

Pensieve header: The 3-Heisenberg Formalism for Θ . Engine modified from pensieve://Project-s/BabyDoPeGDO/Solving_to_k=3.nb, $\$px\$$ formulas modified from pensieve://Talks/Bonn-2505/. Now in $\$xp\$$ order. Now using the myopic projection $\$M$.

$\mathbb{E}[\omega, Q, P_eSeries]$ represents ωe^{Q+P} , where ω is a scalar, Q is an ϵ -free quadratic, and $P = \sum_{k=0}^{\$k} P[k] \epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}[_]$ // $\mathbb{E}[_]$ calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

```
In[ ]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\3H"];
Once[<< KnotTheory`];
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[ ]:= $k = 1;
```

```
In[ ]:= $M[_] := E; (* The Myopic Projection *)
```

```
In[ ]:= HL[_] := Style[_ , Background -> If[TrueQ[_], Green, Red]];
```

Minor utilities

```
In[ ]:= CCF[_] := ExpandDenominator@ExpandNumerator@Together[_];
(*CoefficientCanonical Form *)
CCF[_] := Factor[_]; (*CoefficientCanonical Form *)
CF[_List] := CF /@ _List;
CF[_eSeries] := CF /@ Take[_ , $k + 1];
CF[_] := Module[{F}, Expand@Collect[_ , {p | x | pi | xi | g}_, F] /. F -> CCF];
(*CF[_] := Module[
  {vs=Cases[_ , {p | x | pi | xi | g}_, Infinity]},
  Total[(CCF[#[2]] (Times@@vs^#[1])) & /@ CoefficientRules[Expand[_], vs]]
];*)
CF[_E] := CF /@ _E;
CF[_E_Sp___][_S___] := CF /@ E_Sp[_S];
CF[a == b_] := CF[a - b] == 0;
```

```

In[*]:= eSeries /: S1_eSeries ≡ S2_eSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
eSeries[] := eSeries @@ Table[0, {k + 1}];
eSeries /: Plus[ss___eSeries] := Module[{l = Min[Length /@ {ss}]],
  eSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
];
(*eSeries/: S1_eSeries+S2_eSeries :=
  eSeries@@Table[S1[[k]]+S2[[k]],{k,Min[Length@S1,Length@S2]}}];*)
eSeries /: S1_eSeries * S2_eSeries := eSeries @@
  Table[Sum[S1[[j + 1]] * S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
eSeries /: c_ * S_eSeries := (c #) & /@ S;
eSeries /: ∂vs___S_eSeries := (s ↦ ∂vss) /@ S;

```

```

In[*]:= Options[PolyPlot1] = {Labeled → False};
PolyPlot1[Δ_, OptionsPattern[]] := Module[{Δ1, crs, m, maxc, minc, s, rect},
  Δ1 = PowerExpand@Expand@Δ;
  rect = {{0, 0}, {1, 0}, {1, 1}, {0, 1}};
  If[Δ1 === 0, Graphics[],
    m = Max[-Exponent[Δ1, T, Min], Exponent[Δ1, T, Max]];
    crs = CoefficientRules[Tm Δ1, {T}];
    maxc = N@Log@Max@Abs[Last /@ crs];
    minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
    If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
    Graphics[crs /. ({x_} → c_) → {
      Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
      Tooltip[Polygon[{x + m - 1/2, 0} + #] & /@ rect, c Tx-m],
      If[Not@OptionValue[Labeled], {},
        {Black, FontSize → Scaled[ $\frac{1}{(m+1)(6+maxc)}$ ], Text[c Tx-m, {x + m, 0.5}]}]
    }, AspectRatio → Min[1/5, 1/√(m+1)],
    ImagePadding → None, PlotRangePadding → None]
  ];
Options[PolyPlot2] = {Labeled → False, Shear → True};
PolyPlot2[θ_, OptionsPattern[]] :=
  Module[{θ1, crs, m1, m2, maxc, minc, s, stone, mat, p},
    θ1 = PowerExpand@Expand@θ;
    If[θ1 === 0, Graphics[{White, Disk[]}],
      If[OptionValue[Shear],
        stone = Table[{Cos[α], Sin[α]} / Cos[2π/12] / 2, {α, 2π/12, 2π, 2π/6}];
        mat =  $\begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$ ,

```

```

stone = {{1, 1}, {-1, 1}, {-1, -1}, {1, -1}} / 2;
mat = IdentityMatrix[2];
];
m1 = Max[-Exponent[ $\theta$ 1, T1, Min], Exponent[ $\theta$ 1, T1, Max]];
m2 = Max[-Exponent[ $\theta$ 1, T2, Min], Exponent[ $\theta$ 1, T2, Max]];
crs = CoefficientRules[T1m1 T2m2  $\theta$ 1, {T1, T2}];
maxc = N@Log@Max@Abs[Last /@ crs];
minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
If[minc == maxc, s[_] = 0, s[_] := s[c] = (maxc - Log@c) / (maxc - minc)];
Graphics[{{(*{Yellow, Disk[{0, 0}, 1 + Cos[2 $\pi$ /12] Norm[{m1, m2}]/ $\sqrt{2}$ ]}, *)
crs /. ({x1_, x2_} → c_) → {
  Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
  p = mat.{x1 - m1, x2 - m2};
  Tooltip[Polygon[(p + #) & /@ stone], c T1x1-m1 T2x2-m2],
  If[Not@OptionValue[Labeled], {},
    {Black, FontSize → Scaled[ $\frac{1}{\text{Max}[m1, m2] (10 + \text{maxc})}$ ], Text[c T1x1-m1 T2x2-m2, p]}]}
}, ImagePadding → None, PlotRangePadding → None]
]];
Options[PolyPlot] = {Labeled → False, ImageSize → Automatic};
PolyPlot[{ $\Delta$ _,  $\theta$ _, opts___Rule] := GraphicsColumn[
  {PolyPlot1[ $\Delta$ _, FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot1]]],
  PolyPlot2[ $\theta$ _, FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot2]]]},
  Spacings → Scaled@0.08, ImagePadding → None, PlotRangePadding → None,
  FilterRules[Join[{opts}, Options[PolyPlot]], Options[GraphicsColumn]]
];

```

The Main Program

Variables and their duals:

```

In[*]:= {p*, x*,  $\pi$ *,  $\xi$ *} = { $\pi$ ,  $\xi$ , p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_<math>\nu</math>_)* := (u*)<math>\nu</math>;

```

Ⓔ operations:

```

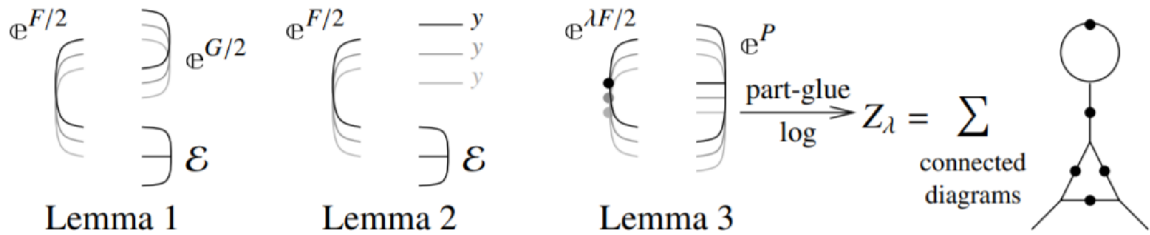
In[*]:= E /: E[ω1_, Q1_, P1_] ≡ E[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
E /: E[ω1_, Q1_, P1_] E[ω2_, Q2_, P2_] := E[ω1 ω2, Q1 + Q2, P1 + P2];
Ed1→r1[ε1s____] ≡ Ed2→r2[ε2s____] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (E[ε1s] ≡ E[ε2s]);
Ed1→r1[ε1s____] Ed2→r2[ε2s____] ^:= E(d1∪d2)→(r1∪r2) @@ (E[ε1s] E[ε2s]);
Edr[εs____]$k := Edr @@ E[εs]$k;

```

```

In[*]:= Ed1→r1[ε1s____] // Ed2→r2[ε2s____] := Module[{is = r1 ∩ d2, lvs},
  lvs = Flatten@Table[{xv, $ei, pv, $ei}, {v, 3}, {i, is}];
  E(d1∪Complement[d2, is])→(r2∪Complement[r1, is]) @@ (Ziplvs∪lvs[lvs*.lvs, Times[
    E[ε1s] /. Table[{u : (x | p)}v,i → uv, $ei, {i, is}],
    E[ε2s] /. Table[{u : (ξ | π)}v,i → uv, $ei, {i, is}]
  ]])
]

```



```

In[*]:= Zipvs[f_, ε_] := (*EchoLabel["EZip called with {vs,f,ε} = "] [{vs,f,ε}];*)
  <f, ε> // Zip1vs // EZip23vs;
Zipvs[f_, ε_] := (*EchoLabel["EZip called with {vs,f,ε} = "] [{vs,f,ε}];*)
  <f, ε> // Zip1vs // Zip2vs // Zip3vs;

```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \oplus^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```

In[*]:= Zip1{} = Identity;
Zip1vs@<f_, E[ω_, Q_, P_]> := Module[{I, F, G, u, v},
  (*EchoLabel["EZip1 called with {vs,f,ω,Q,P} = "] [{vs,f,ω,Q,P}];*)
  I = IdentityMatrix@Length@vs;
  F = Table[∂u,v f, {u, vs*}, {v, vs*}];
  G = Table[∂u,v Q, {u, vs}, {v, vs}];
  <CF[vs*.F.Inverse[I - G.F].vs* / 2],
  E[CF@PowerExpand@Factor[ω Det[I - G.F]^{-1/2}, CF[Q - vs.G.vs / 2], P]>
]

```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E} \oplus^{\sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{P}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```

In[ ]:= Zip2_{ } = Identity;
Zip2_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[{F, Y, u, v},
  (*EchoLabel["Zip2 called with {vs,F,ω,Q,P} = "] [{vs,F,ω,Q,P}];*)
  F = Table[∂_{u,v} F, {u, vs*}, {v, vs*}];
  Y = Table[∂_v Q, {v, vs}];
  CF /@ <F, E[ω, Q - Y.v + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]] >
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \oplus^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[ ]:= Zip3_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[{Z, u, v, m, j},
  (*EchoLabel["Zip3 called with {vs,F,ω,Q,P} = "] [{vs,F,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k + 5, ++m,
    Z[m + 1] = $M @ CF [
      1
      2 (m + 1)
      Sum[∂_{u,v} F (∂_{u,v} Z[m] + Sum[(∂_u Z[j]) (∂_v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]
    ];
  E[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k + 5}] /. Table[v → 0, {v, vs}]]]
]

```

```

In[ ]:= EZip23_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[
  {nP, nF, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
    CoefficientRules[#, vs] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^{ps}))
  ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^{ps}))];
  nF = Total[CoefficientRules[F, vs*] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^{ps}))];
  CF[Expand[<nF, E[ω, nQ, nP]> // Zip2_{vs} // Zip3_{vs} /. rr]
]

```

```

In[ ]:= m_{i_, j_ → k_} := E[_{i,j} → {k}] [1, ∑_{v=1}^3 (π_{v,i} ξ_{v,j} + (π_{v,i} + π_{v,j}) p_{v,k} + (ξ_{v,i} + ξ_{v,j}) x_{v,k}), eSeries[]]

```

```

In[*]:= m_{a,b \rightarrow c}
Out[*]:=
E_{\{a,b\} \rightarrow \{c\}} [1, p_{1,c} (\pi_{1,a} + \pi_{1,b}) + p_{2,c} (\pi_{2,a} + \pi_{2,b}) + p_{3,c} (\pi_{3,a} + \pi_{3,b}) + \pi_{1,a} \xi_{1,b} +
x_{1,c} (\xi_{1,a} + \xi_{1,b}) + \pi_{2,a} \xi_{2,b} + x_{2,c} (\xi_{2,a} + \xi_{2,b}) + \pi_{3,a} \xi_{3,b} + x_{3,c} (\xi_{3,a} + \xi_{3,b}), \in \text{Series}[0, 0]]

In[*]:= m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}
(m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) \equiv (m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})
Out[*]:=
E_{\{1,2,3\} \rightarrow \{1\}} [1,
p_{1,1} \pi_{1,1} + p_{1,1} \pi_{1,2} + p_{1,1} \pi_{1,3} + p_{2,1} \pi_{2,1} + p_{2,1} \pi_{2,2} + p_{2,1} \pi_{2,3} + p_{3,1} \pi_{3,1} + p_{3,1} \pi_{3,2} + p_{3,1} \pi_{3,3} + x_{1,1} \xi_{1,1} +
\pi_{1,1} \xi_{1,2} + x_{1,1} \xi_{1,2} + \pi_{1,1} \xi_{1,3} + \pi_{1,2} \xi_{1,3} + x_{1,1} \xi_{1,3} + x_{2,1} \xi_{2,1} + \pi_{2,1} \xi_{2,2} + x_{2,1} \xi_{2,2} + \pi_{2,1} \xi_{2,3} +
\pi_{2,2} \xi_{2,3} + x_{2,1} \xi_{2,3} + x_{3,1} \xi_{3,1} + \pi_{3,1} \xi_{3,2} + x_{3,1} \xi_{3,2} + \pi_{3,1} \xi_{3,3} + \pi_{3,2} \xi_{3,3} + x_{3,1} \xi_{3,3}, \in \text{Series}[0, 0]]

Out[*]:=
True

```

Some Knot Theory

```

In[*]:= Kink_i_ := CC_3 R_{1,2} // m_{2,3 \rightarrow 2} // m_{2,1 \rightarrow i};
Kink_i_ := CC_3 \bar{R}_{1,2} // m_{1,3 \rightarrow 1} // m_{1,2 \rightarrow i}

```

```

In[*]:= RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
  n = Length[pd]; rots = Table[0, {2 n}];
  xs = Cases[pd, x_X \Rightarrow {Xp[x[[4]], x[[1]] PositiveQ@x},
    {Xm[x[[2]], x[[1]] True}];
  For[k = 0, k < 2 n, ++k, If[k == 0 \vee FreeQ[front, -k],
    front = Flatten[front /. k \rightarrow {xs /. {
      Xp[k + 1, L_] | Xm[L_, k + 1] \Rightarrow {L, k + 1, 1 - L},
      Xp[L_, k + 1] | Xm[k + 1, L_] \Rightarrow {++rots[[L]]; {1 - L, k + 1, L}}
    }]],
    Cases[front, k | -k] /. {k, -k} \Rightarrow --rots[[k + 1]];
  ];
  RVK[xs, rots];
  RVK[K_] := RVK[PD[K]];

```

```

In[*]:= rot[i_, 0] := E_{\{\} \rightarrow \{i\}} [1, 0, eSeries[]];
rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CC_j, rot[i, n + 1] \bar{CC}_j] // m_{i,j \rightarrow i};

```

```

In[*]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
Module[{todo, n, rots, g, done, st, cx, g1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  g = E[{}->{0}][1, 0, eSeries[]];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{ } != todo,
    {cx} = MaximalBy[todo, Length[done ∩ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    {i, j} = List@@cx;
    g1 = Switch[Head[cx],
      Xp, (Ri,j Kinkk) // mj,k→j,
      Xm, (R̄i,j Kinkk) // mj,k→j,
    ];
    g1 = (rot[k, rots[[i]] g1) // mk,i→i; rots[[i]] = 0;
    g1 = (g1 rot[k, rots[[i + 1]]) // mi,k→i; rots[[i + 1]] = 0;
    g1 = (rot[k, rots[[j]] g1) // mk,j→j; rots[[j]] = 0;
    g1 = (g1 rot[k, rots[[j + 1]]) // mj,k→j; rots[[j + 1]] = 0;
    g *= g1;
    If[MemberQ[done, i], g = g // mi,i+1→i; st = st /. st[[i + 2]] → st[[i + 1]]];
    If[MemberQ[done, i - 1], g = g // mst[[i],i→st[[i]]; st = st /. st[[i + 1]] → st[[i]]];
    If[MemberQ[done, j], g = g // mj,j+1→j; st = st /. st[[j + 2]] → st[[j + 1]]];
    If[MemberQ[done, j - 1], g = g // mst[[j],j→st[[j]]; st = st /. st[[j + 1]] → st[[j]]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, cx];
  ];
  CF /@ (g (* /. {x0→x, y0→y, a0→a} *))
]

```

ρ_1 testing

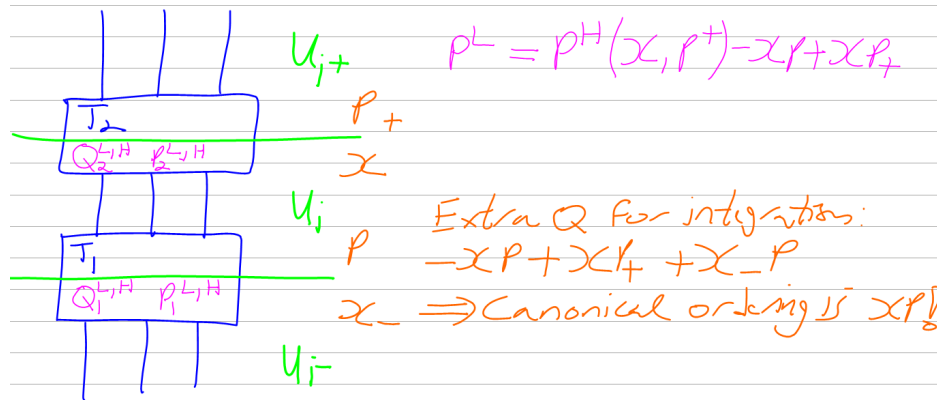
The pPush Lagrangian of Talks/Bonn-2505/Rho11AlternativeIntegrands.nb:

$$\ln[*] := \mathcal{L}[\mathbf{X}_{i,j}[\mathbf{s}_-]] := T^{S/2} \mathbb{E} \left[\left\{ \begin{aligned} & \mathbf{x}_i (\mathbf{p}_{i+1} - \mathbf{p}_i) + \mathbf{x}_j (\mathbf{p}_{j+1} - \mathbf{p}_j) + (T^S - 1) \mathbf{x}_i (\mathbf{p}_{i+1} - \mathbf{p}_{j+1}), \\ & 0, 0, \epsilon \left(-\frac{S}{2} + S T^S \mathbf{p}_{1+j} \mathbf{x}_i + \frac{1}{2} S T^S (-1 + T^S) \mathbf{p}_{1+i} \mathbf{p}_{1+j} \mathbf{x}_i^2 - \right. \right. \\ & \left. \left. \frac{1}{2} S T^S (-1 + T^S) \mathbf{p}_{1+j}^2 \mathbf{x}_i^2 - S \mathbf{p}_{1+j} \mathbf{x}_j - S T^S \mathbf{p}_{1+i} \mathbf{p}_{1+j} \mathbf{x}_i \mathbf{x}_j + S T^S \mathbf{p}_{1+j}^2 \mathbf{x}_i \mathbf{x}_j \right) \right\} \right] \end{aligned}$$

$$\mathcal{L}[\mathbf{C}_i[\varphi_-]] := T^{\varphi/2} \mathbb{E} \left[\left\{ \mathbf{x}_i (\mathbf{p}_{i+1} - \mathbf{p}_i), \epsilon \left(-\frac{\varphi}{2} - \varphi \mathbf{p}_{1+i} \mathbf{x}_i \right) \right\} \right]$$

$$\mathcal{L}[\mathbf{K}_-] := \mathbf{CF}[\mathcal{L} / @ \mathbf{Features}[\mathbf{K}]][[2]]$$

$$\mathbf{vs}[\mathbf{K}_-] := \mathbf{Join} @@ \mathbf{Table}[\{\mathbf{p}_i, \mathbf{x}_i\}, \{\mathbf{i}, \mathbf{Features}[\mathbf{K}][[1]]\}]$$



$$\begin{aligned} \ln[*] := & \mathbf{R}_{i,j} = \mathbf{CF}[\mathcal{L}[\mathbf{X}_{i,j}[1]] / . \omega_- \mathbb{E}[\{\mathbf{Q}_-, \mathbf{P}\theta_-, \mathbf{P1a}_-, \mathbf{P1b}_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i,j\}} [\\ & \omega, \mathbf{Q}, \epsilon \mathbf{Series}[\mathbf{P}\theta, \mathbf{P1a} + \mathbf{P1b} / . \epsilon \rightarrow 1] \\ &] / . \{\mathbf{p}_{i+1} \Rightarrow \mathbf{p}_i, \mathbf{p}_{j+1} \Rightarrow \mathbf{p}_j\} / . \{\mathbf{T} \rightarrow \mathbf{T}_1, \mathbf{x}_{\alpha_-} \Rightarrow \mathbf{x}_{1,\alpha}, \mathbf{p}_{\alpha_-} \Rightarrow \mathbf{p}_{1,\alpha}\}; \\ \bar{\mathbf{R}}_{i,j} = & \mathbf{CF}[\mathcal{L}[\mathbf{X}_{i,j}[-1]] / . \omega_- \mathbb{E}[\{\mathbf{Q}_-, \mathbf{P}\theta_-, \mathbf{P1a}_-, \mathbf{P1b}_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i,j\}} [\\ & \omega, \mathbf{Q}, \epsilon \mathbf{Series}[\mathbf{P}\theta, \mathbf{P1a} + \mathbf{P1b} / . \epsilon \rightarrow 1] \\ &] / . \{\mathbf{p}_{i+1} \Rightarrow \mathbf{p}_i, \mathbf{p}_{j+1} \Rightarrow \mathbf{p}_j\} / . \{\mathbf{T} \rightarrow \mathbf{T}_1, \mathbf{x}_{\alpha_-} \Rightarrow \mathbf{x}_{1,\alpha}, \mathbf{p}_{\alpha_-} \Rightarrow \mathbf{p}_{1,\alpha}\}; \\ \mathbf{CC}_{i_-} = & \mathbf{CF}[\mathcal{L}[\mathbf{C}_i[1]] / . \omega_- \mathbb{E}[\{\mathbf{Q}_-, \mathbf{P}_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i\}} [\\ & \omega, \mathbf{Q}, \epsilon \mathbf{Series}[\mathbf{0}, \mathbf{P} / . \epsilon \rightarrow 1] \\ &] / . \{\mathbf{p}_{i+1} \Rightarrow \mathbf{p}_i\} / . \{\mathbf{T} \rightarrow \mathbf{T}_1, \mathbf{x}_{\alpha_-} \Rightarrow \mathbf{x}_{1,\alpha}, \mathbf{p}_{\alpha_-} \Rightarrow \mathbf{p}_{1,\alpha}\}; \\ \overline{\mathbf{CC}}_{i_-} = & \mathbf{CF}[\mathcal{L}[\mathbf{C}_i[-1]] / . \omega_- \mathbb{E}[\{\mathbf{Q}_-, \mathbf{P}_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i\}} [\\ & \omega, \mathbf{Q}, \epsilon \mathbf{Series}[\mathbf{0}, \mathbf{P} / . \epsilon \rightarrow 1] \\ &] / . \{\mathbf{p}_{i+1} \Rightarrow \mathbf{p}_i\} / . \{\mathbf{T} \rightarrow \mathbf{T}_1, \mathbf{x}_{\alpha_-} \Rightarrow \mathbf{x}_{1,\alpha}, \mathbf{p}_{\alpha_-} \Rightarrow \mathbf{p}_{1,\alpha}\}; \end{aligned}$$

```

In[*]:= $k = 1;
Ri,j
R̄i,j
CCi
CC̄i

Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[ \sqrt{T_1}, -p_{1,i} x_{1,i} + T_1 p_{1,i} x_{1,i} + (1 - T_1) p_{1,j} x_{1,i}, \right.$$


$$\in \text{Series} \left[ 0, -\frac{1}{2} + T_1 p_{1,j} x_{1,i} + \frac{1}{2} (-1 + T_1) T_1 p_{1,i} p_{1,j} x_{1,i}^2 - \right.$$


$$\left. \frac{1}{2} (-1 + T_1) T_1 p_{1,j}^2 x_{1,i}^2 - p_{1,j} x_{1,j} - T_1 p_{1,i} p_{1,j} x_{1,i} x_{1,j} + T_1 p_{1,j}^2 x_{1,i} x_{1,j} \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[ \frac{1}{\sqrt{T_1}}, -p_{1,i} x_{1,i} + \frac{p_{1,i} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,j} x_{1,i}}{T_1}, \right.$$


$$\in \text{Series} \left[ 0, \frac{1}{2} - \frac{p_{1,j} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,i} p_{1,j} x_{1,i}^2}{2 T_1^2} - \right.$$


$$\left. \frac{(-1 + T_1) p_{1,j}^2 x_{1,i}^2}{2 T_1^2} + p_{1,j} x_{1,j} + \frac{p_{1,i} p_{1,j} x_{1,i} x_{1,j}}{T_1} - \frac{p_{1,j}^2 x_{1,i} x_{1,j}}{T_1} \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i\}} \left[ \sqrt{T_1}, 0, \in \text{Series} \left[ 0, -\frac{1}{2} - p_{1,i} x_{1,i} \right] \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i\}} \left[ \frac{1}{\sqrt{T_1}}, 0, \in \text{Series} \left[ 0, \frac{1}{2} + p_{1,i} x_{1,i} \right] \right]$$


In[*]:= $k = 1; {
  {"CC", (CC1 CC̄2 // m1,2→1) ≡ E{ }→{1} [1, 0, eSeries[]]},
  {"An R1", (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (CC̄3 R1,2 // m1,3→1 // m1,2→1)},
  {"R2b", (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ E{ }→{1,2} [1, 0, eSeries[]]},
  {"R3", (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3) }
}

Out[*]=
{{CC, True}, {An R1, True}, {R2b, True}, {R3, True}}

In[*]:= NewBit[K_] := Module[{Alex = Alexander[K][T1]},
  Alex2 Z[K][[3, 2]] // Factor]

In[*]:= NewBit /@ AllKnots[{3, 5}]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\left\{ \frac{(-1 + T_1)^2 (1 + T_1^2)}{T_1^2}, 0, \frac{(-1 + T_1)^2 (1 + T_1^2) (2 + T_1^2 + 2 T_1^4)}{T_1^4}, \frac{(-1 + T_1)^2 (5 - 4 T_1 + 5 T_1^2)}{T_1^2} \right\}$$


```

In papers/APAI, page 7, that was:

$$\left\{ \frac{(-1+T_1)^2 (1+T_1^2)}{T_1^2}, 0, \frac{(-1+T_1)^2 (1+T_1^2) (2+T_1^2+2 T_1^4)}{T_1^4}, \frac{(-1+T_1)^2 (5-4 T_1+5 T_1^2)}{T_1^2} \right\}$$

And now θ

```
In[*]:= $π[ε_] := (CF[ε /. {p_{αβ}__ => B^-1 p_{αβ}, x_{αβ}__ => B x_{αβ}}] /. B -> 0)
$M[eSeries[P0_, P1_]] := eSeries[P0, $π[P1]];
```

The Lagrangian is taken from ThetaAlternativeIntegrands.nb.

```
In[*]:= T3 = T1 T2;
```

```
ℒppush[Xi_, j_ [S_]] :=
```

$$\begin{aligned} & (T_1 T_2)^S \mathbb{E} \left[\frac{S \epsilon}{2} - p_{1,i} x_{1,i} + T_1^S p_{1,1+i} x_{1,i} + (1 - T_1^S) p_{1,1+j} x_{1,i} - p_{1,j} x_{1,j} + p_{1,1+j} x_{1,j} - p_{2,i} x_{2,i} + \right. \\ & T_2^S p_{2,1+i} x_{2,i} - S \epsilon T_2^S p_{2,1+i} x_{2,i} + (1 - T_2^S) p_{2,1+j} x_{2,i} + S \epsilon T_1^S T_2^S p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,i} + \\ & \frac{S \epsilon (-1 + T_1^S) T_2^S p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,i}}{-1 + T_2^S} + (-1 + T_1^S) T_2^S p_{3,1+j} x_{1,i} x_{2,i} - p_{2,j} x_{2,j} + p_{2,1+j} x_{2,j} + \\ & S \epsilon p_{2,1+j} x_{2,j} - S \epsilon T_1^S p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,j} - \frac{S \epsilon (-1 + T_1^S) p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,j}}{-1 + T_2^S} + (1 - T_1^S) p_{3,1+j} \\ & x_{1,i} x_{2,j} + \frac{S \epsilon T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{2,1+i} x_{3,i}}{-1 + T_2^S} - \frac{S \epsilon T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{2,1+j} x_{3,i}}{-1 + T_2^S} - \\ & p_{3,i} x_{3,i} + (T_1 T_2)^S p_{3,1+i} x_{3,i} + S \epsilon (T_1 T_2)^S p_{3,1+i} x_{3,i} + (1 - (T_1 T_2)^S) p_{3,1+j} x_{3,i} - \\ & \frac{S \epsilon T_2^S (-1 + (T_1 T_2)^S) p_{3,1+j} x_{3,i}}{-1 + T_2^S} - \frac{S \epsilon T_1^S T_2^S (-1 + (T_1 T_2)^S) p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^S} + \\ & \frac{S \epsilon (-1 + T_1^S) T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^S} - S \epsilon (T_1 T_2)^S (-1 + T_2^S) p_{2,1+j} p_{3,1+i} x_{2,i} x_{3,i} + \\ & S \epsilon T_2^S (-1 + (T_1 T_2)^S) p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,i} + 2 S \epsilon (T_1 T_2)^S p_{2,1+j} p_{3,1+i} x_{2,j} x_{3,i} + \\ & \frac{S \epsilon T_2^S (-1 + (T_1 T_2)^S) p_{2,1+i} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^S} - \frac{S \epsilon (-1 + 2 T_2^S) (-1 + (T_1 T_2)^S) p_{2,1+j} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^S} - \\ & p_{3,j} x_{3,j} + p_{3,1+j} x_{3,j} + S \epsilon T_1^S p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,j} + \frac{S \epsilon (-1 + T_1^S) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,j}}{-1 + T_2^S} - \\ & \left. S \epsilon T_2^S p_{2,1+i} p_{3,1+j} x_{2,i} x_{3,j} - S \epsilon p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,j} \right] \end{aligned}$$

```
In[*]:= ℒppush[Ci_ [φ_]] :=
```

$$(T_1 T_2)^\varphi \mathbb{E} \left[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \epsilon \left(\frac{1}{2} + p_{3,1+i} x_{3,i} \right) \right]$$

```

In[*]:= $k = 1;

Ri,j = CF[ $\mathcal{L}_{\text{pPush}}[X_{i,j}[1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1,  $P0 = \{0, 0\}$  /. L1,  $P1a = \{1, 2\}$  /. L1,  $P1b = \{1, 1\}$  /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i,j\}\}}[$ 
     $\omega, \sum_{v=1}^3 ((T_v - 1) x_{v,i} (p_{v,i} - p_{v,j}))$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}, p_{v,j+1} \Rightarrow p_{v,j}$ }
];

R̄i,j = CF[ $\mathcal{L}_{\text{pPush}}[X_{i,j}[-1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1,  $P0 = \{0, 0\}$  /. L1,  $P1a = \{1, 2\}$  /. L1,  $P1b = \{1, 1\}$  /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i,j\}\}}[$ 
     $\omega, \sum_{v=1}^3 ((T_v^{-1} - 1) x_{v,i} (p_{v,i} - p_{v,j}))$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}, p_{v,j+1} \Rightarrow p_{v,j}$ }
];

CCi = CF[ $\mathcal{L}_{\text{pPush}}[C_i[1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1,  $P0 = 0, P1a = 0, P1b = \{1, 1\}$  /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i\}\}}[$ 
     $\omega, 0$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}$ }
];

CC̄i = CF[ $\mathcal{L}_{\text{pPush}}[C_i[-1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1,  $P0 = 0, P1a = 0, P1b = \{1, 1\}$  /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i\}\}}[$ 
     $\omega, 0$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}$ }
];

```

In[*]:= **R_{i,j}**

Out[*]=

$$\begin{aligned} & \mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[T_1 T_2, (-1 + T_1) p_{1,i} x_{1,i} + (1 - T_1) p_{1,j} x_{1,i} + \right. \\ & \quad (-1 + T_2) p_{2,i} x_{2,i} + (1 - T_2) p_{2,j} x_{2,i} + (-1 + T_1 T_2) p_{3,i} x_{3,i} + (1 - T_1 T_2) p_{3,j} x_{3,i}, \\ & \left. \in \text{Series} \left[(-1 + T_1) T_2 p_{3,j} x_{1,i} x_{2,i} + (1 - T_1) p_{3,j} x_{1,i} x_{2,j}, \frac{1}{2} - T_2 p_{2,i} x_{2,i} + T_1 T_2 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + \right. \right. \\ & \quad \frac{(-1 + T_1) T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{-1 + T_2} + p_{2,j} x_{2,j} - T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j} - \frac{(-1 + T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + \\ & \quad \frac{T_2 (-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{-1 + T_2} - \frac{T_2 (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{-1 + T_2} + T_1 T_2 p_{3,i} x_{3,i} - \frac{T_2 (-1 + T_1 T_2) p_{3,j} x_{3,i}}{-1 + T_2} - \\ & \quad \frac{T_1 T_2 (-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{-1 + T_2} + \frac{(-1 + T_1) T_2 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{-1 + T_2} - \\ & \quad \frac{T_1 (-1 + T_2) T_2 p_{2,j} p_{3,i} x_{2,i} x_{3,i} + T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + 2 T_1 T_2 p_{2,j} p_{3,i} x_{2,j} x_{3,i} +}{-1 + T_2} \\ & \quad \left. \frac{T_2 (-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_2} - \frac{(-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_2} + \right. \\ & \quad \left. T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j} + \frac{(-1 + T_1) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_2} - T_2 p_{2,i} p_{3,j} x_{2,i} x_{3,j} - p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right] \end{aligned}$$

In[*]:= **$\bar{R}_{i,j}$**

Out[*]=

$$\begin{aligned} & \mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[\frac{1}{T_1 T_2}, -\frac{(-1 + T_1) p_{1,i} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,j} x_{1,i}}{T_1} - \right. \\ & \quad \frac{(-1 + T_2) p_{2,i} x_{2,i}}{T_2} + \frac{(-1 + T_2) p_{2,j} x_{2,i}}{T_2} - \frac{(-1 + T_1 T_2) p_{3,i} x_{3,i}}{T_1 T_2} + \frac{(-1 + T_1 T_2) p_{3,j} x_{3,i}}{T_1 T_2}, \\ & \left. \in \text{Series} \left[-\frac{(-1 + T_1) p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2} + \frac{(-1 + T_1) p_{3,j} x_{1,i} x_{2,j}}{T_1}, \right. \right. \\ & \quad -\frac{1}{2} + \frac{p_{2,i} x_{2,i}}{T_2} - \frac{p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{T_1 T_2} - \frac{(-1 + T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1 (-1 + T_2)} - p_{2,j} x_{2,j} + \frac{p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{T_1} + \\ & \quad \frac{(-1 + T_1) T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1 (-1 + T_2)} - \frac{(-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{T_1 (-1 + T_2) T_2} + \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \frac{p_{3,i} x_{3,i}}{T_1 T_2} + \\ & \quad \frac{(-1 + T_1 T_2) p_{3,j} x_{3,i}}{T_1 (-1 + T_2) T_2} + \frac{(-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 (-1 + T_2) T_2} + \frac{(-1 + T_1) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 (-1 + T_2) T_2} - \\ & \quad \frac{(-1 + T_2) p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_1 T_2^2} + \frac{(-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2^2} - \frac{2 p_{2,j} p_{3,i} x_{2,j} x_{3,i}}{T_1 T_2} - \\ & \quad \frac{(-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \frac{(-2 + T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \\ & \quad \left. \frac{p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{T_1} - \frac{(-1 + T_1) T_2 p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1 (-1 + T_2)} + \frac{p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{T_2} + p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right] \end{aligned}$$

In[*]:= **$\mathcal{L}_{\text{pPush}}[\mathbf{C}_i[1]]$**

Out[*]=

$$T_1 T_2 \mathbb{E} \left[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \left(\frac{1}{2} + p_{3,1+i} x_{3,i} \right) \right]$$

```

In[*]:= CC1 // CF
Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{1\}} \left[ T_1 T_2, 0, \text{Series} \left[ 0, \frac{1}{2} + p_{3,1} x_{3,1} \right] \right]$$


In[*]:= CC1 // CF
Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{1\}} \left[ \frac{1}{T_1 T_2}, 0, \text{Series} \left[ 0, -\frac{1}{2} - p_{3,1} x_{3,1} \right] \right]$$


In[*]:= $k = 0; {
  {"CC", (CC1 CC2 // m1,2→1) ≡ E_{\{\} \rightarrow \{1\}} [1, 0, Series[]]},
  {"An R1", (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (CC3 R1,2 // m1,3→1 // m1,2→1)},
  {"R2b", (R1,2 R3,4 // m1,3→1 // m2,4→2) ≡ E_{\{\} \rightarrow \{1,2\}} [1, 0, Series[]]},
  {"R3", (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)}
}
Out[*]=
{{CC, True}, {An R1, True}, {R2b, True}, {R3, True}}

In[*]:= $k = 1; HL[ (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (CC3 R1,2 // m1,3→1 // m1,2→1) ]
Out[*]=
True

In[*]:= R1,2 R3,4 // m1,3→1 // m2,4→2
Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{1,2\}} [1, 0, \text{Series}[0, 0]]$$


In[*]:= $k = 1; HL[ (R1,2 R3,4 // m1,3→1 // m2,4→2) ≡ E_{\{\} \rightarrow \{1,2\}} [1, 0, Series[]] ]
Out[*]=
True

In[*]:= $k = 1; HL[ (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3) ]
Out[*]=
True

In[*]:= Z[Knot[3, 1]]
Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[ \frac{T_1^2 T_2^2}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)}, 0, \right. \\ \left. \text{Series} \left[ 0, -\frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)} \right] \right]$$


In[*]:= Θ[K_] := Z[K] /. E_{\{\} \rightarrow \{\emptyset\}} [d_, 0, Series[0, th_]] => CCF@Module[{d1},
  d1 = PowerExpand[d^{1/2} /. {T1 -> T, T2 -> 1}];
  {(d1 /. T -> 1) / d1, th / d}
]

```

```
In[*]:=  $\Theta$ [Knot[3, 1]]
```

```
Out[*]=
```

$$\left\{ \frac{1 - T + T^2}{T}, -\frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{T_1^2 T_2^2} \right\}$$

```
In[*]:=  $\Theta$ [Knot[4, 1]]
```

```
Out[*]=
```

$$\left\{ -\frac{1 - 3T + T^2}{T}, 0 \right\}$$

```
In[*]:= Table[
```

```
  Echo@AbsoluteTiming@PolyPlot[Th =  $\Theta$ [K]];
```

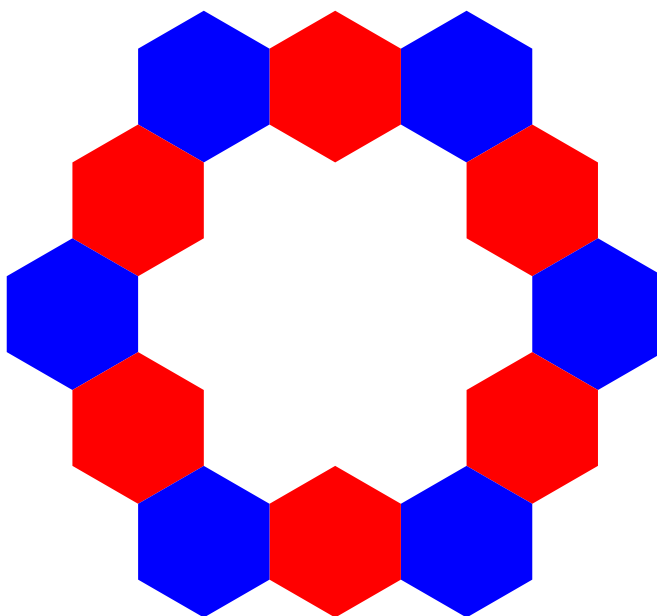
```
  K → Th,
```

```
  {K, AllKnots[{3, 7]}}
```

```
]
```



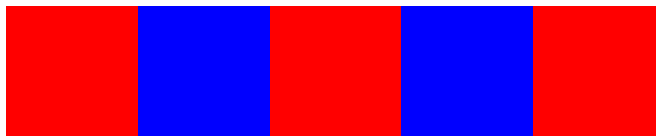
```
» {204.75,
```



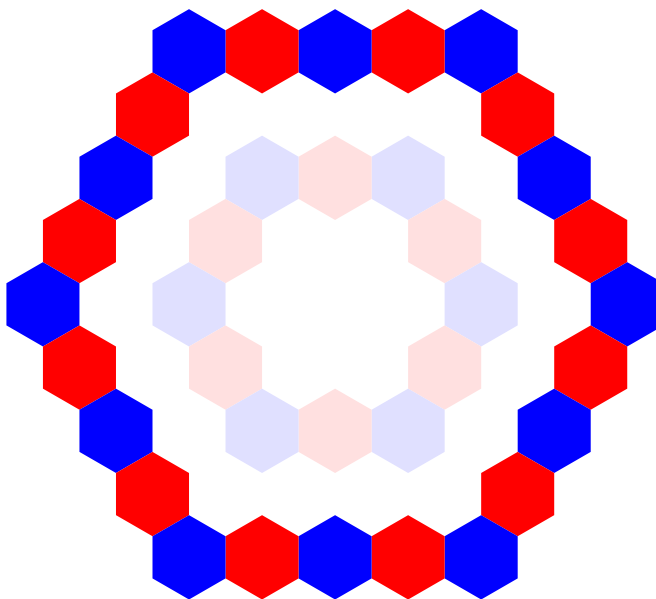
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}
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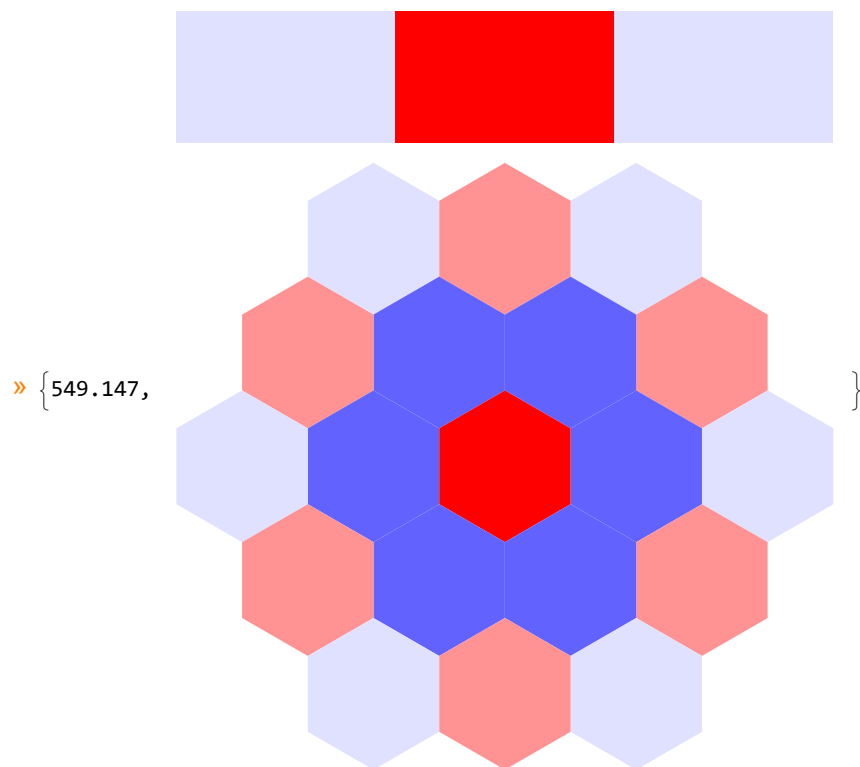
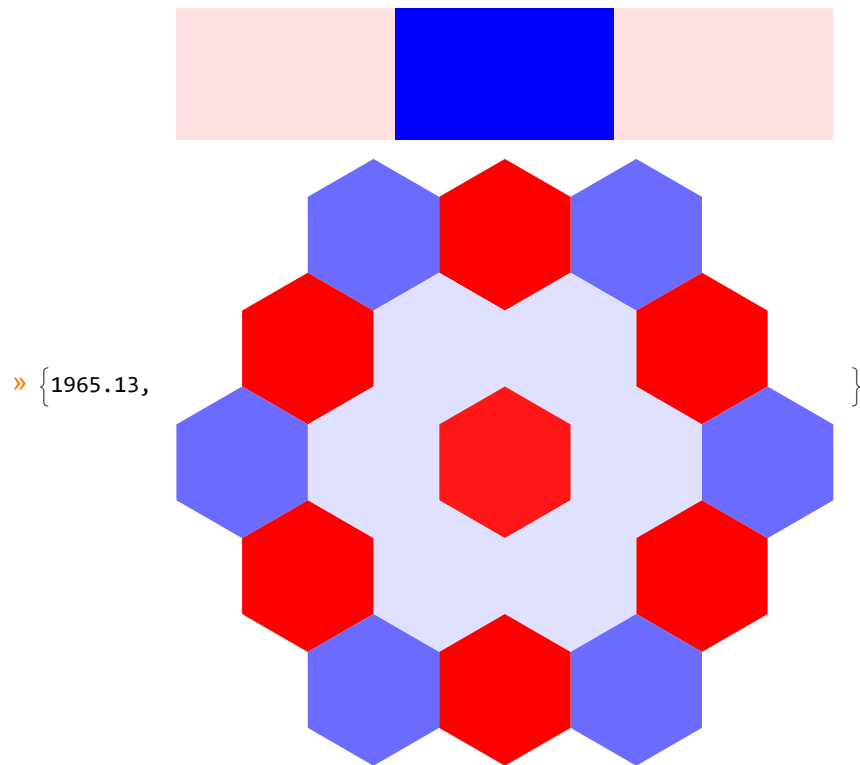


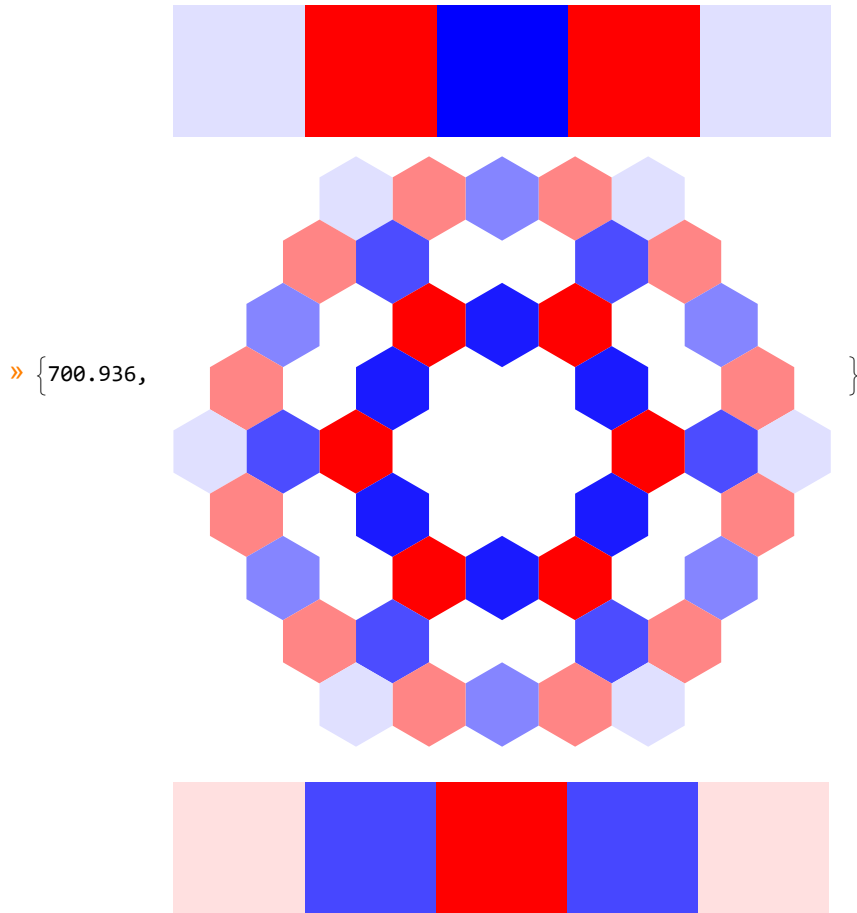
» {140.015, }



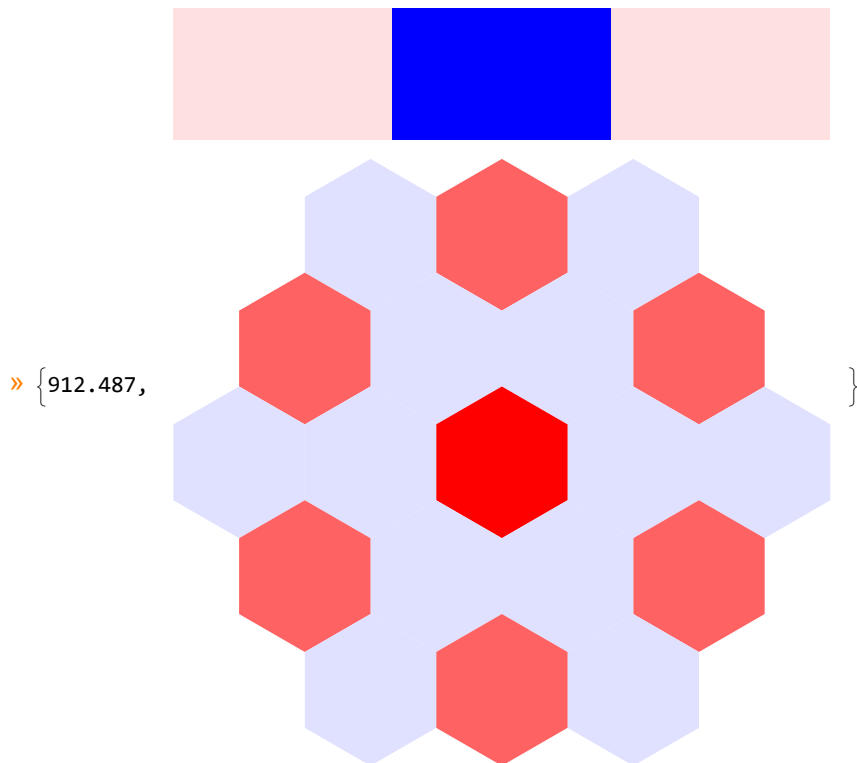
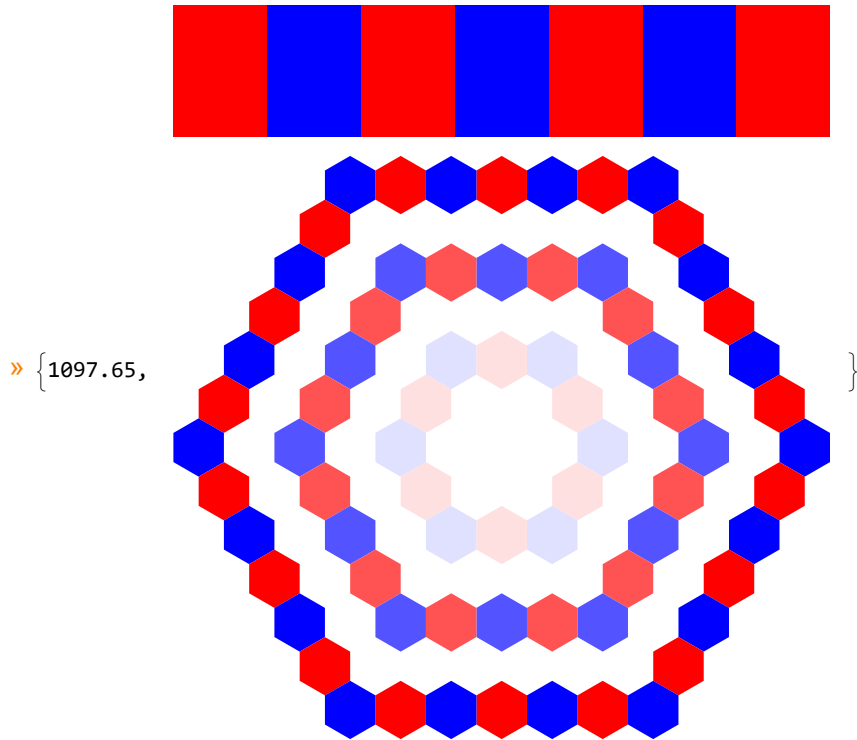
» {537.548, }

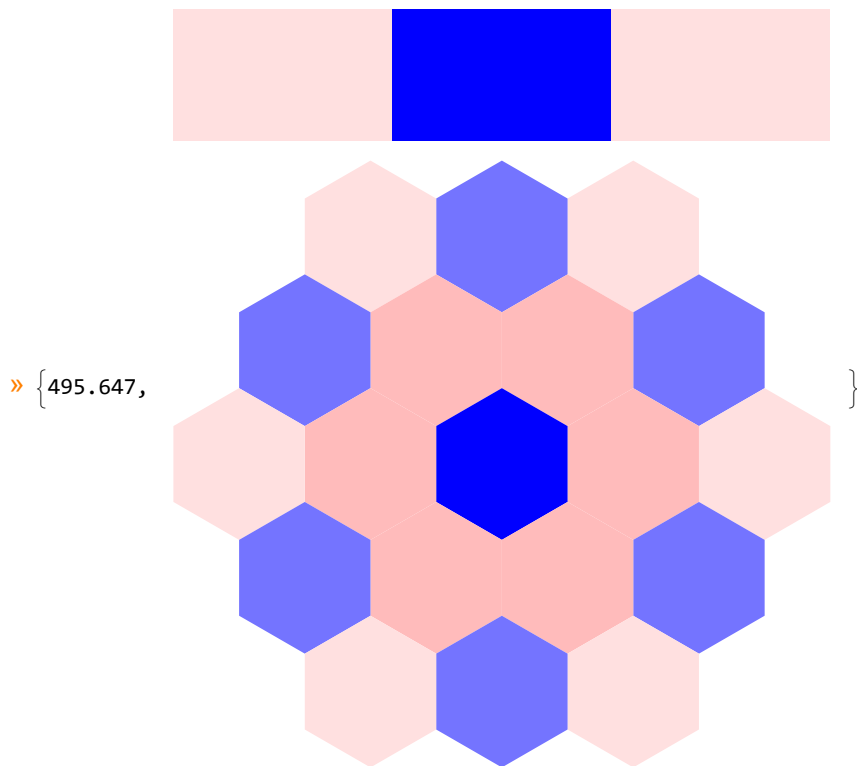
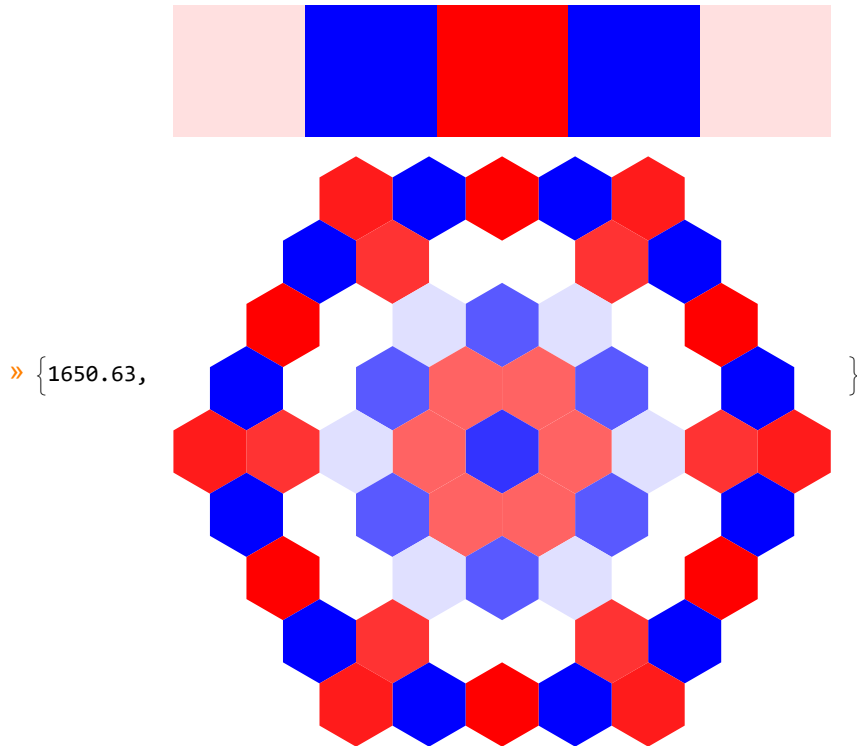


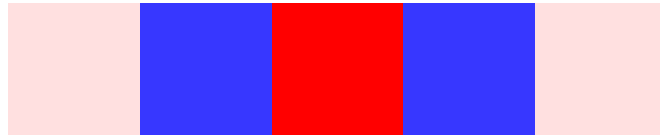




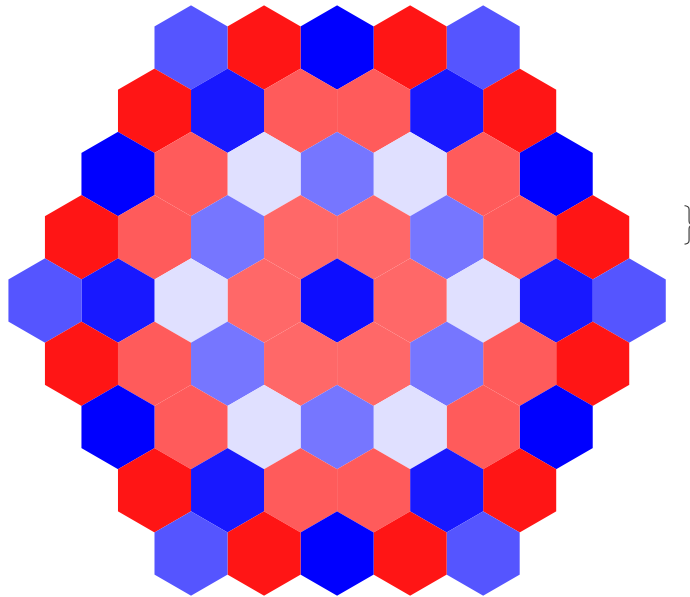
» {513.319, }



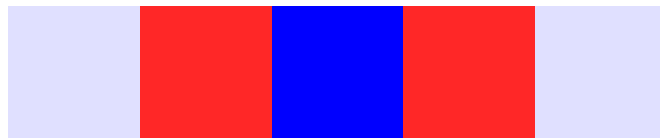




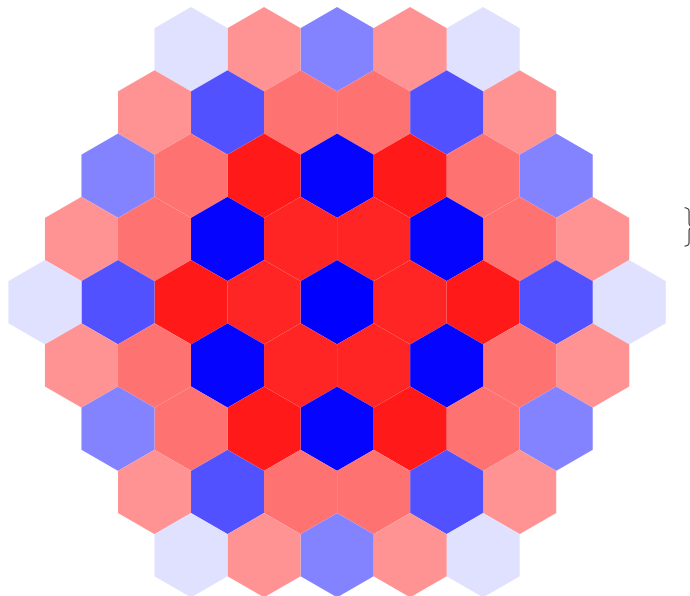
» {1195.17,



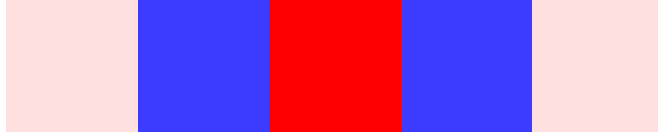
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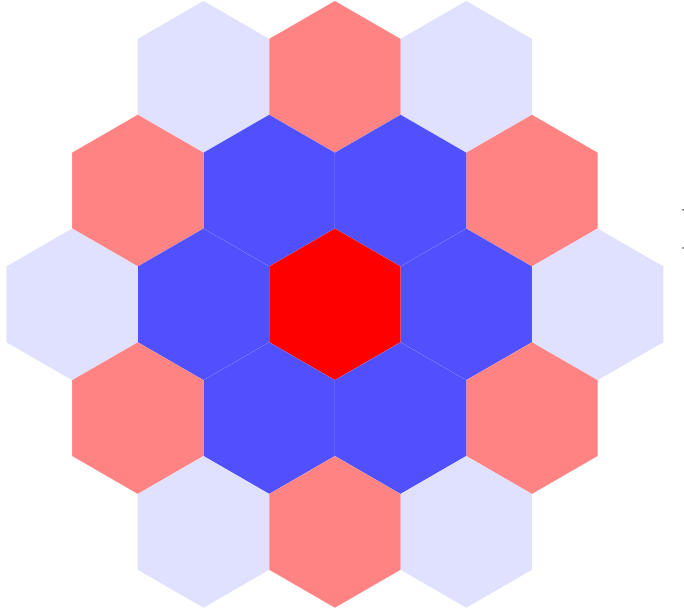
» {1749.92,



}



» {2270.15,



Out[8]=

$$\left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1 - T + T^2}{T}, -\frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{T_1^2 T_2^2} \right\}, \right.$$

$$\left. \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1 - 3T + T^2}{T}, \emptyset \right\}, \text{Knot}[5, 1] \rightarrow \left\{ \frac{1 - T + T^2 - T^3 + T^4}{T^2}, \right.$$

$$\left. -\frac{1}{T_1^4 T_2^4} \left(2 - 2T_1 + 2T_1^2 - 2T_1^3 + 2T_1^4 - 2T_2 - 2T_1^5 T_2 + 2T_2^2 + T_1^2 T_2^2 - T_1^3 T_2^2 + T_1^4 T_2^2 + 2T_1^6 T_2^2 - 2T_2^3 - T_1^2 T_2^3 - \right. \right.$$

$$\left. T_1^5 T_2^3 - 2T_1^7 T_2^3 + 2T_2^4 + T_1^2 T_2^4 + T_1^6 T_2^4 + 2T_1^8 T_2^4 - 2T_1 T_2^5 - T_1^3 T_2^5 - T_1^6 T_2^5 - 2T_1^8 T_2^5 + 2T_1^2 T_2^6 + T_1^4 T_2^6 - \right.$$

$$\left. T_1^5 T_2^6 + T_1^6 T_2^6 + 2T_1^8 T_2^6 - 2T_1^3 T_2^7 - 2T_1^8 T_2^7 + 2T_1^4 T_2^8 - 2T_1^5 T_2^8 + 2T_1^6 T_2^8 - 2T_1^7 T_2^8 + 2T_1^8 T_2^8 \right) \left. \right\},$$

$$\left. \text{Knot}[5, 2] \rightarrow \left\{ \frac{2 - 3T + 2T^2}{T}, -\frac{1}{T_1^2 T_2^2} \left(9 - 13T_1 + 9T_1^2 - 13T_2 + 6T_1 T_2 + 6T_1^2 T_2 - 13T_1^3 T_2 + 9T_2^2 + 6T_1 T_2^2 - \right. \right.$$

$$\left. 12T_1^2 T_2^2 + 6T_1^3 T_2^2 + 9T_1^4 T_2^2 - 13T_1 T_2^3 + 6T_1^2 T_2^3 + 6T_1^3 T_2^3 - 13T_1^4 T_2^3 + 9T_1^2 T_2^4 - 13T_1^3 T_2^4 + 9T_1^4 T_2^4 \right) \right\},$$

$$\left. \text{Knot}[6, 1] \rightarrow \left\{ -\frac{(-2 + T)(-1 + 2T)}{T}, -\frac{1}{T_1^2 T_2^2} \left(1 - 3T_1 + T_1^2 - 3T_2 + 6T_1 T_2 + 6T_1^2 T_2 - 3T_1^3 T_2 + T_2^2 + \right. \right.$$

$$\left. 6T_1 T_2^2 - 24T_1^2 T_2^2 + 6T_1^3 T_2^2 + T_1^4 T_2^2 - 3T_1 T_2^3 + 6T_1^2 T_2^3 + 6T_1^3 T_2^3 - 3T_1^4 T_2^3 + T_1^2 T_2^4 - 3T_1^3 T_2^4 + T_1^4 T_2^4 \right) \right\},$$

$$\left. \text{Knot}[6, 2] \rightarrow \left\{ -\frac{1 - 3T + 3T^2 - 3T^3 + T^4}{T^2}, -\frac{1}{T_1^4 T_2^4} \left(1 - 3T_1 + 3T_1^2 - 3T_1^3 + T_1^4 - 3T_2 + 6T_1 T_2 + \right. \right.$$

$$\left. 6T_1^4 T_2 - 3T_1^5 T_2 + 3T_2^2 - 15T_1^2 T_2^2 + 11T_1^3 T_2^2 - 15T_1^4 T_2^2 + 3T_1^6 T_2^2 - 3T_2^3 + 11T_1^2 T_2^3 + \right.$$

$$\left. 11T_1^5 T_2^3 - 3T_1^7 T_2^3 + T_2^4 + 6T_1 T_2^4 - 15T_1^2 T_2^4 - 15T_1^6 T_2^4 + 6T_1^7 T_2^4 + T_1^8 T_2^4 - 3T_1 T_2^5 + \right.$$

$$\left. 11T_1^3 T_2^5 + 11T_1^6 T_2^5 - 3T_1^8 T_2^5 + 3T_1^2 T_2^6 - 15T_1^4 T_2^6 + 11T_1^5 T_2^6 - 15T_1^6 T_2^6 + 3T_1^8 T_2^6 - \right.$$

$$\left. 3T_1^3 T_2^7 + 6T_1^4 T_2^7 + 6T_1^7 T_2^7 - 3T_1^8 T_2^7 + T_1^4 T_2^8 - 3T_1^5 T_2^8 + 3T_1^6 T_2^8 - 3T_1^7 T_2^8 + T_1^8 T_2^8 \right) \left. \right\},$$

$$\begin{aligned}
& \text{Knot}[6, 3] \rightarrow \left\{ \frac{1 - 3T + 5T^2 - 3T^3 + T^4}{T^2}, 0 \right\}, \text{Knot}[7, 1] \rightarrow \\
& \left\{ \frac{1 - T + T^2 - T^3 + T^4 - T^5 + T^6}{T^3}, \right. \\
& - \frac{1}{T_1^6 T_2^6} \left(3 - 3T_1 + 3T_1^2 - 3T_1^3 + 3T_1^4 - 3T_1^5 + 3T_1^6 - 3T_2 - 3T_1^7 T_2 + 3T_2^2 + 2T_1^2 T_2^2 - 2T_1^3 T_2^2 + 2T_1^4 T_2^2 - \right. \\
& 2T_1^5 T_2^2 + 2T_1^6 T_2^2 + 3T_1^8 T_2^2 - 3T_2^3 - 2T_1^2 T_2^3 - 2T_1^7 T_2^3 - 3T_1^9 T_2^3 + 3T_2^4 + 2T_1^2 T_2^4 + T_1^4 T_2^4 - T_1^5 T_2^4 + \\
& T_1^6 T_2^4 + 2T_1^8 T_2^4 + 3T_1^{10} T_2^4 - 3T_2^5 - 2T_1^2 T_2^5 - T_1^4 T_2^5 - T_1^7 T_2^5 - 2T_1^9 T_2^5 - 3T_1^{11} T_2^5 + 3T_2^6 + 2T_1^2 T_2^6 + \\
& T_1^4 T_2^6 + T_1^8 T_2^6 + 2T_1^{10} T_2^6 + 3T_1^{12} T_2^6 - 3T_1^7 T_2^7 - 2T_1^3 T_2^7 - T_1^5 T_2^7 - T_1^8 T_2^7 - 2T_1^{10} T_2^7 - 3T_1^{12} T_2^7 + 3T_1^2 T_2^8 + \\
& 2T_1^4 T_2^8 + T_1^6 T_2^8 - T_1^7 T_2^8 + T_1^8 T_2^8 + 2T_1^{10} T_2^8 + 3T_1^{12} T_2^8 - 3T_1^3 T_2^9 - 2T_1^5 T_2^9 - 2T_1^{10} T_2^9 - 3T_1^{12} T_2^9 + 3T_1^4 T_2^{10} + \\
& 2T_1^6 T_2^{10} - 2T_1^7 T_2^{10} + 2T_1^8 T_2^{10} - 2T_1^9 T_2^{10} + 2T_1^{10} T_2^{10} + 3T_1^{12} T_2^{10} - 3T_1^5 T_2^{11} - 3T_1^{12} T_2^{11} + 3T_1^6 T_2^{12} - \\
& \left. \left. 3T_1^7 T_2^{12} + 3T_1^8 T_2^{12} - 3T_1^9 T_2^{12} + 3T_1^{10} T_2^{12} - 3T_1^{11} T_2^{12} + 3T_1^{12} T_2^{12} \right) \right\}, \text{Knot}[7, 2] \rightarrow \left\{ \frac{3 - 5T + 3T^2}{T}, \right. \\
& - \frac{1}{T_1^2 T_2^2} 2 \left(18 - 29T_1 + 18T_1^2 - 29T_2 + 18T_1 T_2 + 18T_1^2 T_2 - 29T_1^3 T_2 + 18T_2^2 + 18T_1 T_2^2 - 42T_1^2 T_2^2 + \right. \\
& \left. 18T_1^3 T_2^2 + 18T_1^4 T_2^2 - 29T_1 T_2^3 + 18T_1^2 T_2^3 + 18T_1^3 T_2^3 - 29T_1^4 T_2^3 + 18T_1^2 T_2^4 - 29T_1^3 T_2^4 + 18T_1^4 T_2^4 \right) \left. \right\}, \\
& \text{Knot}[7, 3] \rightarrow \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, \frac{1}{T_1^4 T_2^4} \left(17 - 25T_1 + 25T_1^2 - 25T_1^3 + 17T_1^4 - 25T_2 + \right. \right. \\
& 12T_1 T_2 + 12T_1^4 T_2 - 25T_1^5 T_2 + 25T_2^2 - T_1^2 T_2^2 - 7T_1^3 T_2^2 - T_1^4 T_2^2 + 25T_1^6 T_2^2 - 25T_2^3 - 7T_1^2 T_2^3 + \\
& 6T_1^3 T_2^3 + 6T_1^4 T_2^3 - 7T_1^5 T_2^3 - 25T_1^7 T_2^3 + 17T_2^4 + 12T_1 T_2^4 - T_1^2 T_2^4 + 6T_1^3 T_2^4 - 12T_1^4 T_2^4 + 6T_1^5 T_2^4 - \\
& T_1^6 T_2^4 + 12T_1^7 T_2^4 + 17T_1^8 T_2^4 - 25T_1 T_2^5 - 7T_1^3 T_2^5 + 6T_1^4 T_2^5 + 6T_1^5 T_2^5 - 7T_1^6 T_2^5 - 25T_1^8 T_2^5 + \\
& 25T_1^2 T_2^6 - T_1^4 T_2^6 - 7T_1^5 T_2^6 - T_1^6 T_2^6 + 25T_1^8 T_2^6 - 25T_1^3 T_2^7 + 12T_1^4 T_2^7 + 12T_1^7 T_2^7 - 25T_1^8 T_2^7 + \\
& \left. 17T_1^4 T_2^8 - 25T_1^5 T_2^8 + 25T_1^6 T_2^8 - 25T_1^7 T_2^8 + 17T_1^8 T_2^8 \right) \left. \right\}, \text{Knot}[7, 4] \rightarrow \left\{ \frac{4 - 7T + 4T^2}{T}, \right. \\
& \frac{1}{T_1^2 T_2^2} 8 \left(10 - 17T_1 + 10T_1^2 - 17T_2 + 12T_1 T_2 + 12T_1^2 T_2 - 17T_1^3 T_2 + 10T_2^2 + 12T_1 T_2^2 - 30T_1^2 T_2^2 + \right. \\
& \left. 12T_1^3 T_2^2 + 10T_1^4 T_2^2 - 17T_1 T_2^3 + 12T_1^2 T_2^3 + 12T_1^3 T_2^3 - 17T_1^4 T_2^3 + 10T_1^2 T_2^4 - 17T_1^3 T_2^4 + 10T_1^4 T_2^4 \right) \left. \right\}, \\
& \text{Knot}[7, 5] \rightarrow \left\{ \frac{2 - 4T + 5T^2 - 4T^3 + 2T^4}{T^2}, -\frac{1}{T_1^4 T_2^4} \left(17 - 33T_1 + 41T_1^2 - 33T_1^3 + 17T_1^4 - 33T_2 + \right. \right. \\
& 32T_1 T_2 - 16T_1^2 T_2 - 16T_1^3 T_2 + 32T_1^4 T_2 - 33T_1^5 T_2 + 41T_2^2 - 16T_1 T_2^2 + 4T_1^2 T_2^2 + 12T_1^3 T_2^2 + \\
& 4T_1^4 T_2^2 - 16T_1^5 T_2^2 + 41T_1^6 T_2^2 - 33T_2^3 - 16T_1 T_2^3 + 12T_1^2 T_2^3 - 14T_1^3 T_2^3 - 14T_1^4 T_2^3 + 12T_1^5 T_2^3 - \\
& 16T_1^6 T_2^3 - 33T_1^7 T_2^3 + 17T_2^4 + 32T_1 T_2^4 + 4T_1^2 T_2^4 - 14T_1^3 T_2^4 + 36T_1^4 T_2^4 - 14T_1^5 T_2^4 + 4T_1^6 T_2^4 + 32T_1^7 T_2^4 + \\
& 17T_1^8 T_2^4 - 33T_1 T_2^5 - 16T_1^2 T_2^5 + 12T_1^3 T_2^5 - 14T_1^4 T_2^5 - 14T_1^5 T_2^5 + 12T_1^6 T_2^5 - 16T_1^7 T_2^5 - 33T_1^8 T_2^5 + \\
& 41T_1^2 T_2^6 - 16T_1^3 T_2^6 + 4T_1^4 T_2^6 + 12T_1^5 T_2^6 + 4T_1^6 T_2^6 - 16T_1^7 T_2^6 + 41T_1^8 T_2^6 - 33T_1^3 T_2^7 + 32T_1^4 T_2^7 - \\
& \left. 16T_1^5 T_2^7 - 16T_1^6 T_2^7 + 32T_1^7 T_2^7 - 33T_1^8 T_2^7 + 17T_1^4 T_2^8 - 33T_1^5 T_2^8 + 41T_1^6 T_2^8 - 33T_1^7 T_2^8 + 17T_1^8 T_2^8 \right) \left. \right\}, \\
& \text{Knot}[7, 6] \rightarrow \left\{ -\frac{1 - 5T + 7T^2 - 5T^3 + T^4}{T^2}, -\frac{1}{T_1^4 T_2^4} \left(1 - 5T_1 + 7T_1^2 - 5T_1^3 + T_1^4 - 5T_2 + 20T_1 T_2 - \right. \right. \\
& 10T_1^2 T_2 - 10T_1^3 T_2 + 20T_1^4 T_2 - 5T_1^5 T_2 + 7T_2^2 - 10T_1 T_2^2 - 64T_1^2 T_2^2 + 98T_1^3 T_2^2 - 64T_1^4 T_2^2 - \\
& 10T_1^5 T_2^2 + 7T_1^6 T_2^2 - 5T_2^3 - 10T_1 T_2^3 + 98T_1^2 T_2^3 - 50T_1^3 T_2^3 - 50T_1^4 T_2^3 + 98T_1^5 T_2^3 - 10T_1^6 T_2^3 - \\
& 5T_1^7 T_2^3 + T_2^4 + 20T_1 T_2^4 - 64T_1^2 T_2^4 - 50T_1^3 T_2^4 + 108T_1^4 T_2^4 - 50T_1^5 T_2^4 - 64T_1^6 T_2^4 + 20T_1^7 T_2^4 + \\
& \left. T_1^8 T_2^4 - 5T_1 T_2^5 - 10T_1^2 T_2^5 + 98T_1^3 T_2^5 - 50T_1^4 T_2^5 - 50T_1^5 T_2^5 + 98T_1^6 T_2^5 - 10T_1^7 T_2^5 - 5T_1^8 T_2^5 + \right. \\
& \left. 10T_1^2 T_2^6 - 10T_1^3 T_2^6 + 20T_1^4 T_2^6 - 5T_1^5 T_2^6 + 7T_2^7 - 10T_1 T_2^7 - 64T_1^2 T_2^7 + 98T_1^3 T_2^7 - 64T_1^4 T_2^7 - \right. \\
& \left. 10T_1^5 T_2^7 + 7T_1^6 T_2^7 - 5T_2^8 - 10T_1 T_2^8 + 98T_1^2 T_2^8 - 50T_1^3 T_2^8 - 50T_1^4 T_2^8 + 98T_1^5 T_2^8 - 10T_1^6 T_2^8 - \right. \\
& \left. 5T_1^7 T_2^8 + T_2^9 + 20T_1 T_2^9 - 64T_1^2 T_2^9 + 98T_1^3 T_2^9 - 50T_1^4 T_2^9 - 50T_1^5 T_2^9 + 98T_1^6 T_2^9 - 10T_1^7 T_2^9 - 5T_1^8 T_2^9 + \right. \\
& \left. 10T_1^2 T_2^{10} - 10T_1^3 T_2^{10} + 20T_1^4 T_2^{10} - 5T_1^5 T_2^{10} + 7T_2^{11} - 10T_1 T_2^{11} - 64T_1^2 T_2^{11} + 98T_1^3 T_2^{11} - 64T_1^4 T_2^{11} - \right. \\
& \left. 10T_1^5 T_2^{11} + 7T_1^6 T_2^{11} - 5T_2^{12} - 10T_1 T_2^{12} + 98T_1^2 T_2^{12} - 50T_1^3 T_2^{12} - 50T_1^4 T_2^{12} + 98T_1^5 T_2^{12} - 10T_1^6 T_2^{12} - \right. \\
& \left. 5T_1^7 T_2^{12} + T_2^{13} + 20T_1 T_2^{13} - 64T_1^2 T_2^{13} + 98T_1^3 T_2^{13} - 50T_1^4 T_2^{13} - 50T_1^5 T_2^{13} + 98T_1^6 T_2^{13} - 10T_1^7 T_2^{13} - 5T_1^8 T_2^{13} + \right. \\
& \left. 10T_1^2 T_2^{14} - 10T_1^3 T_2^{14} + 20T_1^4 T_2^{14} - 5T_1^5 T_2^{14} + 7T_2^{15} - 10T_1 T_2^{15} - 64T_1^2 T_2^{15} + 98T_1^3 T_2^{15} - 64T_1^4 T_2^{15} - \right. \\
& \left. 10T_1^5 T_2^{15} + 7T_1^6 T_2^{15} - 5T_2^{16} - 10T_1 T_2^{16} + 98T_1^2 T_2^{16} - 50T_1^3 T_2^{16} - 50T_1^4 T_2^{16} + 98T_1^5 T_2^{16} - 10T_1^6 T_2^{16} - \right. \\
& \left. 5T_1^7 T_2^{16} + T_2^{17} + 20T_1 T_2^{17} - 64T_1^2 T_2^{17} + 98T_1^3 T_2^{17} - 50T_1^4 T_2^{17} - 50T_1^5 T_2^{17} + 98T_1^6 T_2^{17} - 10T_1^7 T_2^{17} - 5T_1^8 T_2^{17} + \right. \\
& \left. 10T_1^2 T_2^{18} - 10T_1^3 T_2^{18} + 20T_1^4 T_2^{18} - 5T_1^5 T_2^{18} + 7T_2^{19} - 10T_1 T_2^{19} - 64T_1^2 T_2^{19} + 98T_1^3 T_2^{19} - 64T_1^4 T_2^{19} - \right. \\
& \left. 10T_1^5 T_2^{19} + 7T_1^6 T_2^{19} - 5T_2^{20} - 10T_1 T_2^{20} + 98T_1^2 T_2^{20} - 50T_1^3 T_2^{20} - 50T_1^4 T_2^{20} + 98T_1^5 T_2^{20} - 10T_1^6 T_2^{20} - \right. \\
& \left. 5T_1^7 T_2^{20} + T_2^{21} + 20T_1 T_2^{21} - 64T_1^2 T_2^{21} + 98T_1^3 T_2^{21} - 50T_1^4 T_2^{21} - 50T_1^5 T_2^{21} + 98T_1^6 T_2^{21} - 10T_1^7 T_2^{21} - 5T_1^8 T_2^{21} + \right. \\
& \left. 10T_1^2 T_2^{22} - 10T_1^3 T_2^{22} + 20T_1^4 T_2^{22} - 5T_1^5 T_2^{22} + 7T_2^{23} - 10T_1 T_2^{23} - 64T_1^2 T_2^{23} + 98T_1^3 T_2^{23} - 64T_1^4 T_2^{23} - \right. \\
& \left. 10T_1^5 T_2^{23} + 7T_1^6 T_2^{23} - 5T_2^{24} - 10T_1 T_2^{24} + 98T_1^2 T_2^{24} - 50T_1^3 T_2^{24} - 50T_1^4 T_2^{24} + 98T_1^5 T_2^{24} - 10T_1^6 T_2^{24} - \right. \\
& \left. 5T_1^7 T_2^{24} + T_2^{25} + 20T_1 T_2^{25} - 64T_1^2 T_2^{25} + 98T_1^3 T_2^{25} - 50T_1^4 T_2^{25} - 50T_1^5 T_2^{25} + 98T_1^6 T_2^{25} - 10T_1^7 T_2^{25} - 5T_1^8 T_2^{25} + \right. \\
& \left. 10T_1^2 T_2^{26} - 10T_1^3 T_2^{26} + 20T_1^4 T_2^{26} - 5T_1^5 T_2^{26} + 7T_2^{27} - 10T_1 T_2^{27} - 64T_1^2 T_2^{27} + 98T_1^3 T_2^{27} - 64T_1^4 T_2^{27} - \right. \\
& \left. 10T_1^5 T_2^{27} + 7T_1^6 T_2^{27} - 5T_2^{28} - 10T_1 T_2^{28} + 98T_1^2 T_2^{28} - 50T_1^3 T_2^{28} - 50T_1^4 T_2^{28} + 98T_1^5 T_2^{28} - 10T_1^6 T_2^{28} - \right. \\
& \left. 5T_1^7 T_2^{28} + T_2^{29} + 20T_1 T_2^{29} - 64T_1^2 T_2^{29} + 98T_1^3 T_2^{29} - 50T_1^4 T_2^{29} - 50T_1^5 T_2^{29} + 98T_1^6 T_2^{29} - 10T_1^7 T_2^{29} - 5T_1^8 T_2^{29} + \right. \\
& \left. 10T_1^2 T_2^{30} - 10T_1^3 T_2^{30} + 20T_1^4 T_2^{30} - 5T_1^5 T_2^{30} + 7T_2^{31} - 10T_1 T_2^{31} - 64T_1^2 T_2^{31} + 98T_1^3 T_2^{31} - 64T_1^4 T_2^{31} - \right. \\
& \left. 10T_1^5 T_2^{31} + 7T_1^6 T_2^{31} - 5T_2^{32} - 10T_1 T_2^{32} + 98T_1^2 T_2^{32} - 50T_1^3 T_2^{32} - 50T_1^4 T_2^{32} + 98T_1^5 T_2^{32} - 10T_1^6 T_2^{32} - \right. \\
& \left. 5T_1^7 T_2^{32} + T_2^{33} + 20T_1 T_2^{33} - 64T_1^2 T_2^{33} + 98T_1^3 T_2^{33} - 50T_1^4 T_2^{33} - 50T_1^5 T_2^{33} + 98T_1^6 T_2^{33} - 10T_1^7 T_2^{33} - 5T_1^8 T_2^{33} + \right. \\
& \left. 10T_1^2 T_2^{34} - 10T_1^3 T_2^{34} + 20T_1^4 T_2^{34} - 5T_1^5 T_2^{34} + 7T_2^{35} - 10T_1 T_2^{35} - 64T_1^2 T_2^{35} + 98T_1^3 T_2^{35} - 64T_1^4 T_2^{35} - \right. \\
& \left. 10T_1^5 T_2^{35} + 7T_1^6 T_2^{35} - 5T_2^{36} - 10T_1 T_2^{36} + 98T_1^2 T_2^{36} - 50T_1^3 T_2^{36} - 50T_1^4 T_2^{36} + 98T_1^5 T_2^{36} - 10T_1^6 T_2^{36} - \right. \\
& \left. 5T_1^7 T_2^{36} + T_2^{37} + 20T_1 T_2^{37} - 64T_1^2 T_2^{37} + 98T_1^3 T_2^{37} - 50T_1^4 T_2^{37} - 50T_1^5 T_2^{37} + 98T_1^6 T_2^{37} - 10T_1^7 T_2^{37} - 5T_1^8 T_2^{37} + \right. \\
& \left. 10T_1^2 T_2^{38} - 10T_1^3 T_2^{38} + 20T_1^4 T_2^{38} - 5T_1^5 T_2^{38} + 7T_2^{39} - 10T_1 T_2^{39} - 64T_1^2 T_2^{39} + 98T_1^3 T_2^{39} - 64T_1^4 T_2^{39} - \right. \\
& \left. 10T_1^5 T_2^{39} + 7T_1^6 T_2^{39} - 5T_2^{40} - 10T_1 T_2^{40} + 98T_1^2 T_2^{40} - 50T_1^3 T_2^{40} - 50T_1^4 T_2^{40} + 98T_1^5 T_2^{40} - 10T_1^6 T_2^{40} - \right. \\
& \left. 5T_1^7 T_2^{40} + T_2^{41} + 20T_1 T_2^{41} - 64T_1^2 T_2^{41} + 98T_1^3 T_2^{41} - 50T_1^4 T_2^{41} - 50T_1^5 T_2^{41} + 98T_1^6 T_2^{41} - 10T_1^7 T_2^{41} - 5T_1^8 T_2^{41} + \right. \\
& \left. 10T_1^2 T_2^{42} - 10T_1^3 T_2^{42} + 20T_1^4 T_2^{42} - 5T_1^5 T_2^{42} + 7T_2^{43} - 10T_1 T_2^{43} - 64T_1^2 T_2^{43} + 98T_1^3 T_2^{43} - 64T_1^4 T_2^{43} - \right. \\
& \left. 10T_1^5 T_2^{43} + 7T_1^6 T_2^{43} - 5T_2^{44} - 10T_1 T_2^{44} + 98T_1^2 T_2^{44} - 50T_1^3 T_2^{44} - 50T_1^4 T_2^{44} + 98T_1^5 T_2^{44} - 10T_1^6 T_2^{44} - \right. \\
& \left. 5T_1^7 T_2^{44} + T_2^{45} + 20T_1 T_2^{45} - 64T_1^2 T_2^{45} + 98T_1^3 T_2^{45} - 50T_1^4 T_2^{45} - 50T_1^5 T_2^{45} + 98T_1^6 T_2^{45} - 10T_1^7 T_2^{45} - 5T_1^8 T_2^{45} + \right. \\
& \left. 10T_1^2 T_2^{46} - 10T_1^3 T_2^{46} + 20T_1^4 T_2^{46} - 5T_1^5 T_2^{46} + 7T_2^{47} - 10T_1 T_2^{47} - 64T_1^2 T_2^{47} + 98T_1^3 T_2^{47} - 64T_1^4 T_2^{47} - \right. \\
& \left. 10T_1^5 T_2^{47} + 7T_1^6 T_2^{47} - 5T_2^{48} - 10T_1 T_2^{48} + 98T_1^2 T_2^{48} - 50T_1^3 T_2^{48} - 50T_1^4 T_2^{48} + 98T_1^5 T_2^{48} - 10T_1^6 T_2^{48} - \right. \\
& \left. 5T_1^7 T_2^{48} + T_2^{49} + 20T_1 T_2^{49} - 64T_1^2 T_2^{49} + 98T_1^3 T_2^{49} - 50T_1^4 T_2^{49} - 50T_1^5 T_2^{49} + 98T_1^6 T_2^{49} - 10T_1^7 T_2^{49} - 5T_1^8 T_2^{49} + \right. \\
& \left. 10T_1^2 T_2^{50} - 10T_1^3 T_2^{50} + 20T_1^4 T_2^{50} - 5T_1^5 T_2^{50} + 7T_2^{51} - 10T_1 T_2^{51} - 64T_1^2 T_2^{51} + 98T_1^3 T_2^{51} - 64T_1^4 T_2^{51} - \right. \\
& \left. 10T_1^5 T_2^{51} + 7T_1^6 T_2^{51} - 5T_2^{52} - 10T_1 T_2^{52} + 98T_1^2 T_2^{52} - 50T_1^3 T_2^{52} - 50T_1^4 T_2^{52} + 98T_1^5 T_2^{52} - 10T_1^6 T_2^{52} - \right. \\
& \left. 5T_1^7 T_2^{52} + T_2^{53} + 20T_1 T_2^{53} - 64T_1^2 T_2^{53} + 98T_1^3 T_2^{53} - 50T_1^4 T_2^{53} - 50T_1^5 T_2^{53} + 98T_1^6 T_2^{53} - 10T_1^7 T_2^{53} - 5T_1^8 T_2^{53} + \right. \\
& \left. 10T_1^2 T_2^{54} - 10T_1^3 T_2^{54} + 20T_1^4 T_2^{54} - 5T_1^5 T_2^{54} + 7T_2^{55} - 10T_1 T_2^{55} - 64T_1^2 T_2^{55} + 98T_1^3 T_2^{55} - 64T_1^4 T_2^{55} - \right. \\
& \left. 10T_1^5 T_2^{55} + 7T_1^6 T_2^{55} - 5T_2^{56} - 10T_1 T_2^{56} + 98T_1^2 T_2^{56} - 50T_1^3 T_2^{56} - 50T_1^4 T_2^{56} + 98T_1^5 T_2^{56} - 10T_1^6 T_2^{56} - \right. \\
& \left. 5T_1^7 T_2^{56} + T_2^{57} + 20T_1 T_2^{57} - 64T_1^2 T_2^{57} + 98T_1^3 T_2^{57} - 50T_1^4 T_2^{57} - 50T_1^5 T_2^{57} + 98T_1^6 T_2^{57} - 10T_1^7 T_2^{57} - 5T_1^8 T_2^{57} + \right. \\
& \left. 10T_1^2 T_2^{58} - 10T_1^3 T_2^{58} + 20T_1^4 T_2^{58} - 5T_1^5 T_2^{58} + 7T_2^{59} - 10T_1 T_2^{59} - 64T_1^2 T_2^{59} + 98T_1^3 T_2^{59} - 64T_1^4 T_2^{59} - \right. \\
& \left. 10T_1^5 T_2^{59} + 7T_1^6 T_2^{59} - 5T_2^{60} - 10T_1 T_2^{60} + 98T_1^2 T_2^{60} - 50T_1^3 T_2^{60} - 50T_1^4 T_2^{60} + 98T_1^5 T_2^{60} - 10T_1^6 T_2^{60} - \right. \\
& \left. 5T_1^7 T_2^{60} + T_2^{61} + 20T_1 T_2^{61} - 64T_1^2 T_2^{61} + 98T_1^3 T_2^{61} - 50T_1^4 T_2^{61} - 50T_1^5 T_2^{61} + 98T_1^6 T_2^{61} - 10T_1^7 T_2^{61} - 5T_1^8 T_2^{61} + \right. \\
& \left. 10T_1^2 T_2^{62} - 10T_1^3 T_2^{62} + 20T_1^4 T_2^{62} - 5T_1^5 T_2^{62} + 7T_2^{63} - 10T_1 T_2^{63} - 64T_1^2 T_2^{63} + 98T_1^3 T_2^{63} - 64T_1^4 T_2^{63} - \right. \\
& \left. 10T_1^5 T_2^{63} + 7T_1^6 T_2^{63} - 5T_2^{64} - 10T_1 T_2^{64} + 98T_1^2 T_2^{64} - 50T_1^3 T_2^{64} - 50T_1^4 T_2^{64} + 98T_1^5 T_2^{64} - 10T_1^6 T_2^{64} - \right. \\
& \left. 5T_1^7 T_2^{64} + T_2^{65} + 20T_1 T_2^{65} - 64T_1^2 T_2^{65} + 98T_1^3 T_2^{65} - 50T_1^4 T_2^{65} - 50T_1^5 T_2^{65} + 98T_1^6 T_2^{65} - 10T_1^7 T_2^{65} - 5T_1^8 T_2^{65} + \right. \\
& \left. 10T_1^2 T_2^{66} - 10T_1^3 T_2^{66} + 20T_1^4 T_2^{66} - 5T_1^5 T_2^{66} + 7T_2^{67} - 10T_1 T_2^{67} - 64T_1^2 T_2^{67} + 98T_1^3 T_2^{67} - 64T_1^4 T_2^{67} - \right. \\
& \left. 10T_1^5 T_2^{67} + 7T_1^6 T_2^{67} - 5T_2^{68} - 10T_1 T_2^{68} + 98T_1^2 T_2^{68} - 50T_1^3 T_2^{68} - 50T_1^4 T_2^{68} + 98T_1^5 T_2^{68} - 10T_1^6 T_2^{68} - \right. \\
& \left. 5T_1^7 T_2^{68} + T_2^{69} + 20T_1 T_2^{69} - 64T_1^2 T_2^{69} + 98T_1^3 T_2^{69} - 50T_1^4 T_2^{69} - 50T_1^5 T_2^{69} + 98T_1^6 T_2^{69} - 10T_1^7 T_2^{69} - 5T_1^8 T_2^{69} + \right. \\
& \left. 10T_1^2 T_2^{70} - 10T_1^3 T_2^{70} + 20T_1^4 T_2^{70} - 5T_1^5 T_2^{70} + 7T_2^{71} - 10T_1 T_2^{71} - 64T_1^2 T_2^{71} + 98T_1^3 T_2^{71} - 64T_1^4 T_2^{71} - \right. \\
& \left. 10T_1^5 T_2^{71} + 7T_1^6 T_2^{71} - 5T_2^{72} - 10T_1 T_2^{72} + 98T_1^2 T_2^{72} - 50T_1^3 T_2^{72} - 50T_1^4 T_2^{72} + 98T_1^5 T_2^{72} - 10T_1^6 T_2^{72} - \right. \\
& \left. 5T_1^7 T_2^{72} + T_2^{73} + 20T_1 T_2^{73} - 64T_1^2 T_2^{73} + 98T_1^3 T_2^{73} - 50T_1^4 T_2^{73} - 50T_1^5 T_2^{73} + 98T_1^6 T_2^{73} - 10T_1^7 T_2^{73} - 5T_1^8 T_2^{73} + \right. \\
& \left. 10T_1^2 T_2^{74} - 10T_1^3 T_2^{74} + 20T_1^4 T_2^{74} - 5T_1^5 T_2^{74} + 7T_2^{75} - 10T_1 T_2^{75} - 64T_1^2 T_2^{75} + 98T_1^3 T_2^{75} - 64T_1^4 T_2^{75} - \right. \\
& \left. 10T_1^5 T_2^{75} + 7T_1^6 T_2^{75} - 5T_2^{76} - 10T_1 T_2^{76} + 98T_1^2 T_2^{76} - 50T_1^3 T_2^{76} - 50T_1^4 T_2^{76} + 98T_1^5 T_2^{76} - 10T_1^6 T_2^{76} - \right. \\
& \left. 5T_1^7 T_2^{76} + T_2^{77} + 20T_1 T_2^{77} - 64T_1^2 T_2^{77} + 98T_1^3 T_2^{77} - 50T_1^4 T_2^{77} - 50T_1^5 T_2^{77} + 98T_1^6 T_2^{77} - 10T_1^7 T_2^{77} - 5T_1^8 T_2^{77} + \right. \\
& \left. 10T_1^2 T_2^{78} - 10T_1^3 T_2^{78} + 20T_1^4 T_2^{78} - 5T_1^5 T_2^{78} + 7T_2^{79} - 10T_1 T_2^{79} - 64T_1^2 T_2^{79} + 98T_1^3 T_2^{79} - 64T_1^4 T_2^{79} - \right. \\
& \left. 10T_1^5 T_2^{79} + 7T_1^6 T_2^{79} - 5T_2^{80} - 10T_1 T_2^{80} + 98T_1^2 T_2^{80} - 50T_1^3 T_2^{80} - 50T_1^4 T_2^{80} + 98T_1^5 T_2^{80} - 10T_1^6 T_2^{80} - \right. \\
& \left. 5T_1^7 T_2^{80} + T_2^{81} + 20T_1 T_2^{81} - 64T_1^2 T_2^{81} + 98T_1^3 T_2^{81} -$$

$$\begin{aligned}
& 7 T_1^2 T_2^6 - 10 T_1^3 T_2^6 - 64 T_1^4 T_2^6 + 98 T_1^5 T_2^6 - 64 T_1^6 T_2^6 - 10 T_1^7 T_2^6 + 7 T_1^8 T_2^6 - 5 T_1^3 T_2^7 + 20 T_1^4 T_2^7 - \\
& 10 T_1^5 T_2^7 - 10 T_1^6 T_2^7 + 20 T_1^7 T_2^7 - 5 T_1^8 T_2^7 + T_1^4 T_2^8 - 5 T_1^5 T_2^8 + 7 T_1^6 T_2^8 - 5 T_1^7 T_2^8 + T_1^8 T_2^8 \Big\}, \\
\text{Knot}[7, 7] \rightarrow & \left\{ \frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^2}, -\frac{1}{T_1^2 T_2^2} \left(1 - 5 T_1 + T_1^2 - 5 T_2 + 12 T_1 T_2 + 12 T_1^2 T_2 - 5 T_1^3 T_2 + T_2^2 + \right. \right. \\
& \left. \left. 12 T_1 T_2^2 - 48 T_1^2 T_2^2 + 12 T_1^3 T_2^2 + T_1^4 T_2^2 - 5 T_1 T_2^3 + 12 T_1^2 T_2^3 + 12 T_1^3 T_2^3 - 5 T_1^4 T_2^3 + T_1^2 T_2^4 - 5 T_1^3 T_2^4 + T_1^4 T_2^4 \right) \right\}
\end{aligned}$$