Pensieve header: The 3-Heisenberg Formalism for Θ. Engine modified from pensieve://Projects/BabyDoPeGDO/Solving_to_k=3.nb, px formulas modified from pensieve://Talks/Bonn-2505/. Now in xp order. Now using the myopic projection $M.

𝔼[ω, Q, P_ϵSeries] represents $\omega\, e^{Q+P}$, where ω is a scalar, Q is an ϵ-free quadratic, and $P=\sum_{k=0}^{\$k} P[\![k]\!]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: 𝔼_[_] // 𝔼_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\3H"];
        Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[∘]:= $k = 1;
```

```
In[∘]:= $M[ℰ_] := ℰ; (* The Myopic Projection *)
```

```
In[∘]:= HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Minor utilities

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
        (*CoefficientCanonical Form *)
        CCF[ℰ_] := Factor[ℰ]; (*CoefficientCanonical Form *)
        CF[ℰ_List] := CF /@ ℰ;
        CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
        CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, (p | x | π | ξ | g)__, F] /. F → CCF];
        (*CF[ℰ_] := Module[
           {vs=Cases[ℰ, (p|x|π|ξ)__,∞]},
           Total[(CCF[#[[2]]] (Times@@vs^#[[1]]))& /@ CoefficientRules[Expand[ℰ], vs]]
          ];*)
        CF[ℰ_𝔼] := CF /@ ℰ;
        CF[𝔼sp___[ℰs___]] := CF /@ 𝔼sp[ℰs];
        CF[a_ == b_] := CF[a - b] == 0;
```

In[•]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: Plus[ss___ϵSeries] := Module[{l = Min[Length /@ {ss}]},
   ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
  ];
(*ϵSeries/: S1_ϵSeries+S2_ϵSeries :=
  ϵSeries@@Table[S1⟦k⟧+S2⟦k⟧,{k,Min[Length@S1,Length@S2]}];*)
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1⟦j + 1⟧ * S2⟦k - j + 1⟧, {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂vs__ S_ϵSeries := (s ↦ ∂vs s) /@ S;
```

In[•]:=
```
Options[PolyPlot1] = {Labeled → False};
PolyPlot1[Δ_, OptionsPattern[]] := Module[{Δ1, crs, m, maxc, minc, s, rect},
  Δ1 = PowerExpand@Expand@Δ;
  rect = {{0, 0}, {1, 0}, {1, 1}, {0, 1}};
  If[Δ1 === 0, Graphics[],
   m = Max[-Exponent[Δ1, T, Min], Exponent[Δ1, T, Max]];
   crs = CoefficientRules[T^m Δ1, {T}];
   maxc = N@Log@Max@Abs[Last /@ crs];
   minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
   If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
   Graphics[crs /. ({x_} → c_) :> {
       Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
       Tooltip[Polygon[({x + m - 1/2, 0} + #) & /@ rect], c T^(x-m)],
       If[Not@OptionValue[Labeled], {},
        {Black, FontSize → Scaled[1/((m + 1) (6 + maxc))], Text[c T^(x-m), {x + m, 0.5}]}]
      }, AspectRatio → Min[1/5, 1/Sqrt[m + 1]],
    ImagePadding → None, PlotRangePadding → None]
  ]];
Options[PolyPlot2] = {Labeled → False, Shear → True};
PolyPlot2[θ_, OptionsPattern[]] :=
  Module[{θ1, crs, m1, m2, maxc, minc, s, stone, mat, p},
   θ1 = PowerExpand@Expand@θ;
   If[θ1 === 0, Graphics[{White, Disk[]}],
    If[OptionValue[Shear],
     stone = Table[{Cos[α], Sin[α]} / Cos[2π/12] / 2, {α, 2π/12, 2π, 2π/6}];
     mat = {{1, -1/2}, {0, Sqrt[3]/2}},
```

```
      stone = {{1, 1}, {-1, 1}, {-1, -1}, {1, -1}} / 2;
      mat = IdentityMatrix[2];
     ];
    m1 = Max[-Exponent[θ1, T_1, Min], Exponent[θ1, T_1, Max]];
    m2 = Max[-Exponent[θ1, T_2, Min], Exponent[θ1, T_2, Max]];
    crs = CoefficientRules[T_1^m1 T_2^m2 θ1, {T_1, T_2}];
    maxc = N@Log@Max@Abs[Last /@ crs];
    minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
    If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
    Graphics[{(*{Yellow,Disk[{0,0},1+Cos[2π/12]Norm[{m1,m2}]/√2]},*)
      crs /. ({x1_, x2_} → c_) :→ {
         Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
         p = mat.{x1 - m1, x2 - m2};
         Tooltip[Polygon[(p + #) & /@ stone], c T_1^(x1-m1) T_2^(x2-m2)],
         If[Not@OptionValue[Labeled], {},
          {Black, FontSize → Scaled[1 / (Max[m1, m2] (10 + maxc))], Text[c T_1^(x1-m1) T_2^(x2-m2), p]}]
        }
     }, ImagePadding → None, PlotRangePadding → None]
   ]];
Options[PolyPlot] = {Labeled → False, ImageSize → Automatic};
PolyPlot[{Δ_, θ_}, opts___Rule] := GraphicsColumn[
   {PolyPlot1[Δ, FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot1]]],
    PolyPlot2[θ, FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot2]]]},
   Spacings → Scaled@0.08, ImagePadding → None, PlotRangePadding → None,
   FilterRules[Join[{opts}, Options[PolyPlot]], Options[GraphicsColumn]]
  ];
```

The Main Program

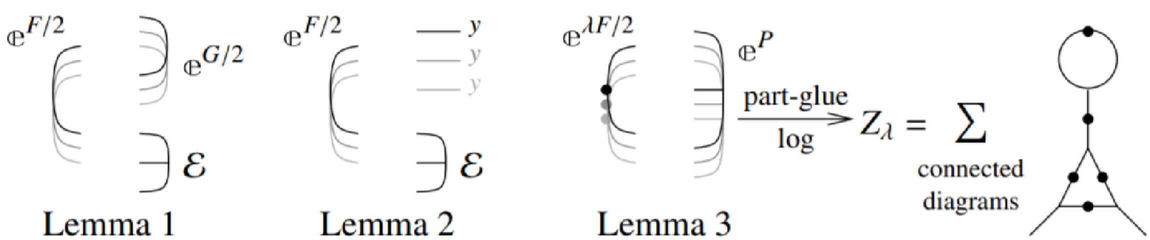
Variables and their duals:

```
In[]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(Subscript[u_, vi__])* := Subscript[(u*), vi];
```

$\mathbb{E}$ operations:

```
In[∘]:= 𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
        𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
        𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
        𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
        𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

```
In[∘]:= 𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
          lvs = Flatten@Table[{x_{v,$@i}, p_{v,$@i}}, {v, 3}, {i, is}];
          𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
              𝔼[ℰ1s] /. Table[(u : (x | p))_{v_,i} → u_{v,$@i}, {i, is}],
              𝔼[ℰ2s] /. Table[(u : (ξ | π))_{v_,i} → u_{v,$@i}, {i, is}]
            ]])
        ]
```




```
In[∘]:= Zip_{vs_}[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
           ⟨ℱ, ℰ⟩ // Zip1_{vs} // EZip23_{vs});
        Zip_{vs_}[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
           ⟨ℱ, ℰ⟩ // Zip1_{vs} // Zip2_{vs} // Zip3_{vs});
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

```
In[∘]:= Zip1_{} = Identity;
        Zip1_{vs_}@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{ℐ, F, G, u, v},
          (*EchoLabel["EZip1 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
          ℐ = IdentityMatrix@Length@vs;
          F = Table[∂_{u,v} ℱ, {u, vs*}, {v, vs*}];
          G = Table[∂_{u,v} Q, {u, vs}, {v, vs}];
          ⟨CF[vs*.F.Inverse[ℐ - G.F].vs* / 2],
            𝔼[CF@PowerExpand@Factor[ω Det[ℐ - G.F]^(-1/2)], CF[Q - vs.G.vs / 2], P]⟩
        ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\, e^{\sum_{i\in B} y_i z_i} \right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F :\ \mathcal{E}|_{z_B \to z_B + F y_B} \right\rangle_B$.

```
In[∘]:= Zip2{} = Identity;
Zip2vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{F, Y, u, v},
  (*EchoLabel["EZip2 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
  Y = Table[∂v Q, {v, vs}];
  CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
 ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda \coloneqq \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

```
In[∘]:= Zip3vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{Z, u, v, m, j},
  (*EchoLabel["Zip3 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k + 5, ++m,
   Z[m + 1] = $M@CF[1/(2 (m + 1))
      Sum[∂u*,v* ℱ (∂u,v Z[m] + Sum[(∂u Z[j]) (∂v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
  ];
  𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k + 5}] /. Table[v → 0, {v, vs}]]]
 ]
```

```
In[∘]:= EZip23vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[
  {nP, nℱ, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
      CoefficientRules[#, vs] /.
       (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))
     ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))];
  nℱ = Total[CoefficientRules[ℱ, vs*] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^ps))];
  CF[Expand[⟨nℱ, 𝔼[ω, nQ, nP]⟩ // Zip2vs // Zip3vs] /. rr]
 ]
```

```
In[∘]:= mi_,j_→k_ := 𝔼{i,j}→{k}[1, Sum_{ν=1}^{3} (πν,i ξν,j + (πν,i + πν,j) pν,k + (ξν,i + ξν,j) xν,k), ϵSeries[]]
```

In[∘]:= $m_{a,b\to c}$

Out[∘]=

$\mathbb{E}_{\{a,b\}\to\{c\}}[1, p_{1,c}(\pi_{1,a}+\pi_{1,b}) + p_{2,c}(\pi_{2,a}+\pi_{2,b}) + p_{3,c}(\pi_{3,a}+\pi_{3,b}) + \pi_{1,a}\xi_{1,b} + x_{1,c}(\xi_{1,a}+\xi_{1,b}) + \pi_{2,a}\xi_{2,b} + x_{2,c}(\xi_{2,a}+\xi_{2,b}) + \pi_{3,a}\xi_{3,b} + x_{3,c}(\xi_{3,a}+\xi_{3,b}),\ \epsilon\text{Series}[0, 0]]$

In[∘]:= $m_{1,2\to 1}$ // $m_{1,3\to 1}$

($m_{1,2\to 1}$ // $m_{1,3\to 1}$) ≡ ($m_{2,3\to 2}$ // $m_{1,2\to 1}$)

Out[∘]=

$\mathbb{E}_{\{1,2,3\}\to\{1\}}[1, p_{1,1}\pi_{1,1} + p_{1,1}\pi_{1,2} + p_{1,1}\pi_{1,3} + p_{2,1}\pi_{2,1} + p_{2,1}\pi_{2,2} + p_{2,1}\pi_{2,3} + p_{3,1}\pi_{3,1} + p_{3,1}\pi_{3,2} + p_{3,1}\pi_{3,3} + x_{1,1}\xi_{1,1} + \pi_{1,1}\xi_{1,2} + x_{1,1}\xi_{1,2} + \pi_{1,1}\xi_{1,3} + \pi_{1,2}\xi_{1,3} + x_{1,1}\xi_{1,3} + x_{2,1}\xi_{2,1} + \pi_{2,1}\xi_{2,2} + x_{2,1}\xi_{2,2} + \pi_{2,1}\xi_{2,3} + \pi_{2,2}\xi_{2,3} + x_{2,1}\xi_{2,3} + x_{3,1}\xi_{3,1} + \pi_{3,1}\xi_{3,2} + x_{3,1}\xi_{3,2} + \pi_{3,1}\xi_{3,3} + \pi_{3,2}\xi_{3,3} + x_{3,1}\xi_{3,3},\ \epsilon\text{Series}[0, 0]]$

Out[∘]=

True

Some Knot Theory

In[∘]:=
```
Kink_i_ := CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→i};
\overline{Kink}_i_ := CC_3 \overline{R}_{1,2} // m_{1,3→1} // m_{1,2→i}
```

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]   PositiveQ@x
                           Xm[x[[2]], x[[1]]]   True       ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → (xs /. {
             Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
             Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
           })],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := 𝔼_{{}→{i}}[1, 0, εSeries[]];
rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CC_j, rot[i, n + 1] \overline{CC}_j] // m_{i,j→i}];
```

```
In[•]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{{}→{0}}[1, 0, εSeries[]];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= todo,
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (R_{i,j} OverBar[Kink]_k) // m_{j,k→j},
     Xm, (OverBar[R]_{i,j} Kink_k) // m_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // m_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // m_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ (*/. {x_0→x, y_0→y, a_0→a}*))
 ]
```

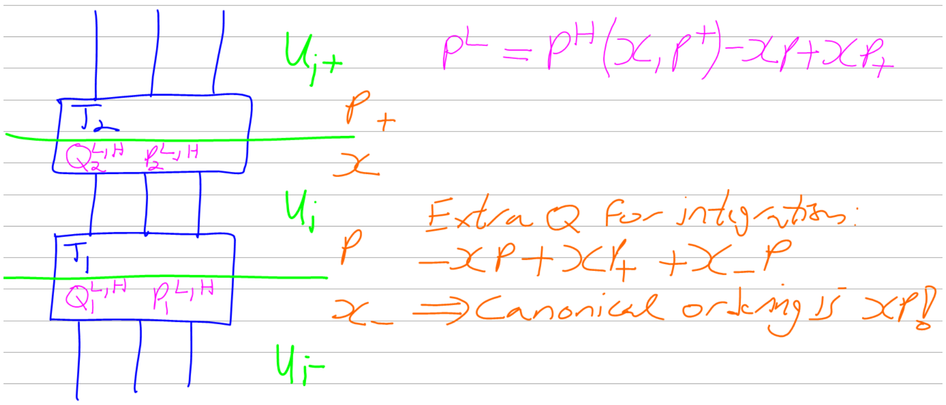
ρ_1 testing

The pPush Lagrangian of Talks/Bonn-2505/Rho11AlternativeIntegrands.nb:

In[∘]:=
$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T^{s/2}\,\mathbb{E}\Big[\Big\{$$
$$x_i\,(p_{i+1}-p_i) + x_j\,(p_{j+1}-p_j) + \left(T^s-1\right) x_i\,(p_{i+1}-p_{j+1}),$$
$$0,\ 0,\ \epsilon\left(-\frac{s}{2} + s\,T^s\,p_{1+j}\,x_i + \frac{1}{2}\,s\,T^s\left(-1+T^s\right) p_{1+i}\,p_{1+j}\,x_i^2 - \frac{1}{2}\,s\,T^s\left(-1+T^s\right) p_{1+j}^2\,x_i^2 - s\,p_{1+j}\,x_j - s\,T^s\,p_{1+i}\,p_{1+j}\,x_i\,x_j + s\,T^s\,p_{1+j}^2\,x_i\,x_j\right)$$
$$\Big\}\Big]$$

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T^{\varphi/2}\,\mathbb{E}\left[\left\{x_i\,(p_{i+1}-p_i),\ \epsilon\left(-\frac{\varphi}{2} - \varphi\,p_{1+i}\,x_i\right)\right\}\right]$$

```
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧]
vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K]⟦1⟧}]
```



In[∘]:=
```
R_{i_,j_} = CF[ℒ[X_{i,j}[1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] :> 𝔼_{{}→{i,j}}[
      ω, Q, ϵSeries[P0, P1a + P1b /. ϵ → 1]
    ]] /. {p_{i+1} :> p_i, p_{j+1} → p_j} /. {T → T_1, x_α_ :> x_{1,α}, p_α_ :> p_{1,α}};
R̄_{i_,j_} = CF[ℒ[X_{i,j}[-1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] :> 𝔼_{{}→{i,j}}[
      ω, Q, ϵSeries[P0, P1a + P1b /. ϵ → 1]
    ]] /. {p_{i+1} :> p_i, p_{j+1} → p_j} /. {T → T_1, x_α_ :> x_{1,α}, p_α_ :> p_{1,α}};
CC_{i_} = CF[ℒ[C_i[1]] /. ω_ 𝔼[{Q_, P_}] :> 𝔼_{{}→{i}}[
      ω, Q, ϵSeries[0, P /. ϵ → 1]
    ]] /. {p_{i+1} :> p_i} /. {T → T_1, x_α_ :> x_{1,α}, p_α_ :> p_{1,α}};
CC̄_{i_} = CF[ℒ[C_i[-1]] /. ω_ 𝔼[{Q_, P_}] :> 𝔼_{{}→{i}}[
      ω, Q, ϵSeries[0, P /. ϵ → 1]
    ]] /. {p_{i+1} :> p_i} /. {T → T_1, x_α_ :> x_{1,α}, p_α_ :> p_{1,α}};
```

In[∘]:=
```
$k = 1;
```
$R_{i,j}$

$\overline{R}_{i,j}$

CC_i

$\overline{CC}_i$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[\sqrt{T_1},\ -p_{1,i}\,x_{1,i}+T_1\,p_{1,i}\,x_{1,i}+(1-T_1)\,p_{1,j}\,x_{1,i},\ \epsilon\text{Series}\left[0,\ -\frac{1}{2}+T_1\,p_{1,j}\,x_{1,i}+\frac{1}{2}(-1+T_1)\,T_1\,p_{1,i}\,p_{1,j}\,x_{1,i}^2-\frac{1}{2}(-1+T_1)\,T_1\,p_{1,j}^2\,x_{1,i}^2-p_{1,j}\,x_{1,j}-T_1\,p_{1,i}\,p_{1,j}\,x_{1,i}\,x_{1,j}+T_1\,p_{1,j}^2\,x_{1,i}\,x_{1,j}\right]\right]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[\frac{1}{\sqrt{T_1}},\ -p_{1,i}\,x_{1,i}+\frac{p_{1,i}\,x_{1,i}}{T_1}+\frac{(-1+T_1)\,p_{1,j}\,x_{1,i}}{T_1},\ \epsilon\text{Series}\left[0,\ \frac{1}{2}-\frac{p_{1,j}\,x_{1,i}}{T_1}+\frac{(-1+T_1)\,p_{1,i}\,p_{1,j}\,x_{1,i}^2}{2\,T_1^2}-\frac{(-1+T_1)\,p_{1,j}^2\,x_{1,i}^2}{2\,T_1^2}+p_{1,j}\,x_{1,j}+\frac{p_{1,i}\,p_{1,j}\,x_{1,i}\,x_{1,j}}{T_1}-\frac{p_{1,j}^2\,x_{1,i}\,x_{1,j}}{T_1}\right]\right]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i\}}\left[\sqrt{T_1},\ 0,\ \epsilon\text{Series}\left[0,\ -\frac{1}{2}-p_{1,i}\,x_{1,i}\right]\right]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}\left[0,\ \frac{1}{2}+p_{1,i}\,x_{1,i}\right]\right]$$

In[∘]:=

$k = 1; {
 {"C$\overline{C}$", ($CC_1\,\overline{CC}_2$ // $m_{1,2\to1}$) ≡ $\mathbb{E}_{\{\}\to\{1\}}$[1, 0, ϵSeries[]]},
 {"An R1", ($CC_3\,R_{1,2}$ // $m_{2,3\to2}$ // $m_{2,1\to1}$) ≡ ($\overline{CC}_3\,R_{1,2}$ // $m_{1,3\to1}$ // $m_{1,2\to1}$)},
 {"R2b", ($R_{1,2}\,\overline{R}_{3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$) ≡ $\mathbb{E}_{\{\}\to\{1,2\}}$[1, 0, ϵSeries[]]},
 {"R3", ($R_{1,2}\,R_{4,3}\,R_{5,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$) ≡ ($R_{2,3}\,R_{4,5}\,R_{1,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$)}
}

Out[∘]=

{{C$\overline{C}$, True}, {An R1, True}, {R2b, True}, {R3, True}}

In[∘]:=

NewBit[K_] := Module[{Alex = Alexander[K][T_1]},
 Alex^2 Z[K][[3, 2]] // Factor]

In[∘]:=
```
NewBit /@ AllKnots[{3, 5}]
```
KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\left\{\frac{(-1+T_1)^2\,(1+T_1^2)}{T_1^2},\ 0,\ \frac{(-1+T_1)^2\,(1+T_1^2)\,(2+T_1^2+2\,T_1^4)}{T_1^4},\ \frac{(-1+T_1)^2\,(5-4\,T_1+5\,T_1^2)}{T_1^2}\right\}$$

In papers/APAI, page 7, that was:

$$\left\{\frac{(-1+T_1)^2\ (1+T_1^2)}{T_1^2},\ 0,\ \frac{(-1+T_1)^2\ (1+T_1^2)\ (2+T_1^2+2\,T_1^4)}{T_1^4},\ \frac{(-1+T_1)^2\ (5-4\,T_1+5\,T_1^2)}{T_1^2}\right\}$$

And now θ

In[∘]:=
```
$π[ℰ_] := (CF[ℰ /. {p_{αβ__} :> B^-1 p_{αβ}, x_{αβ__} :> B x_{αβ}}] /. B → 0)
$M[ϵSeries[P0_, P1_]] := ϵSeries[P0, $π[P1]];
```

The Lagrangian is taken from ThetaAlternativeIntegrands.nb.

In[∘]:=

$$T_3 = T_1\,T_2;$$

$$\mathcal{L}_{\text{pPush}}[X_{i\_,j\_}[s\_]] :=$$

$$\begin{aligned}
&(T_1 T_2)^s\ \mathbb{E}\Big[\frac{s\,\epsilon}{2} - p_{1,i}\,x_{1,i} + T_1^s\,p_{1,1+i}\,x_{1,i} + (1-T_1^s)\,p_{1,1+j}\,x_{1,i} - p_{1,j}\,x_{1,j} + p_{1,1+j}\,x_{1,j} - p_{2,i}\,x_{2,i} +\\
&\quad T_2^s\,p_{2,1+i}\,x_{2,i} - s\,\epsilon\,T_2^s\,p_{2,1+i}\,x_{2,i} + (1-T_2^s)\,p_{2,1+j}\,x_{2,i} + s\,\epsilon\,T_1^s\,T_2^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,i} +\\
&\quad \frac{s\,\epsilon\,(-1+T_1^s)\,T_2^s\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,i}}{-1+T_2^s} + (-1+T_1^s)\,T_2^s\,p_{3,1+j}\,x_{1,i}\,x_{2,i} - p_{2,j}\,x_{2,j} + p_{2,1+j}\,x_{2,j} +\\
&\quad s\,\epsilon\,p_{2,1+j}\,x_{2,j} - s\,\epsilon\,T_1^s\,p_{1,1+i}\,p_{2,1+j}\,x_{1,i}\,x_{2,j} - \frac{s\,\epsilon\,(-1+T_1^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{1,i}\,x_{2,j}}{-1+T_2^s} + (1-T_1^s)\,p_{3,1+j}\\
&\quad x_{1,i}\,x_{2,j} + \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{2,1+i}\,x_{3,i}}{-1+T_2^s} - \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{2,1+j}\,x_{3,i}}{-1+T_2^s} -\\
&\quad p_{3,i}\,x_{3,i} + (T_1 T_2)^s\,p_{3,1+i}\,x_{3,i} + s\,\epsilon\,(T_1 T_2)^s\,p_{3,1+i}\,x_{3,i} + (1-(T_1 T_2)^s)\,p_{3,1+j}\,x_{3,i} -\\
&\quad \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{3,1+j}\,x_{3,i}}{-1+T_2^s} - \frac{s\,\epsilon\,T_1^s\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s} +\\
&\quad \frac{s\,\epsilon\,(-1+T_1^s)\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,i}}{-1+T_2^s} - s\,\epsilon\,(T_1 T_2)^s\,(-1+T_2^s)\,p_{2,1+j}\,p_{3,1+i}\,x_{2,i}\,x_{3,i} +\\
&\quad s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,i} + 2\,s\,\epsilon\,(T_1 T_2)^s\,p_{2,1+j}\,p_{3,1+i}\,x_{2,j}\,x_{3,i} +\\
&\quad \frac{s\,\epsilon\,T_2^s\,(-1+(T_1 T_2)^s)\,p_{2,1+i}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s} - \frac{s\,\epsilon\,(-1+2\,T_2^s)\,(-1+(T_1 T_2)^s)\,p_{2,1+j}\,p_{3,1+j}\,x_{2,j}\,x_{3,i}}{-1+T_2^s} -\\
&\quad p_{3,j}\,x_{3,j} + p_{3,1+j}\,x_{3,j} + s\,\epsilon\,T_1^s\,p_{1,1+i}\,p_{3,1+j}\,x_{1,i}\,x_{3,j} + \frac{s\,\epsilon\,(-1+T_1^s)\,p_{1,1+j}\,p_{3,1+j}\,x_{1,i}\,x_{3,j}}{-1+T_2^s} -\\
&\quad s\,\epsilon\,T_2^s\,p_{2,1+i}\,p_{3,1+j}\,x_{2,i}\,x_{3,j} - s\,\epsilon\,p_{2,1+j}\,p_{3,1+j}\,x_{2,i}\,x_{3,j}\Big]
\end{aligned}$$

In[∘]:=

$$\mathcal{L}_{\text{pPush}}[C_{i\_}[\varphi\_]] :=$$

$$(T_1 T_2)^{\varphi}\ \mathbb{E}\Big[(-p_{1,i}+p_{1,1+i})\,x_{1,i} + (-p_{2,i}+p_{2,1+i})\,x_{2,i} + (-p_{3,i}+p_{3,1+i})\,x_{3,i} + \epsilon\,\varphi\left(\frac{1}{2}+p_{3,1+i}\,x_{3,i}\right)\Big]$$

```
In[•]:= $k = 1;
R_{i_,j_} = CF[ℒ_pPush[X_{i,j}[1]] /. ω_ 𝔼[L_] :> Module[{B, L1, Q, P0, P1a, P1b},
      L1 = CoefficientRules[B L /. {p_{αβ__} :> B p_{αβ}, x_{αβ__} :> B^-1 x_{αβ}}, {ϵ, B}];
      {ω, Q = {0, 1} /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
      𝔼_{{}→{i,j}}[
        ω, Sum_{ν=1}^{3} ((T_ν - 1) x_{ν,i} (p_{ν,i} - p_{ν,j})),
        ϵSeries[P0, P1a + P1b]
       ] /. {p_{ν_,i+1} :> p_{ν,i}, p_{ν_,j+1} → p_{ν,j}}
     ]];
R̄_{i_,j_} = CF[ℒ_pPush[X_{i,j}[-1]] /. ω_ 𝔼[L_] :> Module[{B, L1, Q, P0, P1a, P1b},
      L1 = CoefficientRules[B L /. {p_{αβ__} :> B p_{αβ}, x_{αβ__} :> B^-1 x_{αβ}}, {ϵ, B}];
      {ω, Q = {0, 1} /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
      𝔼_{{}→{i,j}}[
        ω, Sum_{ν=1}^{3} ((T_ν^-1 - 1) x_{ν,i} (p_{ν,i} - p_{ν,j})),
        ϵSeries[P0, P1a + P1b]
       ] /. {p_{ν_,i+1} :> p_{ν,i}, p_{ν_,j+1} → p_{ν,j}}
     ]];
CC_{i_} = CF[ℒ_pPush[C_i[1]] /. ω_ 𝔼[L_] :> Module[{B, L1, Q, P0, P1a, P1b},
      L1 = CoefficientRules[B L /. {p_{αβ__} :> B p_{αβ}, x_{αβ__} :> B^-1 x_{αβ}}, {ϵ, B}];
      {ω, Q = {0, 1} /. L1, P0 = 0, P1a = 0, P1b = {1, 1} /. L1};
      𝔼_{{}→{i}}[
        ω, 0,
        ϵSeries[P0, P1a + P1b]
       ] /. {p_{ν_,i+1} :> p_{ν,i}}
     ]];
C̄C̄_{i_} = CF[ℒ_pPush[C_i[-1]] /. ω_ 𝔼[L_] :> Module[{B, L1, Q, P0, P1a, P1b},
      L1 = CoefficientRules[B L /. {p_{αβ__} :> B p_{αβ}, x_{αβ__} :> B^-1 x_{αβ}}, {ϵ, B}];
      {ω, Q = {0, 1} /. L1, P0 = 0, P1a = 0, P1b = {1, 1} /. L1};
      𝔼_{{}→{i}}[
        ω, 0,
        ϵSeries[P0, P1a + P1b]
       ] /. {p_{ν_,i+1} :> p_{ν,i}}
     ]];
```

In[∘]:= $\mathbf{R_{i,j}}$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[T_1 T_2,\ (-1+T_1)\,p_{1,i}\,x_{1,i}+(1-T_1)\,p_{1,j}\,x_{1,i}+(-1+T_2)\,p_{2,i}\,x_{2,i}+(1-T_2)\,p_{2,j}\,x_{2,i}+(-1+T_1 T_2)\,p_{3,i}\,x_{3,i}+(1-T_1 T_2)\,p_{3,j}\,x_{3,i},$$
$$\epsilon\text{Series}\Big[(-1+T_1)\,T_2\,p_{3,j}\,x_{1,i}\,x_{2,i}+(1-T_1)\,p_{3,j}\,x_{1,i}\,x_{2,j},\ \frac{1}{2}-T_2\,p_{2,i}\,x_{2,i}+T_1 T_2\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}+$$
$$\frac{(-1+T_1)\,T_2\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{-1+T_2}+p_{2,j}\,x_{2,j}-T_1\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}-\frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{-1+T_2}+$$
$$\frac{T_2\,(-1+T_1 T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{-1+T_2}-\frac{T_2\,(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{-1+T_2}+T_1 T_2\,p_{3,i}\,x_{3,i}-\frac{T_2\,(-1+T_1 T_2)\,p_{3,j}\,x_{3,i}}{-1+T_2}-$$
$$\frac{T_1 T_2\,(-1+T_1 T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{-1+T_2}+\frac{(-1+T_1)\,T_2\,(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{-1+T_2}-$$
$$T_1\,(-1+T_2)\,T_2\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}+T_2\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}+2\,T_1 T_2\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}+$$
$$\frac{T_2\,(-1+T_1 T_2)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{-1+T_2}-\frac{(-1+2\,T_2)\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{-1+T_2}+$$
$$T_1\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}+\frac{(-1+T_1)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{-1+T_2}-T_2\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}-p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\Big]\Big]$$

In[∘]:= $\mathbf{\overline{R}_{i,j}}$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[\frac{1}{T_1 T_2},\ -\frac{(-1+T_1)\,p_{1,i}\,x_{1,i}}{T_1}+\frac{(-1+T_1)\,p_{1,j}\,x_{1,i}}{T_1}-$$
$$\frac{(-1+T_2)\,p_{2,i}\,x_{2,i}}{T_2}+\frac{(-1+T_2)\,p_{2,j}\,x_{2,i}}{T_2}-\frac{(-1+T_1 T_2)\,p_{3,i}\,x_{3,i}}{T_1 T_2}+\frac{(-1+T_1 T_2)\,p_{3,j}\,x_{3,i}}{T_1 T_2},$$
$$\epsilon\text{Series}\Big[-\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2}+\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,j}}{T_1},$$
$$-\frac{1}{2}+\frac{p_{2,i}\,x_{2,i}}{T_2}-\frac{p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2}-\frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_1\,(-1+T_2)}-p_{2,j}\,x_{2,j}+\frac{p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1}+$$
$$\frac{(-1+T_1)\,T_2\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1\,(-1+T_2)}-\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}+\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}-\frac{p_{3,i}\,x_{3,i}}{T_1 T_2}+$$
$$\frac{(-1+T_1 T_2)\,p_{3,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}+\frac{(-1+T_1 T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1^2\,(-1+T_2)\,T_2}+\frac{(-1+T_1)\,(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1^2\,(-1+T_2)\,T_2}-$$
$$\frac{(-1+T_2)\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}}{T_1 T_2^2}+\frac{(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}}{T_1 T_2^2}-\frac{2\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}}{T_1 T_2}-$$
$$\frac{(-1+T_1 T_2)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}-\frac{(-2+T_2)\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}-$$
$$\frac{p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{T_1}-\frac{(-1+T_1)\,T_2\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{T_1\,(-1+T_2)}+\frac{p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{T_2}+p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\Big]\Big]$$

In[∘]:= $\mathcal{L}_{\text{pPush}}[\mathbf{C_i}[1]]$

Out[∘]=

$$T_1 T_2\,\mathbb{E}\Big[(-p_{1,i}+p_{1,1+i})\,x_{1,i}+(-p_{2,i}+p_{2,1+i})\,x_{2,i}+(-p_{3,i}+p_{3,1+i})\,x_{3,i}+\epsilon\left(\frac{1}{2}+p_{3,1+i}\,x_{3,i}\right)\Big]$$

In[•]:= CC_1 // CF

Out[•]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[T_1 T_2,\ 0,\ \epsilon\text{Series}\left[0,\ \frac{1}{2}+p_{3,1}\,x_{3,1}\right]\right]$$

In[•]:= $\overline{CC}_1$ // CF

Out[•]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{T_1 T_2},\ 0,\ \epsilon\text{Series}\left[0,\ -\frac{1}{2}-p_{3,1}\,x_{3,1}\right]\right]$$

In[•]:= $k = 0; {
{"C$\overline{\text{C}}$", ($CC_1\,\overline{CC}_2$ // $m_{1,2\to1}$) ≡ $\mathbb{E}_{\{\}\to\{1\}}$[1, 0, ϵSeries[]]},
{"An R1", ($CC_3\,R_{1,2}$ // $m_{2,3\to2}$ // $m_{2,1\to1}$) ≡ ($\overline{CC}_3\,R_{1,2}$ // $m_{1,3\to1}$ // $m_{1,2\to1}$)},
{"R2b", ($R_{1,2}\,\overline{R}_{3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$) ≡ $\mathbb{E}_{\{\}\to\{1,2\}}$[1, 0, ϵSeries[]]},
{"R3", ($R_{1,2}\,R_{4,3}\,R_{5,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$) ≡ ($R_{2,3}\,R_{4,5}\,R_{1,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$)}
}

Out[•]=

{{C$\overline{\text{C}}$, True}, {An R1, True}, {R2b, True}, {R3, True}}

In[•]:= $k = 1; HL[($CC_3\,R_{1,2}$ // $m_{2,3\to2}$ // $m_{2,1\to1}$) ≡ ($\overline{CC}_3\,R_{1,2}$ // $m_{1,3\to1}$ // $m_{1,2\to1}$)]

Out[•]=

True

In[•]:= $R_{1,2}\,\overline{R}_{3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$

Out[•]=

$$\mathbb{E}_{\{\}\to\{1,2\}}[1,\ 0,\ \epsilon\text{Series}[0,\ 0]]$$

In[•]:= $k = 1; HL[($R_{1,2}\,\overline{R}_{3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$) ≡ $\mathbb{E}_{\{\}\to\{1,2\}}$[1, 0, ϵSeries[]]]

Out[•]=

True

In[•]:= $k = 1; HL[($R_{1,2}\,R_{4,3}\,R_{5,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$) ≡ ($R_{2,3}\,R_{4,5}\,R_{1,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$)]

Out[•]=

True

In[•]:= Z[Knot[3, 1]]

Out[•]=

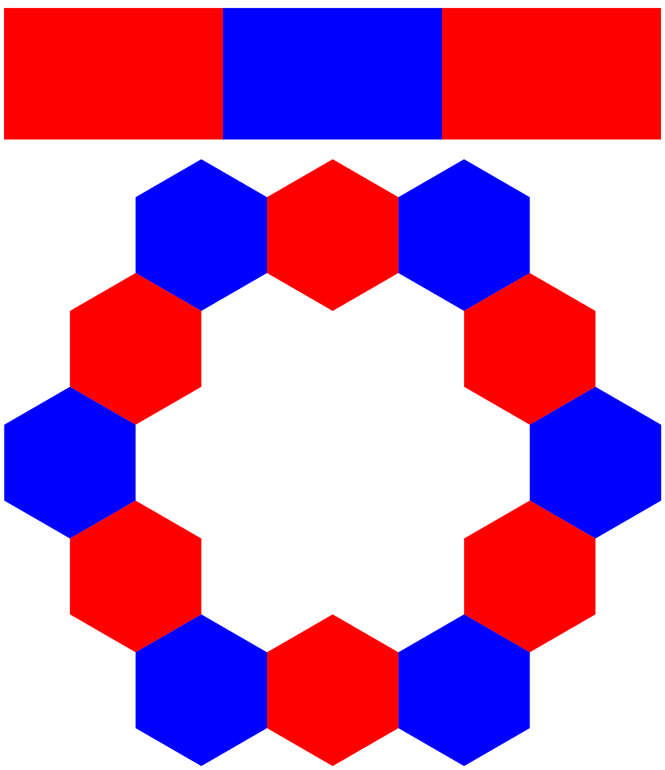
$$\mathbb{E}_{\{\}\to\{0\}}\left[\frac{T_1^2 T_2^2}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)},\ 0,\right.$$
$$\left.\epsilon\text{Series}\left[0,\ -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4}{\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1 T_2+T_1^2 T_2^2\right)}\right]\right]$$

In[•]:= Θ[K_] := Z[K] /. $\mathbb{E}_{\{\}\to\{0\}}$[d_, 0, ϵSeries[0, th_]] :→ CCF@Module[{d1},
d1 = PowerExpand[$d^{1/2}$ /. {$T_1\to T$, $T_2\to 1$}];
{(d1 /. T → 1) / d1, th / d}
]

In[ ]:=
```
Θ[Knot[3, 1]]
```
Out[ ]=
$$\left\{\frac{1-T+T^2}{T},\ -\frac{1-T_1+T_1^2-T_2-T_1^3\,T_2+T_2^2+T_1^4\,T_2^2-T_1\,T_2^3-T_1^4\,T_2^3+T_1^2\,T_2^4-T_1^3\,T_2^4+T_1^4\,T_2^4}{T_1^2\,T_2^2}\right\}$$

In[ ]:=
```
Θ[Knot[4, 1]]
```
Out[ ]=
$$\left\{-\frac{1-3\,T+T^2}{T},\ 0\right\}$$

In[ ]:=
```
Table[
 Echo@AbsoluteTiming@PolyPlot[Th = Θ[K]];
 K → Th,
 {K, AllKnots[{3, 7}]}
]
```

» {204.75,  }

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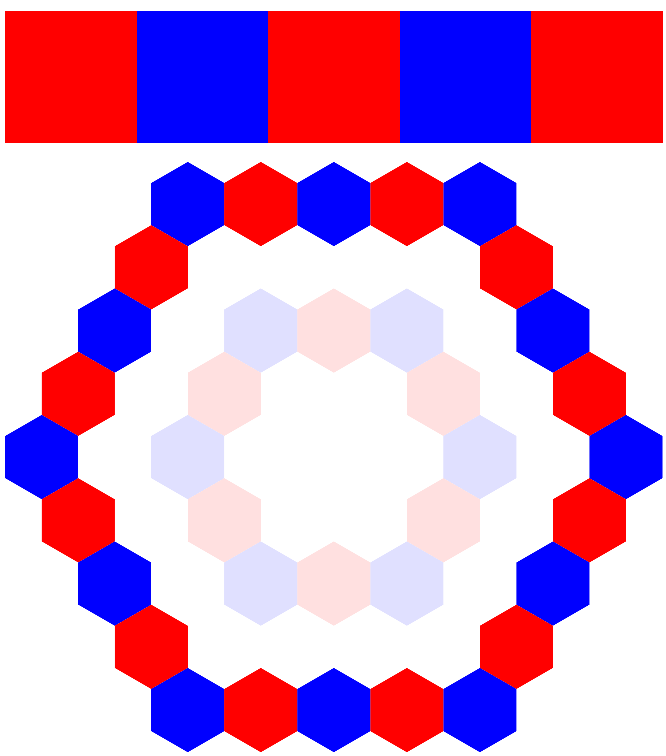

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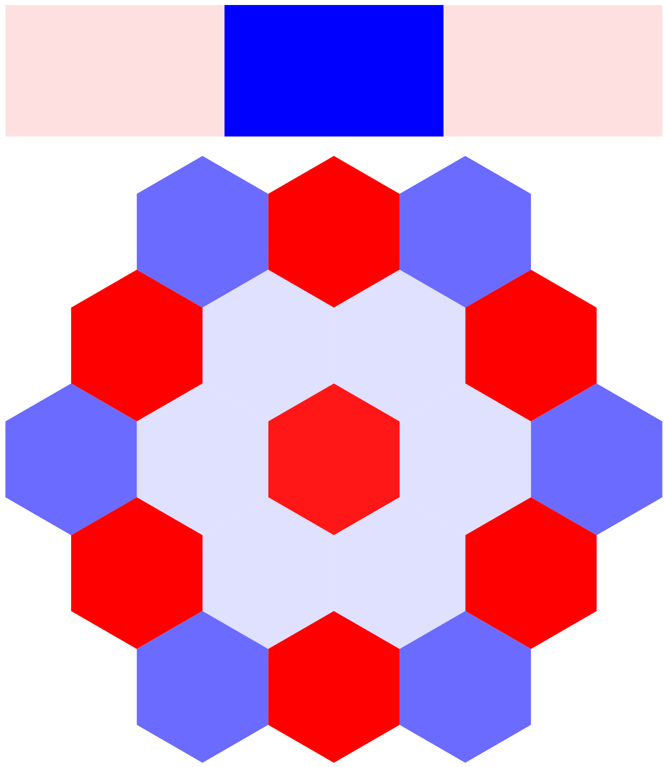
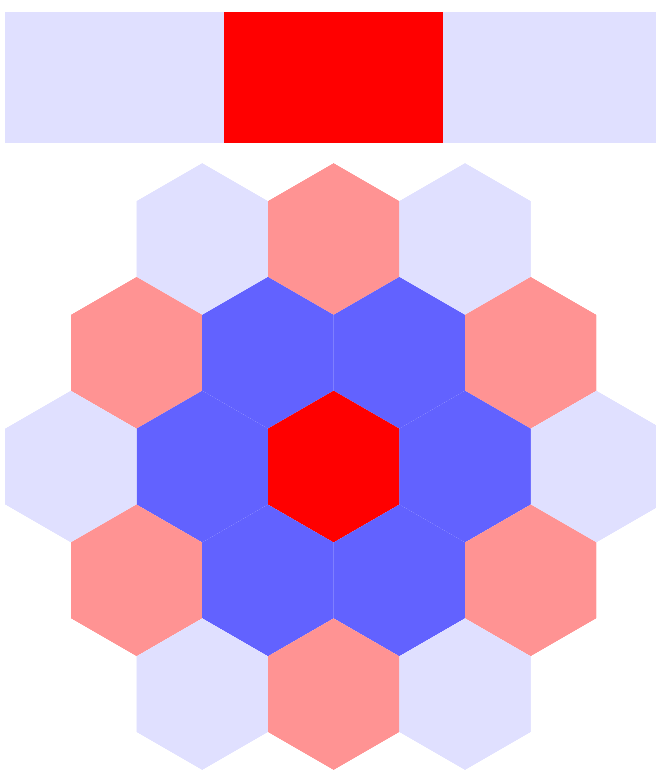
» {1965.13,  }

» {549.147,  }

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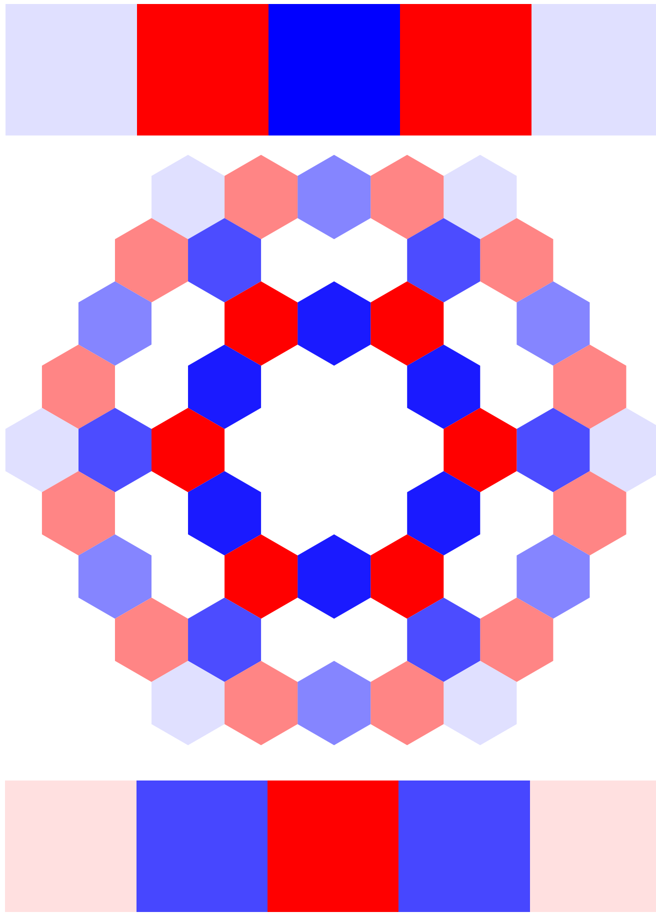

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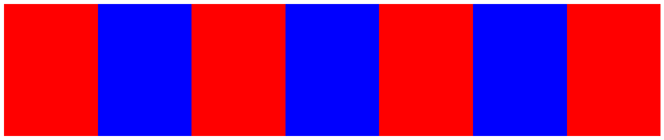
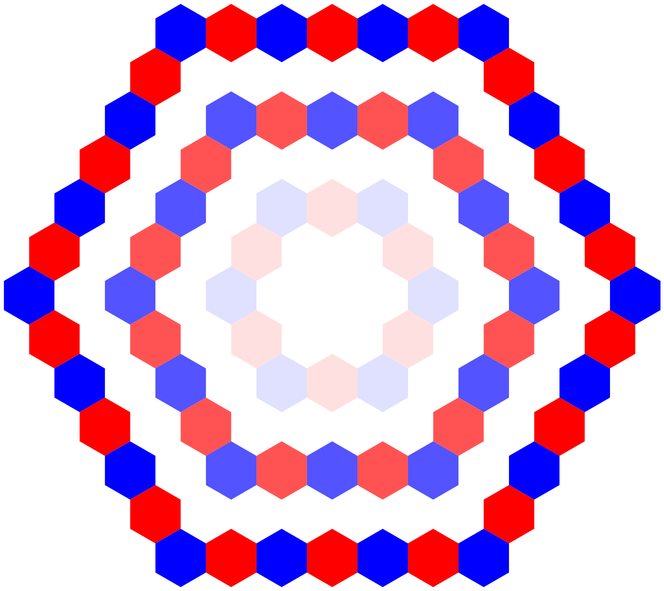
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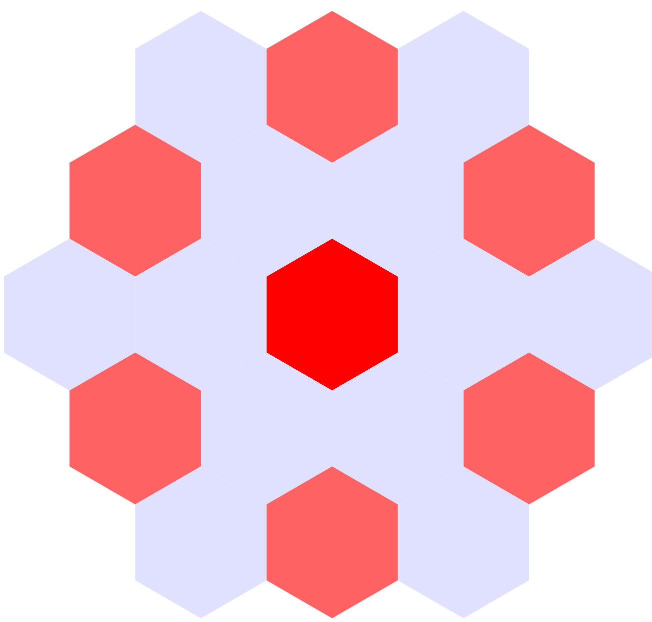
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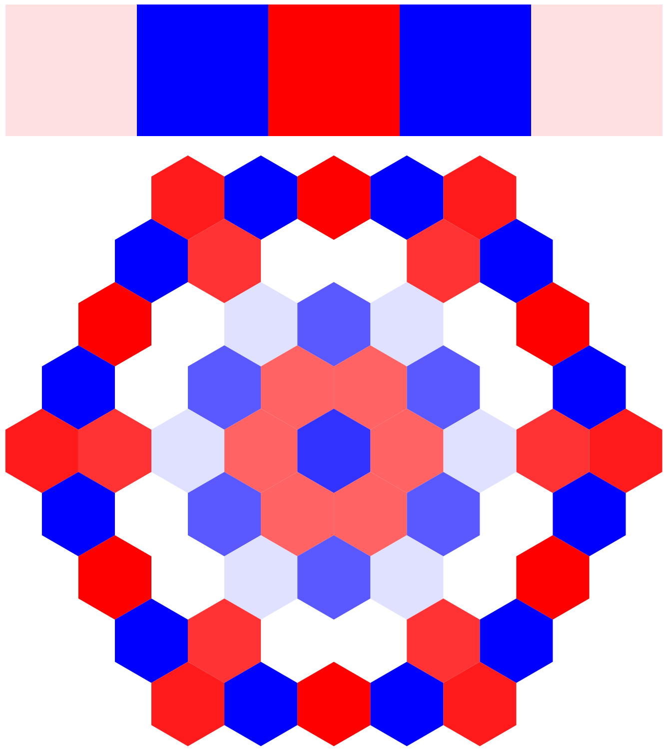
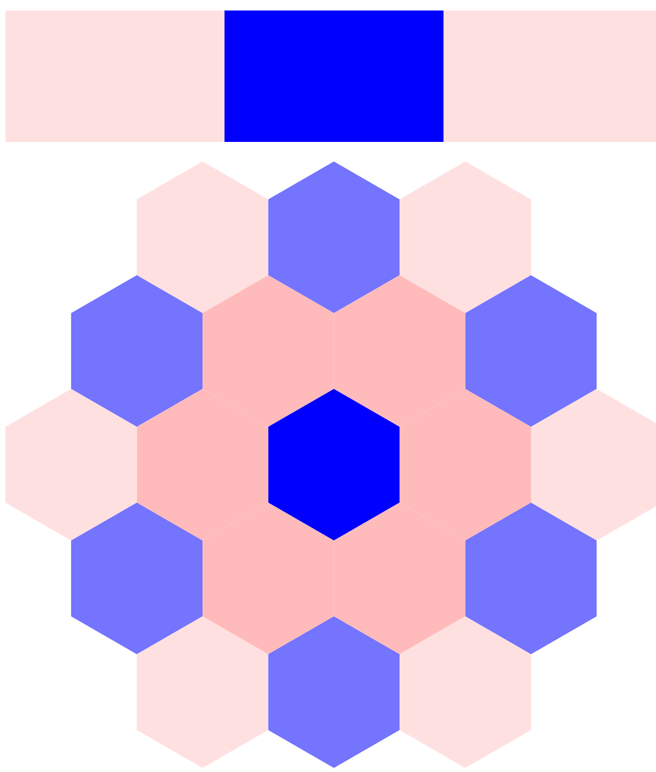
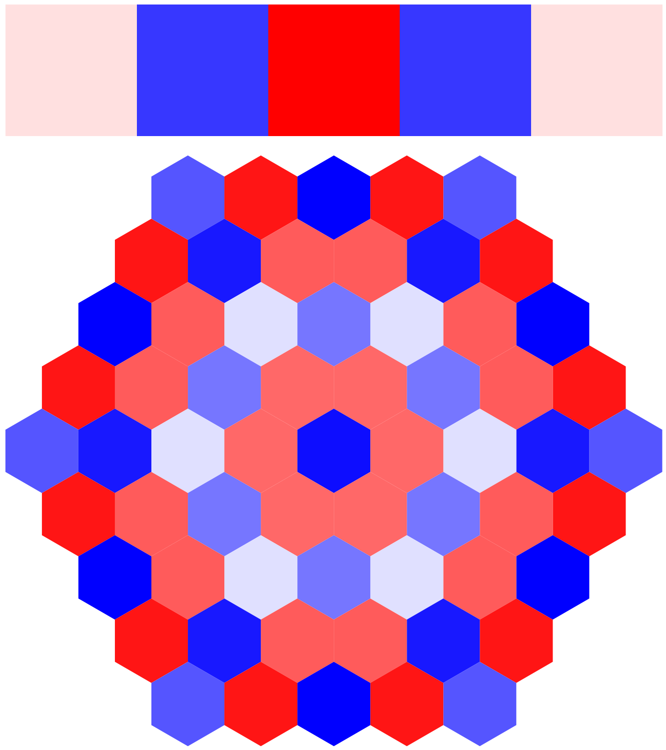
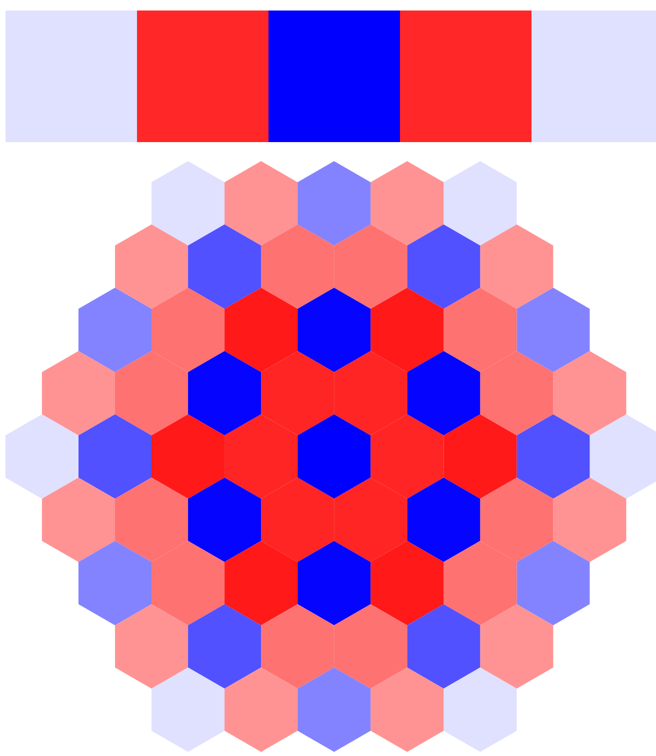
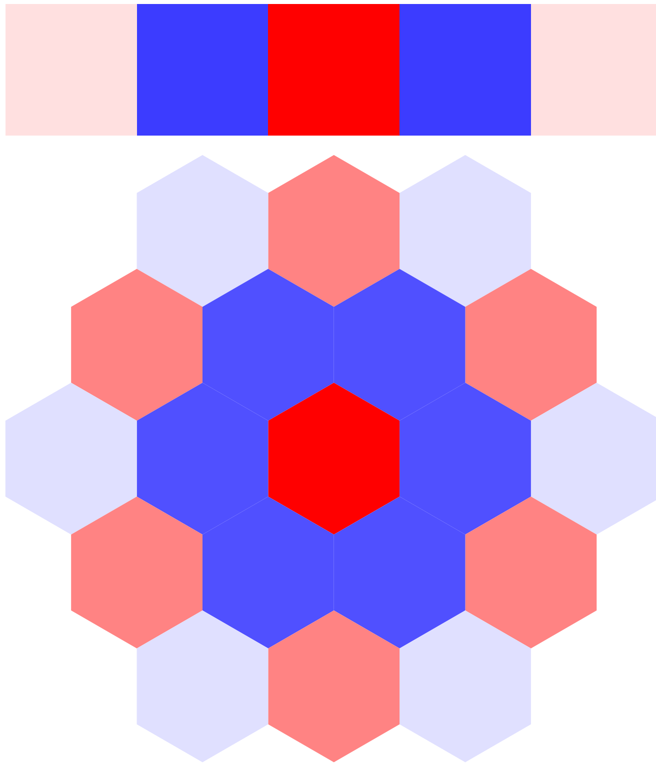
» {1650.63,  }

» {495.647,  }

» {1195.17,  }

» {1749.92,  }

» {2270.15,  }

Out[ ]=

$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},\ -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4}{T_1^2 T_2^2}\right\},\right.$$

$$\text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},\ 0\right\},\ \text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},\right.$$

$$-\frac{1}{T_1^4 T_2^4}\left(2-2T_1+2T_1^2-2T_1^3+2T_1^4-2T_2-2T_1^5 T_2+2T_2^2+T_1^2 T_2^2-T_1^3 T_2^2+T_1^4 T_2^2+2T_1^6 T_2^2-2T_2^3-T_1^2 T_2^3-\right.$$
$$T_1^5 T_2^3-2T_1^7 T_2^3+2T_2^4+T_1^2 T_2^4+T_1^6 T_2^4+2T_1^8 T_2^4-2T_1 T_2^5-T_1^3 T_2^5-T_1^6 T_2^5-2T_1^8 T_2^5+2T_1^2 T_2^6+T_1^4 T_2^6-$$
$$\left.\left.T_1^5 T_2^6+T_1^6 T_2^6+2T_1^8 T_2^6-2T_1^3 T_2^7-2T_1^8 T_2^7+2T_1^4 T_2^8-2T_1^5 T_2^8+2T_1^6 T_2^8-2T_1^7 T_2^8+2T_1^8 T_2^8\right)\right\},$$

$$\text{Knot}[5,2]\to\left\{\frac{2-3T+2T^2}{T},\ -\frac{1}{T_1^2 T_2^2}\left(9-13T_1+9T_1^2-13T_2+6T_1 T_2+6T_1^2 T_2-13T_1^3 T_2+9T_2^2+6T_1 T_2^2-\right.\right.$$
$$\left.\left.12T_1^2 T_2^2+6T_1^3 T_2^2+9T_1^4 T_2^2-13T_1 T_2^3+6T_1^2 T_2^3+6T_1^3 T_2^3-13T_1^4 T_2^3+9T_1^2 T_2^4-13T_1^3 T_2^4+9T_1^4 T_2^4\right)\right\},$$

$$\text{Knot}[6,1]\to\left\{-\frac{(-2+T)\,(-1+2T)}{T},\ -\frac{1}{T_1^2 T_2^2}\left(1-3T_1+T_1^2-3T_2+6T_1 T_2+6T_1^2 T_2-3T_1^3 T_2+T_2^2+\right.\right.$$
$$\left.\left.6T_1 T_2^2-24T_1^2 T_2^2+6T_1^3 T_2^2+T_1^4 T_2^2-3T_1 T_2^3+6T_1^2 T_2^3+6T_1^3 T_2^3-3T_1^4 T_2^3+T_1^2 T_2^4-3T_1^3 T_2^4+T_1^4 T_2^4\right)\right\},$$

$$\text{Knot}[6,2]\to\left\{-\frac{1-3T+3T^2-3T^3+T^4}{T^2},\ -\frac{1}{T_1^4 T_2^4}\left(1-3T_1+3T_1^2-3T_1^3+T_1^4-3T_2+6T_1 T_2+\right.\right.$$
$$6T_1^4 T_2-3T_1^5 T_2+3T_2^2-15T_1^2 T_2^2+11T_1^3 T_2^2-15T_1^4 T_2^2+3T_1^6 T_2^2-3T_2^3+11T_1^2 T_2^3+$$
$$11T_1^5 T_2^3-3T_1^7 T_2^3+T_2^4+6T_1 T_2^4-15T_1^2 T_2^4-15T_1^6 T_2^4+6T_1^7 T_2^4+T_1^8 T_2^4-3T_1 T_2^5+$$
$$11T_1^3 T_2^5+11T_1^6 T_2^5-3T_1^8 T_2^5+3T_1^2 T_2^6-15T_1^4 T_2^6+11T_1^5 T_2^6-15T_1^6 T_2^6+3T_1^8 T_2^6-$$
$$\left.\left.3T_1^3 T_2^7+6T_1^4 T_2^7+6T_1^7 T_2^7-3T_1^8 T_2^7+T_1^4 T_2^8-3T_1^5 T_2^8+3T_1^6 T_2^8-3T_1^7 T_2^8+T_1^8 T_2^8\right)\right\},$$

$$\text{Knot}[6,3]\to\left\{\frac{1-3\,T+5\,T^2-3\,T^3+T^4}{T^2},\,0\right\},\ \text{Knot}[7,1]\to$$
$$\left\{\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3},\right.$$
$$-\frac{1}{T_1^6\,T_2^6}\,\left(3-3\,T_1+3\,T_1^2-3\,T_1^3+3\,T_1^4-3\,T_1^5+3\,T_1^6-3\,T_2-3\,T_1^7\,T_2+3\,T_2^2+2\,T_1^2\,T_2^2-2\,T_1^3\,T_2^2+2\,T_1^4\,T_2^2-\right.$$
$$2\,T_1^5\,T_2^2+2\,T_1^6\,T_2^2+3\,T_1^8\,T_2^2-3\,T_2^3-2\,T_1^2\,T_2^3-2\,T_1^7\,T_2^3-3\,T_1^9\,T_2^3+3\,T_2^4+2\,T_1^2\,T_2^4+T_1^4\,T_2^4-T_1^5\,T_2^4+$$
$$T_1^6\,T_2^4+2\,T_1^8\,T_2^4+3\,T_1^{10}\,T_2^4-3\,T_2^5-2\,T_1^2\,T_2^5-T_1^4\,T_2^5-T_1^7\,T_2^5-2\,T_1^9\,T_2^5-3\,T_1^{11}\,T_2^5+3\,T_2^6+2\,T_1^2\,T_2^6+$$
$$T_1^4\,T_2^6+T_1^8\,T_2^6+2\,T_1^{10}\,T_2^6+3\,T_1^{12}\,T_2^6-3\,T_1\,T_2^7-2\,T_1^3\,T_2^7-T_1^5\,T_2^7-T_1^8\,T_2^7-2\,T_1^{10}\,T_2^7-3\,T_1^{12}\,T_2^7+3\,T_1^2\,T_2^8+$$
$$2\,T_1^4\,T_2^8+T_1^6\,T_2^8-T_1^7\,T_2^8+T_1^8\,T_2^8+2\,T_1^{10}\,T_2^8+3\,T_1^{12}\,T_2^8-3\,T_1^3\,T_2^9-2\,T_1^5\,T_2^9-2\,T_1^{10}\,T_2^9-3\,T_1^{12}\,T_2^9+3\,T_1^4\,T_2^{10}+$$
$$2\,T_1^6\,T_2^{10}-2\,T_1^7\,T_2^{10}+2\,T_1^8\,T_2^{10}-2\,T_1^9\,T_2^{10}+2\,T_1^{10}\,T_2^{10}+3\,T_1^{12}\,T_2^{10}-3\,T_1^5\,T_2^{11}-3\,T_1^{12}\,T_2^{11}+3\,T_1^6\,T_2^{12}-$$
$$\left.\left.3\,T_1^7\,T_2^{12}+3\,T_1^8\,T_2^{12}-3\,T_1^9\,T_2^{12}+3\,T_1^{10}\,T_2^{12}-3\,T_1^{11}\,T_2^{12}+3\,T_1^{12}\,T_2^{12}\right)\right\},\ \text{Knot}[7,2]\to\left\{\frac{3-5\,T+3\,T^2}{T},\right.$$
$$-\frac{1}{T_1^2\,T_2^2}\,2\,\left(18-29\,T_1+18\,T_1^2-29\,T_2+18\,T_1\,T_2+18\,T_1^2\,T_2-29\,T_1^3\,T_2+18\,T_2^2+18\,T_1\,T_2^2-42\,T_1^2\,T_2^2+\right.$$
$$\left.\left.18\,T_1^3\,T_2^2+18\,T_1^4\,T_2^2-29\,T_1\,T_2^3+18\,T_1^2\,T_2^3+18\,T_1^3\,T_2^3-29\,T_1^4\,T_2^3+18\,T_1^2\,T_2^4-29\,T_1^3\,T_2^4+18\,T_1^4\,T_2^4\right)\right\},$$
$$\text{Knot}[7,3]\to\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\,\frac{1}{T_1^4\,T_2^4}\,\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+\right.\right.$$
$$12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+$$
$$6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-$$
$$T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+$$
$$25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+$$
$$\left.\left.17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\},\ \text{Knot}[7,4]\to\left\{\frac{4-7\,T+4\,T^2}{T},\right.$$
$$\frac{1}{T_1^2\,T_2^2}\,8\,\left(10-17\,T_1+10\,T_1^2-17\,T_2+12\,T_1\,T_2+12\,T_1^2\,T_2-17\,T_1^3\,T_2+10\,T_2^2+12\,T_1\,T_2^2-30\,T_1^2\,T_2^2+\right.$$
$$\left.\left.12\,T_1^3\,T_2^2+10\,T_1^4\,T_2^2-17\,T_1\,T_2^3+12\,T_1^2\,T_2^3+12\,T_1^3\,T_2^3-17\,T_1^4\,T_2^3+10\,T_1^2\,T_2^4-17\,T_1^3\,T_2^4+10\,T_1^4\,T_2^4\right)\right\},$$
$$\text{Knot}[7,5]\to\left\{\frac{2-4\,T+5\,T^2-4\,T^3+2\,T^4}{T^2},\,-\frac{1}{T_1^4\,T_2^4}\,\left(17-33\,T_1+41\,T_1^2-33\,T_1^3+17\,T_1^4-33\,T_2+\right.\right.$$
$$32\,T_1\,T_2-16\,T_1^2\,T_2-16\,T_1^3\,T_2+32\,T_1^4\,T_2-33\,T_1^5\,T_2+41\,T_2^2-16\,T_1\,T_2^2+4\,T_1^2\,T_2^2+12\,T_1^3\,T_2^2+$$
$$4\,T_1^4\,T_2^2-16\,T_1^5\,T_2^2+41\,T_1^6\,T_2^2-33\,T_2^3-16\,T_1\,T_2^3+12\,T_1^2\,T_2^3-14\,T_1^3\,T_2^3-14\,T_1^4\,T_2^3+12\,T_1^5\,T_2^3-$$
$$16\,T_1^6\,T_2^3-33\,T_1^7\,T_2^3+17\,T_2^4+32\,T_1\,T_2^4+4\,T_1^2\,T_2^4-14\,T_1^3\,T_2^4+36\,T_1^4\,T_2^4-14\,T_1^5\,T_2^4+4\,T_1^6\,T_2^4+32\,T_1^7\,T_2^4+$$
$$17\,T_1^8\,T_2^4-33\,T_1\,T_2^5-16\,T_1^2\,T_2^5+12\,T_1^3\,T_2^5-14\,T_1^4\,T_2^5-14\,T_1^5\,T_2^5+12\,T_1^6\,T_2^5-16\,T_1^7\,T_2^5-33\,T_1^8\,T_2^5+$$
$$41\,T_1^2\,T_2^6-16\,T_1^3\,T_2^6+4\,T_1^4\,T_2^6+12\,T_1^5\,T_2^6+4\,T_1^6\,T_2^6-16\,T_1^7\,T_2^6+41\,T_1^8\,T_2^6-33\,T_1^3\,T_2^7+32\,T_1^4\,T_2^7-$$
$$\left.\left.16\,T_1^5\,T_2^7-16\,T_1^6\,T_2^7+32\,T_1^7\,T_2^7-33\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-33\,T_1^5\,T_2^8+41\,T_1^6\,T_2^8-33\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\},$$
$$\text{Knot}[7,6]\to\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\,-\frac{1}{T_1^4\,T_2^4}\,\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-\right.\right.$$
$$10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-$$
$$10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-$$
$$5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+$$
$$T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+$$

$$7 T_1^2 T_2^6 - 10 T_1^3 T_2^6 - 64 T_1^4 T_2^6 + 98 T_1^5 T_2^6 - 64 T_1^6 T_2^6 - 10 T_1^7 T_2^6 + 7 T_1^8 T_2^6 - 5 T_1^3 T_2^7 + 20 T_1^4 T_2^7 - 10 T_1^5 T_2^7 - 10 T_1^6 T_2^7 + 20 T_1^7 T_2^7 - 5 T_1^8 T_2^7 + T_1^4 T_2^8 - 5 T_1^5 T_2^8 + 7 T_1^6 T_2^8 - 5 T_1^7 T_2^8 + T_1^8 T_2^8\Big)\Big\},$$

$$\text{Knot}[7, 7] \to \Big\{\frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^2}, -\frac{1}{T_1^2 T_2^2} \Big(1 - 5 T_1 + T_1^2 - 5 T_2 + 12 T_1 T_2 + 12 T_1^2 T_2 - 5 T_1^3 T_2 + T_2^2 + 12 T_1 T_2^2 - 48 T_1^2 T_2^2 + 12 T_1^3 T_2^2 + T_1^4 T_2^2 - 5 T_1 T_2^3 + 12 T_1^2 T_2^3 + 12 T_1^3 T_2^3 - 5 T_1^4 T_2^3 + T_1^2 T_2^4 - 5 T_1^3 T_2^4 + T_1^4 T_2^4\Big)\Big\}\Big\}$$