideal of $P[[h]]$ generated by

$$(27) \quad [H, X^+] - 2X^+, \quad [H, X^-] + 2X^-, \quad [X^+, X^-] - \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}},$$

and let \bar{I} be the closure of I in the h -adic topology. Then, $U_h(sl_2(\mathbb{C})) = P[[h]]/\bar{I}$ as an algebra over $\mathbb{C}[[h]]$.

There are homomorphisms of $\mathbb{C}[[h]]$ -algebras

$$\Delta_h: U_h(sl_2(\mathbb{C})) \rightarrow U_h(sl_2(\mathbb{C})) \otimes U_h(sl_2(\mathbb{C})),$$

$$\epsilon_h: U_h(sl_2(\mathbb{C})) \rightarrow \mathbb{C}[[h]], \quad S_h: U_h(sl_2(\mathbb{C})) \rightarrow U_h(sl_2(\mathbb{C})),$$

(an anti-homomorphism in the case S_h) given on generators by (19) and (21)–(24), which define the structure of a topological Hopf algebra on $U_h(sl_2(\mathbb{C}))$.

Moreover, $U_h(sl_2(\mathbb{C}))$ is a QUE algebra whose classical limit is the Lie bialgebra structure on $sl_2(\mathbb{C})$ defined by $r = X^+ \wedge X^-$.

$$\Delta_h(H) = H \otimes 1 + 1 \otimes H; \quad \Delta_h(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+ \quad (19;21)$$

$$S_h(H) = -H, \quad S_h(X^+) = -X^+ e^{-hH}, \quad \epsilon_h(H) = \epsilon_h(X^+) = h \quad (22)$$

$$\Delta_h(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-; \quad S_h(X^-) = -e^{hH} X^-, \quad \epsilon_h(X^-) = 0 \quad (23;24)$$

[CP, Proposition 6.4.6], **The $U_h \simeq U[[h]]$ isomorphism for sl_2 :**

With $\bar{\Omega} = (\bar{H} + 1)^2/4 + \bar{X}^- \bar{X}^+ = (\bar{H} - 1)^2/4 + \bar{X}^+ \bar{X}^-$,

PROPOSITION 6.4.6 There is an isomorphism

$$\varphi: U_h(sl_2(\mathbb{C})) \rightarrow U(sl_2(\mathbb{C}))[[h]]$$

of algebras over $\mathbb{C}[[h]]$ such that

$$\begin{aligned} \varphi(H) &= \bar{H}, & \varphi(X^-) &= \bar{X}^-, \\ \varphi(X^+) &= 2 \left(\frac{\cosh h(\bar{H} - 1) - \cosh 2h\sqrt{\bar{\Omega}}}{(\bar{H} - 1)^2 - 4\bar{\Omega}} \sinh^2 h \right) \bar{X}^+. \end{aligned}$$

[CP, Proposition 6.4.8], **R for $U_h(sl_2)$:** With $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and $[n]_q! = [n]_q \cdots [1]_q$,

PROPOSITION 6.4.8 The Hopf algebra $U_h(sl_2(\mathbb{C}))$ is topologically quasitriangular with universal R-matrix

$$(30) \quad \mathcal{R}_h = \sum_{n=0}^{\infty} R_n(h) e^{\frac{1}{2}h(H \otimes H)} (X^+)^n \otimes (X^-)^n,$$

where

$$R_n(h) = \frac{q^{\frac{1}{2}n(n+1)}(1 - q^{-2})^n}{[n]_q!}, \quad (q = e^h).$$

[Ma2, Example 18.8], **the QG structure on a general SS \mathfrak{g} .** Here i 's are simple roots, μ 's are positive roots:

Example 18.8 Given a Cartan and root datum and H, B, \tilde{B} as above, and $q \in k^*$ such that $q_i^2 \neq 1$ for all i , we have an ordinary Hopf algebra $U_q = B' \triangleright \mathfrak{H} \triangleleft B^{\text{op}}$. It consists of generators e_i, f_i, K_μ , the relations

$$\begin{aligned} K_\mu K_\nu &= K_{\mu+\nu}, & K_\mu e^i &= q^{\langle \mu, i \rangle} e^i K_\mu, & f_i K_\mu &= q^{\langle \mu, i \rangle} K_\mu f_i, \\ [e^i, f_j] &= \frac{K_i^{\frac{i_i}{2}} - K_i^{-\frac{i_i}{2}}}{q_i - q_i^{-1}} \delta_j^i \end{aligned}$$

and the relations coming from the kernels of ev . The coalgebra is

$$\Delta e^i = e^i \otimes K_i^{\frac{i_i}{2}} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + K_i^{-\frac{i_i}{2}} \otimes f_i,$$

$$\epsilon(K_\mu) = 1, \quad \epsilon(e^i) = \epsilon(f_i) = 0.$$

The antipode has the usual form uniquely determined by the above.

([Ma2] seems to have no universal R formula).

[CP, Theorem 8.3.9], **the universal R for a general SS \mathfrak{g} :**

THEOREM 8.3.9 For any finite-dimensional complex simple Lie algebra \mathfrak{g} , $U_h(\mathfrak{g})$ is topologically quasitriangular with universal R-matrix

$$\mathcal{R}_h = \exp \left[h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j \right] \prod_{\beta} \exp_{q_\beta} [(1 - q_\beta^{-2}) X_\beta^+ \otimes X_\beta^-],$$

where the product is over all the positive roots of \mathfrak{g} , and the order of the terms is such that the β_r -term appears to the left of the β_s -term if $r > s$ (see 8.1.4). ■

[Ma2, Section 8], **the quantum double:** $\mathcal{DA} := A^{*,\text{op}} \otimes A$ with $(\phi a)(\psi b) := \langle S a_1, \psi_1 \rangle \langle a_3, \psi_3 \rangle (\psi_2 \phi)(a_2 b)$, $R = Id \in A^* \otimes A \subset \mathcal{DA} \otimes \mathcal{DA}$. What problem does it solve?

In QGverse, why start by deforming the co-product? And why identify the two Cartans?

[ES, Section 3.3] For $(\mathfrak{g}, [,])$ and $r \in \Lambda^2(\mathfrak{g})$, $\delta a := [r, a \otimes 1 + 1 \otimes a]$ makes \mathfrak{g} a ‘‘coboundary’’ Lie bialgebra iff $\text{CYB}(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$ is \mathfrak{g} -invariant. ‘‘Triangular’’ means $\text{CYB}(r) = 0$.

References.

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