The full sl_3 invariant using the Drinfel'd double. For compatibility reasons, we use XX instead of the generator X. This program continues sl2invariant.nb by Dror Bar-Natan and Roland van der Veen.

Profiling

In[∘]:=
```
PP_ = Identity;
(*BeginProfile[];*)
```

External Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

In[∘]:=
```
MaxBy[list_, fun_, n_] := list[[Ordering[fun /@ list, -n]]];
```

Canonical Form:

Program

In[∘]:=
```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
        Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (XX | Y | Z | A | B | b | s | t | a | x |
            y | z | XX* | Y* | Z* | A* | B* | s* | t* | b* | a* | x* | y* | z*)_, ∞] ∪
        {XX, Y, Z, A, B, b, s, t, a, x, y, z, XX*, Y*, Z*, A*, B*, s*, t*, b*, a*, x*, y*, z*}},
    Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
   ];
```

Program

The Kronecker δ:

Program

In[∘]:=
```
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q}\,P$:

Program

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

In[∘]:=
```
𝔼3@𝔼sp___[ω_, L_, Q_, P_] := Module[
   {NP = Normal[P]},
   𝔼sp[L, ω^-1 Q, (ω^-1 NP /. ϵ → ω^-4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
𝔼4@𝔼sp___[L_, Q_, P_] := Module[
   {NP = Normal[P], ω},
   ω = (NP /. ϵ → 0)^-1;
   𝔼sp[ω, L, ω Q, (ω NP /. ϵ → ω^4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[∘]:=
```
(u_i_)* := (u*)i;
((u_i_)*)* := ui;
(((u_)*)i_)* := ui;
((u_)*)* := u;
```

Program

Finite Zips:

Program

In[∘]:=
```
collect[sd_SeriesData, ξ_] := MapAt[collect[#, ξ] &, sd, 3];
collect[ℰ_, ξ_] := PPCollect@Collect[ℰ, ξ];
Zip{}[P_] := P; Zip{ξ_,ξs___}[P_] := PPZip[
  (collect[P // Zip{ξs}, ξ] /. f_. ξ^d_. :→ ∂{ξ*,d} f) /. ξ* → 0]
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars. $\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a,b's, Q is a combination of $x_i X_j$ (possibly starred and/or mixed with other variables), and P is a perturbation polynomial. It should be interpreted via $\mathbb{O}[\mathbb{E}[\ldots], \{X_1, Y_1, Z_1, A_1, B_1, b_1, a_1, z_1, y_1, x_1\}_i, \ldots]$, with an assumed standard ordering on the generators for an interpretation of the tensor as an expression in $U_q(\mathrm{sl}_3{}^{\epsilon})$.

Program

In[∘]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := PP_{QZip}@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

Program

Upper to lower and lower to Upper:

Program

In[∘]:=
```
U2l = {𝕓_{i_·}^{p_·} → 𝕖^{-p b_i}, 𝕓^{p_·} → 𝕖^{-p b}, 𝕒_{i_·}^{p_·} → 𝕖^{-p a_i}, 𝕒^{p_·} → 𝕖^{-p a}, T_{i_·}^{p_·} → 𝕖^{p t_i}, T^{p_·} → 𝕖^{p t}, S_{i_·}^{p_·} → 𝕖^{p s_i},
   S^{p_·} → 𝕖^{p s}, a_{i_·}^{p_·} → 𝕖^{p a*_i}, a^{p_·} → 𝕖^{p a*}, b_{i_·}^{p_·} → 𝕖^{p b*_i}, b^{p_·} → 𝕖^{p b*}, 𝔹_{i_·}^{p_·} → 𝕖^{-p B_i}, 𝔹^{p_·} → 𝕖^{-p B},
   𝔸_{i_·}^{p_·} → 𝕖^{-p A_i}, 𝔸^{p_·} → 𝕖^{-p A}, 𝒜_{i_·}^{p_·} → 𝕖^{p A*_i}, 𝒜^{p_·} → 𝕖^{p A*}, ℬ_{i_·}^{p_·} → 𝕖^{p B*_i}, ℬ^{p_·} → 𝕖^{p B*}};
l2U = {𝕖^{c_·. b_{i_}+d_·} :→ 𝕓_i^{-c} 𝕖^d, 𝕖^{c_·. b+d_·} :→ 𝕓^{-c} 𝕖^d, 𝕖^{c_·. a_{i_}+d_·} :→ 𝕒_i^{-c} 𝕖^d, 𝕖^{c_·. a+d_·} :→ 𝕒^{-c} 𝕖^d,
   𝕖^{c_·. t_{i_}+d_·} :→ T_i^c 𝕖^d, 𝕖^{c_·. t+d_·} :→ T^c 𝕖^d, 𝕖^{c_·. s_{i_}+d_·} :→ S_i^c 𝕖^d, 𝕖^{c_·. s+d_·} :→ S^c 𝕖^d,
   𝕖^{c_·. a*_{i_}+d_·} :→ a_i^c 𝕖^d, 𝕖^{c_·. a*+d_·} :→ a^c 𝕖^d, 𝕖^{c_·. b*_{i_}+d_·} :→ b_i^c 𝕖^d, 𝕖^{c_·. b*+d_·} :→ b^c 𝕖^d,
   𝕖^{c_·. B_{i_}+d_·} :→ 𝔹_i^{-c} 𝕖^d, 𝕖^{c_·. B+d_·} :→ 𝔹^{-c} 𝕖^d, 𝕖^{c_·. A_{i_}+d_·} :→ 𝔸_i^{-c} 𝕖^d, 𝕖^{c_·. A+d_·} :→ 𝔸^{-c} 𝕖^d,
   𝕖^{c_·. A*_{i_}+d_·} :→ 𝒜_i^c 𝕖^d, 𝕖^{c_·. A*+d_·} :→ 𝒜^c 𝕖^d, 𝕖^{c_·. B*_{i_}+d_·} :→ ℬ_i^c 𝕖^d, 𝕖^{c_·. B*+d_·} :→ ℬ^c 𝕖^d,
   𝕖^{ℰ_·} :→ 𝕖^{Expand@ℰ}};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = \mathrm{P}\mathbb{e}^{L+Q}$. Such zips regard all of $\mathrm{P}\mathbb{e}^{Q}$ as a single"P". Here the z's are A, B and a^*, b^* and the ζ's are A^*, B^* and a, b. s and t are not regarded as scalars for zip-technicalities. DB and STB are variations of B with a different choice for ζs in LZip, to speed up the zipping of tensors with s and t instead of A and B.

Program

In[∘]:=
```
LZip_{ξs_List}@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ξ, z, zs, c, ys, ηs, lt, zrule, Zrule, ξrule, Q1, EEQ, EQ},
    zs = Table[ξ*, {ξ, ξs}];
    c = L /. Alternatives @@ (ξs ⋃ zs) → 0;
    ys = Table[∂_ξ (L /. Alternatives @@ zs → 0), {ξ, ξs}];
    ηs = Table[∂_z (L /. Alternatives @@ ξs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} L, {ξ, ξs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    ξrule = Thread[ξs → ξs + ηs.lt];
    Q1 = Q /. U2l /. (zrule ⋃ ξrule);
    EEQ[ps___] := EEQ[ps] = PP_"EEQ"@
       (CF[ⅇ^(-Q1) D[ⅇ^Q1, Sequence @@ Thread[{zs, {ps}}]]] /. Alternatives @@ zs → 0 //. l2U);
    CF /@ ((*CF/@*)𝔼[
        c + ηs.lt.ys, Q1 /. Alternatives @@ zs → 0,
        Det[lt] (Zip_ξs[(EQ @@ zs) (P /. U2l /. (zrule ⋃ ξrule))] /.
            Derivative[ps___][EQ][___] :→ EEQ[ps] /. _EQ → 1)
       ] //. l2U)
    ];
```

Program

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_Bind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | A | 𝔸 | B | 𝔹 | s | t | S | T | b | (*𝕓|*)a(*|𝕒*) | x | y | z)_i →
           v_{n@i}, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | A* (*|𝒜*) | B* |
                (*ℬ| *)b* | s* | t* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_{n@i}, {i, {is}}]
       ] // LZip_Flatten@Table[{A*_{n@i},B*_{n@i},(s*)_{n@i},(t*)_{n@i},b_{n@i},a_{n@i}},{i,{is}}] //
      QZip_Flatten@Table[{XX_{n@i},Y*_{n@i},y_{n@i},x*_{n@i}},{i,{is}}] // QZip_Flatten@Table[{Z*_{n@i},z_{n@i}},{i,{is}}]];
B_{is___}[L_, R_] := B_{is}[L, R];
```

```
DB_{}[L_, R_] := L R;
DB_{is__}[L_𝔼, R_𝔼] := PP_DBind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | s | t | b | 𝕓 | a | 𝕒 | x | y | z)_i → v_{n@i}, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | s* | t* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_{n@i}, {i, {is}}]
       ] // LZip_Flatten@Table[{(s*)_{n@i},(t*)_{n@i},b_{n@i},a_{n@i}},{i,{is}}] //
      QZip_Flatten@Table[{XX_{n@i},Y*_{n@i},y_{n@i},x*_{n@i}},{i,{is}}] // QZip_Flatten@Table[{Z*_{n@i},z_{n@i}},{i,{is}}]];
DB_{is___}[L_, R_] := DB_{is}[L, R];
```

In[∘]:=
```
STB{}[L_, R_] := L R;
STB{is__}[L_𝔼, R_𝔼] := PPSTBind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | b | 𝕓 | a | 𝕒 | x | y | z)i → vn@i, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | b* | a* | 𝕒 | 𝕓 | x* | y* | z*)i → vn@i, {i, {is}}]
       ] // LZipFlatten@Table[{bn@i,an@i},{i,{is}}] // QZipFlatten@Table[{XXn@i,Y*n@i,yn@i,x*n@i},{i,{is}}] //
     QZipFlatten@Table[{Z*n@i,zn@i},{i,{is}}]];
STBis___[L_, R_] := STB{is}[L, R];
```

Program

𝔼 morphisms with domain and range.

Program

In[∘]:=
```
Bis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
STBis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ STBis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] // 𝔼d2_→r2_[L2_, Q2_, P2_] :=
  Br1⋂d2[𝔼d1→r1[L1, Q1, P1], 𝔼d2→r2[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_[L1_, Q1_, P1_] 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼d_→r_[L_, Q_, P_]$k_ := 𝔼d→r @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{{sis}}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

Program

Booting Up

Program

In[∘]:=
```
$k = 1;
```

The multiplication tensors on both halves of the double are defined here. $k indicates the degree $\epsilon^{\$k+1} = 0$ we are working over. The am and bm tensors given here only work for $k=1 and $k=0.

In[∘]:=
```
Define[am_{i,j→k} =
   𝔼_{{i,j}→{k}}[(a*_i + a*_j) a_k + (b*_i + b*_j) b_k, (ⅇ^{(b*)_j+(a*)_j} z*_i + z*_j + ⅇ^{-(b*)_j+2 (a*)_j} x*_i (y*)_j) z_k +
      (ⅇ^{-(a*)_j+2 (b*)_j} y*_i + y*_j) y_k + (ⅇ^{-(b*)_j+2 (a*)_j} x*_i + x*_j) x_k,
     1 + ϵ (ℏ (z*)_j (x*)_i ⅇ^{-(b*)_j+2 (a*)_j} z_k x_k + ℏ (z*)_j (x*)_i (y*)_j ⅇ^{-(b*)_j+2 (a*)_j} z_k z_k -
         ℏ (z*)_j (y*)_i ⅇ^{2 (b*)_j-(a*)_j} z_k y_k - ℏ (y*)_i (x*)_i (y*)_j ⅇ^{(a*)_j+(b*)_j} z_k y_k -
         ℏ (x*)_i ⅇ^{-(b*)_j+2 (a*)_j} (y*)_j (y*)_j z_k y_k - ℏ (x*)_i ⅇ^{-(b*)_j+2 (a*)_j} (y*)_j y_k x_k) + O[ϵ]^2]_$k,
  bm_{i,j→k} =
   𝔼_{{i,j}→{k}}[A_k A*_i + A_k A*_j + B_k B*_i + B_k B*_j, XX_k XX*_i + XX_k XX*_j + Y_k Y*_i + Y_k Y*_j + Z_k Z*_i + Z_k Z*_j,
     1 + (-XX_k (A*)_i (XX*)_j + ℏ XX_k Y_k (XX*)_j (Y*)_i - Y_k (B*)_i (Y*)_j + 2 Z_k (XX*)_i (Y*)_j - ℏ XX_k Z_k
          (XX*)_j (Z*)_i + ℏ Y_k Z_k (Y*)_j (Z*)_i - Z_k (A*)_i (Z*)_j - Z_k (B*)_i (Z*)_j) ϵ + O[ϵ]^2]_$k];
```

The R-matrix is defined with the Faddeev-Quesne formula.

Program

In[∘]:=

$$\text{Define}\Big[$$

$$R_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar\, A_i\, a_j + \hbar\, B_i\, b_j,\ \hbar\, XX_i\, x_j + \hbar\, Y_i\, y_j + \hbar\, Z_i\, z_j,\ e\,\hat{}\left(\sum_{k=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^k (\hbar\, XX_i\, x_j)^k}{k\left(1-e^{-2k\epsilon\hbar}\right)}\right)$$

$$e\,\hat{}\left(\sum_{l=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^l (\hbar\, Y_i\, y_j)^l}{l\left(1-e^{-2l\epsilon\hbar}\right)}\right) e\,\hat{}\left(\sum_{m=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^m (\hbar\, Z_i\, z_j)^m}{m\left(1-e^{-2m\epsilon\hbar}\right)}\right)\Big]_{\$k},$$

$$\overline{R}_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[-\hbar\, a_j\, A_i - \hbar\, b_j\, B_i,\ -\hbar\, x_j\, XX_i\, \mathbb{A}_i^{2\hbar}\, \mathbb{B}_i^{-\hbar} + \hbar^2\, XX_i\, Y_i\, z_j\, \mathbb{A}_i^{\hbar}\, \mathbb{B}_i^{\hbar} -$$

$$\hbar\, z_j\, Z_i\, \mathbb{A}_i^{\hbar}\, \mathbb{B}_i^{\hbar} - \hbar\, y_j\, Y_i\, \mathbb{A}_i^{-\hbar}\, \mathbb{B}_i^{2\hbar},\ 1 + \text{If}\Big[\$k == 0,\ 0,\ (\overline{R}_{\{i,j\},\$k-1})_{\$k}[3] -$$

$$\Big(\big((\overline{R}_{\{i,j\},0})_{\$k}\, R_{1,2}\, (\overline{R}_{\{3,4\},\$k-1})_{\$k}\big)\ //\ (bm_{i,1\to i}\, am_{j,2\to j})\ //\ (bm_{i,3\to i}\, am_{j,4\to j})\Big)[3]\Big]\Big],$$

$$P_{i,j} = \mathbb{E}_{\{i,j\}\to\{\}}\Big[\frac{1}{\hbar}\, A^*{}_i\, a^*{}_j + \frac{1}{\hbar}\, B^*{}_i\, (b^*)_j,$$

$$\frac{1}{\hbar}\, XX^*{}_i\, x^*{}_j + \frac{1}{\hbar}\, Y^*{}_i\, y^*{}_j + \frac{1}{\hbar}\, Z^*{}_i\, z^*{}_j,\ 1 + \text{If}[\$k == 0,\ 0,\ (P_{\{i,j\},\$k-1})_{\$k}[3] -$$

$$\big(R_{1,2}\ //\ ((P_{\{1,j\},0})_{\$k}\, (P_{\{i,2\},\$k-1})_{\$k})\big)[3]]\Big]\Big]$$

Program

In[∘]:=

$$\text{Define}\Big[aS_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j},$$

$$\overline{aS}_i = \mathbb{E}_{\{i\}\to\{i\}}\Big[-a^*{}_i\, a_i - b^*{}_i\, b_i,\ -e^{-(b^*)_i-(a^*)_i}\, z^*{}_i\, z_i + e^{-(b^*)_i-(a^*)_i}\, y^*{}_i\, (x^*)_i\, z_i -$$

$$e^{-2(b^*)_i+(a^*)_i}\, y^*{}_i\, y_i - e^{(b^*)_i-2(a^*)_i}\, x^*{}_i\, x_i,\ 1 + \text{If}\Big[\$k == 0,\ 0,\ (\overline{aS}_{\{i\},\$k-1})_{\$k}[3] -$$

$$\Big((\overline{aS}_{\{i\},0})_{\$k} \sim B_i \sim aS_i \sim B_i \sim (\overline{aS}_{\{i\},\$k-1})_{\$k}\Big)[3]\Big]\Big]\Big]$$

Program

In[∘]:=

$$\text{Define}\Big[bS_i = R_{i,1} \sim B_1 \sim aS_1 \sim B_1 \sim P_{i,1},$$

$$\overline{bS}_i = R_{i,1} \sim B_1 \sim \overline{aS}_1 \sim B_1 \sim P_{i,1},$$

$$a\Delta_{i\to j,k} = (R_{1,j}\, R_{2,k})\ //\ bm_{1,2\to 3}\ //\ P_{3,i},$$

$$b\Delta_{i\to j,k} = (R_{j,1}\, R_{k,2})\ //\ am_{1,2\to 3}\ //\ P_{i,3}\Big]$$

Program

In[∘]:=

$$\text{Define}\Big[dm_{i,j\to k} = \Big(\mathbb{E}_{\{i,j\}\to\{i,j\}}\Big[(A^*)_i\, A_i + (B^*)_i\, B_i + b^*{}_j\, b_j + a^*{}_j\, a_j,$$

$$y^*{}_j\, y_j + x^*{}_j\, x_j + z^*{}_j\, z_j + (XX^*)_i\, XX_i + (Y^*)_i\, Y_i + Z^*{}_i\, Z_i,\ 1\Big]$$

$$\Big(a\Delta_{i\to 1,2}\ //\ a\Delta_{2\to 2,3}\ //\ \overline{aS}_3\Big)\ \big(b\Delta_{j\to -1,-2}\ //\ b\Delta_{-2\to -2,-3}\big)\Big)\ //\ (P_{-1,3}\, P_{-3,1}\, am_{2,j\to k}\, bm_{i,-2\to k}),$$

$$dS_i = \mathbb{E}_{\{i\}\to\{1,2\}}[(A^*)_i\, A_1 + (B^*)_i\, B_1 + b^*{}_i\, b_2 + a^*{}_i\, a_2,$$

$$y^*{}_i\, y_2 + x^*{}_i\, x_2 + z^*{}_i\, z_2 + (XX^*)_i\, XX_1 + (Y^*)_i\, Y_1 + Z^*{}_i\, Z_1,\ 1]\ //\ \big(\overline{bS}_1\, aS_2\big)\ //\ dm_{2,1\to i},$$

$$d\Delta_{i\to j,k} = (b\Delta_{i\to 3,1}\, a\Delta_{i\to 2,4})\ //\ (dm_{3,4\to k}\, dm_{1,2\to j}),$$

$$\overline{dS}_i = \mathbb{E}_{\{i\}\to\{1,2\}}[(A^*)_i\, A_1 + (B^*)_i\, B_1 + b^*{}_i\, b_2 + a^*{}_i\, a_2,$$

$$y^*{}_i\, y_2 + x^*{}_i\, x_2 + z^*{}_i\, z_2 + (XX^*)_i\, XX_1 + (Y^*)_i\, Y_1 + Z^*{}_i\, Z_1,\ 1]\ //\ \big(bS_1\, \overline{aS}_2\big)\ //\ dm_{2,1\to i}\Big]$$

Program

In[∘]:=

$$\text{Define}\Big[C_i = \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \frac{1}{\mathbb{A}_i^{\hbar}\,\mathbb{B}_i^{\hbar}} - \frac{\hbar\,(a_i + b_i)\,\epsilon}{\mathbb{A}_i^{\hbar}\,\mathbb{B}_i^{\hbar}} + O[\epsilon]^2\Big]_{\$k},$$
$$\overline{C}_i = \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \big(\mathbb{A}_i^{\hbar}\,\mathbb{B}_i^{\hbar} + \big(\hbar\,a_i\,\mathbb{A}_i^{\hbar}\,\mathbb{B}_i^{\hbar} + \hbar\,b_i\,\mathbb{A}_i^{\hbar}\,\mathbb{B}_i^{\hbar}\big)\,\epsilon\big) + O[\epsilon]^2\Big]_{\$k},$$
$$\text{Kink}_i = \big(R_{1,3}\,\overline{C}_2\big)\ //\ \text{dm}_{1,2\to1}\ //\ \text{dm}_{1,3\to i},$$
$$\overline{\text{Kink}}_i = \big(\overline{R}_{1,3}\,C_2\big)\ //\ \text{dm}_{1,2\to1}\ //\ \text{dm}_{1,3\to i}\Big]$$

Program

Note: s=2A-B+ϵa, t=2B-A+ϵb. This substitution is implemented in the following tensors.

Program

In[∘]:=

$$\text{Define}\Big[\text{AB2st}_i =$$
$$\mathbb{E}_{\{i\}\to\{i\}}\Big[b^*{}_i\,b_i + a^*{}_i\,a_i + (A^*)_i\left(\frac{2}{3}\left(s_i + \frac{t_i}{2}\right)\right) + (B^*)_i\left(\frac{2}{3}\left(t_i + \frac{s_i}{2}\right)\right),\ (XX^*)_i\,XX_i + (Y^*)_i\,Y_i +$$
$$Z^*{}_i\,Z_i + y^*{}_i\,y_i + x^*{}_i\,x_i + z^*{}_i\,z_i,\ 1 - \epsilon\,(A^*)_i\,\frac{1}{3}\,(2\,a_i + b_i) - \epsilon\,(B^*)_i\,\frac{1}{3}\,(2\,b_i + a_i) + O[\epsilon]^2\Big]_{\$k},$$
$$\text{st2AB}_i = \mathbb{E}_{\{i\}\to\{i\}}\Big[b^*{}_i\,b_i + a^*{}_i\,a_i + (s^*)_i\,(2\,A_i - B_i) + (t^*)_i\,(2\,B_i - A_i),$$
$$(XX^*)_i\,XX_i + (Y^*)_i\,Y_i + Z^*{}_i\,Z_i + y^*{}_i\,y_i + x^*{}_i\,x_i + z^*{}_i\,z_i,\ 1 + \epsilon\,(s^*)_i\,a_i + \epsilon\,(t^*)_i\,b_i + O[\epsilon]^2\Big]_{\$k}\Big]$$

The following definitions are used for a slightly faster implementation of the quantum-double that leaves out the s and the t from the zip. Since s and t are central, this is well defined and yields the same result. These tensors should be zipped using the STB and DB zip function. As such, we also check the axioms for these tensors.

In[∘]:=
```
Define[stR_{i,j} = (R_{i,j} ~ B_{{i,j}} ~ (AB2st_i AB2st_j)) // Simplify,
  ‾stR‾_{i,j} = (‾R‾_{i,j} ~ B_{{i,j}} ~ (AB2st_i AB2st_j)) // Simplify,
  ‾stC‾_i = (‾C‾_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stC_i = (C_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  ‾stKink‾_i = (‾Kink‾_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stKink_i = (Kink_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stdm_{i,j→k} = ((st2AB_i st2AB_j) ~ B_{i,j} ~ dm_{i,j→k} ~ B_k ~ AB2st_k),
  stdΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)),
  stdS_i = (st2AB_i // dS_i // AB2st_i),
  stP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}),
  ddΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)) /.
    {S_{j|k} → S, T_{j|k} → T, t_{j|k} → t, s_{j|k} → s, (s*)_i → 0, (t*)_i → 0},
  ddS_i = (st2AB_i // dS_i // AB2st_i) /. {S_i → S, T_i → T, s_i → s, t_i → t, (s*)_i → 0, (t*)_i → 0},
  PP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}) /. {(s*)_{i|j} → 0, (t*)_{i|j} → 0},
  RR_{i,j} = (R_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s},
  ‾RR‾_{i,j} = (‾R‾_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T},
  ‾CC‾_i = (‾C‾_i ~ B_{{i}} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
  CC_i = (C_i ~ B_{{i}} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
  ‾KKink‾_i = (‾Kink‾_i ~ B_{{i}} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} // Simplify,
  KKink_i = (Kink_i ~ B_{{i}} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} // Simplify,
  ddm_{i,j→k} = ((st2AB_i st2AB_j) // dm_{i,j→k} // AB2st_k) /.
    {S_k → S, T_k → T, t_k → t, s_k → s, (s*)_{i|j} → 0, (t*)_{i|j} → 0}];
```

```
Define[BB_{i, j→k} = ((dΔ_{i→1,r1} dΔ_{j→2,r2}) // dS_{r1} // ‾dS‾_{r2} // dm_{r1,r2→k} // dm_{k,1→k} // dm_{k,2→k})]
```

In[∘]:= $R_{i,j}$

$IterationLimit: Iteration limit of 4096 exceeded.

Out[∘]=

TerminatedEvaluation[IterationLimit]

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j A_i + \hbar\, b_j B_i,\ \hbar\, x_j XX_i + \hbar\, y_j Y_i + \hbar\, z_j Z_i,\ 1 + \frac{1}{2}\hbar^3\left(x_j^2 XX_i^2 + y_j^2 Y_i^2 + z_j^2 Z_i^2\right)\epsilon + O[\epsilon]^2\right]$$

In[∘]:= $dm_{i,j\to k}$

Part: The expression TerminatedEvaluation[IterationLimit][$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1 + \hbar\, b_2 B_1,\ \hbar\, x_2 XX_1 + \hbar\, y_2 Y_1 + \hbar\, z_2 Z_1,\ 1 + \frac{1}{2}\hbar^3$ (Times[≪2≫] + Times[≪2≫] + Times[≪2≫]) $\epsilon + O[\epsilon]^2\right]$] cannot be used as a part specification.

Part: The expression TerminatedEvaluation[IterationLimit][$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1 + \hbar\, b_2 B_1,\ \hbar\, x_2 XX_1 + \hbar\, y_2 Y_1 + \hbar\, z_2 Z_1,\ 1 + \frac{1}{2}\hbar^3$ (Times[≪2≫] + Times[≪2≫] + Times[≪2≫]) $\epsilon + O[\epsilon]^2\right]$] cannot be used as a part specification.

Part: The expression TerminatedEvaluation[

IterationLimit][$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, A_1 + \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg])\,\epsilon + O[\epsilon]^2\right]$] cannot be used as a part specification.

General: Further output of Part::pkspec1 will be suppressed during this calculation.

SeriesData: Coefficient specification TerminatedEvaluation[

IterationLimit][$\left\{1, \frac{1}{2}\hbar^3\,(\text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]) + \ll 1\gg - \hbar\, \mathbb{A}^{\hbar}_{\ll 1\gg}\, \ll 1\gg\,(\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Power}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg] + \text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\ll 3\gg\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg] + \ll 1\gg - \ll 1\gg + \ll 1\gg - \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg])\right\}$] in

SeriesData[ϵ, 0, TerminatedEvaluation[

IterationLimit][$\left\{1, \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg]) + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg]) - \hbar\,\text{Subscript}[\ll 2\gg]^{\hbar}\,\text{Subscript}[\ll 2\gg]^{\hbar}\,(\text{Times}[\ll 5\gg] + \text{Times}[\ll 7\gg] + \text{Times}[\ll 5\gg] + \text{Times}[\ll 6\gg] + \text{Times}[\ll 5\gg] + \text{Times}[\ll 6\gg])\right\}$], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[

IterationLimit][$\left\{1, \frac{1}{2}\hbar^3\,(\text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]) + \ll 1\gg - \hbar\, \mathbb{A}^{\hbar}_{\ll 1\gg}\, \ll 1\gg\,(\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Power}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg] + \text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\ll 3\gg\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg] + \ll 1\gg - \ll 1\gg + \ll 1\gg - \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg]\,\text{Subscript}[\ll 2\gg])\right\}$] in

SeriesData[ϵ, 0, TerminatedEvaluation[

IterationLimit][$\left\{1, \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg]) + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg]) - \hbar\,\text{Subscript}[\ll 2\gg]^{\hbar}\,\text{Subscript}[\ll 2\gg]^{\hbar}\,(\text{Times}[\ll 5\gg] + \text{Times}[\ll 7\gg] + \text{Times}[\ll 5\gg] + \text{Times}[\ll 6\gg] + \text{Times}[\ll 5\gg] + \text{Times}[\ll 6\gg])\right\}$], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[

IterationLimit][$\left\{1, \frac{1}{2}\hbar^3\,(\text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]) + \frac{1}{2}\hbar^3\,(\text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg] + \text{Power}[\ll 2\gg]\,\text{Power}[\ll 2\gg]) - \hbar\, a_2\, XX_3\, XX^*_{n\$5933[2]} + \hbar\, XX_3\, Y_3\, XX^*_{n\$5933[2]}\, Y^*_{n\$5933[1]} - \ll 1\gg + \ll 1\gg - \hbar\, XX_3\, Z_3\, \ll 1\gg_{\ll 1\gg}\, Z^*_{\ll 1\gg} + \hbar\, Y_3\, Z_3\, Y^*_{n\$5933[2]}\, Z^*_{n\$5933[1]} - \hbar\, a_2\, Z_3\, Z^*_{n\$5933[2]} - \hbar\, b_2\, Z_3\, Z^*_{n\$5933[2]}\right\}$] in SeriesData[$\epsilon$, 0, TerminatedEvaluation[IterationLimit][$\ll 1\gg$], 0, 2, 1] is not a list.

General: Further output of SeriesData::sdatc will be suppressed during this calculation.

Out[•]=

$$\Bigg(\mathbb{E}_{\{-3,1\}\to\{\}}\Bigg[\frac{a^*_1 A^*_{-3}}{\hbar} + \frac{b^*_1 B^*_{-3}}{\hbar},\ \frac{x^*_1 XX^*_{-3}}{\hbar} + \frac{y^*_1 Y^*_{-3}}{\hbar} + \frac{z^*_1 Z^*_{-3}}{\hbar},\ 1 - \Bigg\{\frac{a^*_2 A^*_{-3}}{\hbar} + \frac{a^*_1 A^*_1}{\hbar} + \frac{b^*_2 B^*_{-3}}{\hbar} + \frac{b^*_1 B^*_1}{\hbar},\ \frac{x^*_2 XX^*_{-3}}{\hbar} + \frac{x^*_1 XX^*_1}{\hbar} + \frac{y^*_2 Y^*_{-3}}{\hbar} + \frac{y^*_1 Y^*_1}{\hbar} + \frac{z^*_2 Z^*_{-3}}{\hbar} + \frac{z^*_1 Z^*_1}{\hbar},\ 1\Bigg\}\Bigg[\!\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\, a_2\, A_1 + \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\,\left(x_2^2\, XX_1^2 + y_2^2\, Y_1^2 + z_2^2\, Z_1^2\right)\epsilon + O[\epsilon]^2\Big]\Big]\Big]\!\Big][3] +$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{-3,1\}\to\{\}}\left[\frac{a^*_1 A^*_{-3}}{\hbar}+\frac{b^*_1 B^*_{-3}}{\hbar},\right.\right.$$
$$\left.\left.\frac{x^*_1 XX^*_{-3}}{\hbar}+\frac{y^*_1 Y^*_{-3}}{\hbar}+\frac{z^*_1 Z^*_{-3}}{\hbar},1\right]\right]_1[3]\Big]$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{-1,3\}\to\{\}}\left[\frac{a^*_3 A^*_{-1}}{\hbar}+\frac{b^*_3 B^*_{-1}}{\hbar},\right.\right.$$
$$\frac{x^*_3 XX^*_{-1}}{\hbar}+\frac{y^*_3 Y^*_{-1}}{\hbar}+\frac{z^*_3 Z^*_{-1}}{\hbar},$$
$$1-\left\{\frac{a^*_2 A^*_{-1}}{\hbar}+\frac{a^*_3 A^*_1}{\hbar}+\frac{b^*_2 B^*_{-1}}{\hbar}+\frac{b^*_3 B^*_1}{\hbar},\frac{x^*_2 XX^*_{-1}}{\hbar}+\frac{x^*_3 XX^*_1}{\hbar}+\frac{y^*_2 Y^*_{-1}}{\hbar}+\frac{y^*_3 Y^*_1}{\hbar}+\frac{z^*_2 Z^*_{-1}}{\hbar}+\right.$$
$$\left.\frac{z^*_3 Z^*_1}{\hbar},1\right\}\Big[\!\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\right.\right.$$
$$\left.\left.\hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\right]\right]\Big]\!\Big][3]+$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{-1,3\}\to\{\}}\left[\frac{a^*_3 A^*_{-1}}{\hbar}+\frac{b^*_3 B^*_{-1}}{\hbar},\right.\right.$$
$$\left.\left.\frac{x^*_3 XX^*_{-1}}{\hbar}+\frac{y^*_3 Y^*_{-1}}{\hbar}+\frac{z^*_3 Z^*_{-1}}{\hbar},1\right]\right]_1[3]\Big]\Big]$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{2,j\}\to\{k\}}\left[a_k\,(a^*_2+a^*_j)+b_k\,(b^*_2+b^*_j),\right.\right.$$
$$x_k\left(e^{2a^*_j-b^*_j}x^*_2+x^*_j\right)+y_k\left(e^{-a^*_j+2b^*_j}y^*_2+y^*_j\right)+z_k\left(e^{2a^*_j-b^*_j}x^*_2 y^*_j+e^{a^*_j+b^*_j}z^*_2+z^*_j\right),$$
$$1-e^{-a^*_j-b^*_j}\hbar\left(e^{3a^*_j}x_k y_k x^*_2 y^*_j+e^{2a^*_j+2b^*_j}y_k z_k x^*_2 y^*_2 y^*_j+e^{3a^*_j}y_k z_k x^*_2\,(y^*)^2_j-\right.$$
$$\left.\left.e^{3a^*_j}x_k z_k x^*_2 z^*_j+e^{3b^*_j}y_k z_k y^*_2 z^*_j-e^{3a^*_j}z_k^2 x^*_2 y^*_j z^*_j\right)\epsilon+O[\epsilon]^2\right]\Big]$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{i,-2\}\to\{k\}}\left[A_k A^*_{-2}+A_k A^*_i+B_k B^*_{-2}+B_k B^*_i,\right.\right.$$
$$XX_k XX^*_{-2}+XX_k XX^*_i+Y_k Y^*_{-2}+Y_k Y^*_i+Z_k Z^*_{-2}+Z_k Z^*_i,$$
$$1+(-XX_k A^*_i XX^*_{-2}-Y_k B^*_i Y^*_{-2}+2 Z_k XX^*_i Y^*_{-2}+\hbar\, XX_k Y_k XX^*_{-2} Y^*_i-$$
$$\left.\left.Z_k A^*_i Z^*_{-2}-Z_k B^*_i Z^*_{-2}-\hbar\, XX_k Z_k XX^*_{-2} Z^*_i+\hbar\, Y_k Z_k Y^*_{-2} Z^*_i)\,\epsilon+O[\epsilon]^2\right]\Big]\Big)\Big[$$

$$\mathbb{E}_{\{i,j\}\to\{i,j\}}\left[a_j a^*_j+A_i A^*_i+b_j b^*_j+B_i B^*_i,\ x_j x^*_j+XX_i XX^*_i+y_j y^*_j+Y_i Y^*_i+\right.$$
$$\left.z_j z^*_j+Z_i Z^*_i,1\right]$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\left\{\frac{a^*_3 A^*_{-2}}{\hbar}+\frac{b^*_3 B^*_{-2}}{\hbar},\frac{x^*_3 XX^*_{-2}}{\hbar}+\frac{y^*_3 Y^*_{-2}}{\hbar}+\frac{z^*_3 Z^*_{-2}}{\hbar},\right.$$
$$1-\left\{\frac{a^*_2 A^*_{-2}}{\hbar}+\frac{a^*_3 A^*_1}{\hbar}+\frac{b^*_2 B^*_{-2}}{\hbar}+\frac{b^*_3 B^*_1}{\hbar},\frac{x^*_2 XX^*_{-2}}{\hbar}+\frac{x^*_3 XX^*_1}{\hbar}+\frac{y^*_2 Y^*_{-2}}{\hbar}+\frac{y^*_3 Y^*_1}{\hbar}+\frac{z^*_2 Z^*_{-2}}{\hbar}+\right.$$
$$\left.\frac{z^*_3 Z^*_1}{\hbar},1\right\}\Big[\!\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\right.\right.$$
$$\left.\left.\hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\right]\right]\Big]\!\Big][3]+$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\left[\mathbb{E}_{\{-2,3\}\to\{\}}\left[\frac{a^*_3 A^*_{-2}}{\hbar}+\frac{b^*_3 B^*_{-2}}{\hbar},\frac{x^*_3 XX^*_{-2}}{\hbar}+\right.\right.$$
$$\left.\left.\frac{y^*_3 Y^*_{-2}}{\hbar}+\frac{z^*_3 Z^*_{-2}}{\hbar},1\right]\right]_1[3]\Big\}\Big[\!\Big[\mathbb{E}_{\{\}\to\{3,-2,-3\}}\left[\text{QZip}_{\{Z^*_{n\$5978[1]},z_{n\$5978[1]},Z^*_{n\$5978[2]},z_{n\$5978[2]}\}}\right.\right.$$
$$\text{QZip}_{\{XX_{n\$5978[1]},Y^*_{n\$5978[1]},y_{n\$5978[1]},x^*_{n\$5978[1]},XX_{n\$5978[2]},Y^*_{n\$5978[2]},y_{n\$5978[2]},x^*_{n\$5978[2]}\}}\Big[$$

```
TerminatedEvaluation[IterationLimit][𝔼[TerminatedEvaluation[IterationLimit][
    a3 A_-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     a3 A_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     b3 B_-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     b3 B_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
  TerminatedEvaluation[IterationLimit][x3 x*n$5978[2]
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
     y3 y*n$5978[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
     z3 z*n$5978[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
     xn$5978[2] XX_-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     xn$5978[1] XX_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     yn$5978[2] Y_-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     yn$5978[1] Y_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     zn$5978[2] Z_-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     zn$5978[1] Z_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + y3 y*n$5978[1]
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][𝔸_-3^ℏ 𝔹_-3^(-2 ℏ)]]] + z3 z*n$5978[1]
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][𝔸_-3^(-ℏ) 𝔹_-3^(-ℏ)]]] + x3 x*n$5978[1]
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][𝔸_-3^(-2 ℏ) 𝔹_-3^ℏ]]] + z3 x*n$5978[1]
      y*n$5978[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸_-3^(-2 ℏ) 𝔹_-3^ℏ]]]],
  TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][
          TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
            IterationLimit][TerminatedEvaluation[IterationLimit][
              TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                IterationLimit][TerminatedEvaluation[IterationLimit][
```

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{SeriesData}\Big[$$

$$\epsilon,\ 0,\ \text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\Big\{1,\ \frac{1}{2}\hbar^3\left(x^2_{n\$5978[2]}\,XX^2_{-3}+y^2_{n\$5978[2]}\,Y^2_{-3}+z^2_{n\$5978[2]}\,Z^2_{-3}\right)+\frac{1}{2}\hbar^3\left(x^2_{n\$5978[1]}\,XX^2_{-2}+y^2_{n\$5978[1]}\,Y^2_{-2}+z^2_{n\$5978[1]}\,Z^2_{-2}\right)-$$

$$\hbar\,\mathbb{A}^{\hbar}_{-3}\,\mathbb{B}^{\hbar}_{-3}\Big(x_3\,y_3\,\mathbb{A}^{-3\hbar}_{-3}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{A}^{-2\hbar}_{-3}\,\mathbb{B}^{-2\hbar}_{-3}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{A}^{-3\hbar}_{-3}\,x^*{}_{n\$5978[1]}\,(y^*)^2_{n\$5978[2]}-x_3\,z_3\,\mathbb{A}^{-3\hbar}_{-3}\,x^*{}_{n\$5978[1]}\,z^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{B}^{-3\hbar}_{-3}\,y^*{}_{n\$5978[1]}\,z^*{}_{n\$5978[2]}-z_3^2\,\mathbb{A}^{-3\hbar}_{-3}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}\,z^*{}_{n\$5978[2]}\Big)\Big\}\Big],\ 0,\ 2,$$

$$1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]⟧][$$

$$\Big\{\frac{a^*{}_3\,A^*{}_j}{\hbar}+\frac{b^*{}_3\,B^*{}_j}{\hbar},\ \frac{x^*{}_3\,XX^*{}_j}{\hbar}+\frac{y^*{}_3\,Y^*{}_j}{\hbar}+\frac{z^*{}_3\,Z^*{}_j}{\hbar},\ 1-\Big\{\frac{a^*{}_3\,A^*{}_1}{\hbar}+\frac{a^*{}_2\,A^*{}_j}{\hbar}+\frac{b^*{}_3\,B^*{}_1}{\hbar}+\frac{b^*{}_2\,B^*{}_j}{\hbar},$$

$$\frac{x^*{}_3\,XX^*{}_1}{\hbar}+\frac{x^*{}_2\,XX^*{}_j}{\hbar}+\frac{y^*{}_3\,Y^*{}_1}{\hbar}+\frac{y^*{}_2\,Y^*{}_j}{\hbar}+\frac{z^*{}_3\,Z^*{}_1}{\hbar}+\frac{z^*{}_2\,Z^*{}_j}{\hbar},\ 1\Big\}⟦$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\,a_2\,A_1+\hbar\,b_2\,B_1,\ \hbar\,x_2\,XX_1+\hbar\,y_2\,Y_1+\hbar\,z_2\,Z_1,$$

$$1+\frac{1}{2}\hbar^3\left(x_2^2\,XX_1^2+y_2^2\,Y_1^2+z_2^2\,Z_1^2\right)\epsilon+O[\epsilon]^2\Big]\Big]⟧[3]+$$

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{j,3\}\to\{\}}\Big[\frac{a^*{}_3\,A^*{}_j}{\hbar}+\frac{b^*{}_3\,B^*{}_j}{\hbar},$$

$$\frac{x^*{}_3\,XX^*{}_j}{\hbar}+\frac{y^*{}_3\,Y^*{}_j}{\hbar}+\frac{z^*{}_3\,Z^*{}_j}{\hbar},\ 1\Big]\Big]_1[3]\Big\}⟦$$

$$\mathbb{E}_{\{\}\to\{3,-1,-2\}}\Big[\text{QZip}_{\{z^*{}_{n\$5978[1]},\,z_{n\$5978[1]},\,z^*{}_{n\$5978[2]},\,z_{n\$5978[2]}\}}\Big[$$

```
QZip{XX_{n$5978[1]},Y*_{n$5978[1]},y_{n$5978[1]},x*_{n$5978[1]},XX_{n$5978[2]},Y*_{n$5978[2]},y_{n$5978[2]},x*_{n$5978[2]}}[
 TerminatedEvaluation[IterationLimit][
  𝔼[TerminatedEvaluation[IterationLimit][
    a_3 A_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     a_3 A_-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     b_3 B_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     b_3 B_-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
   TerminatedEvaluation[IterationLimit][x_3 x*_{n$5978[2]} TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[IterationLimit][1]]] +
     y_3 y*_{n$5978[2]} TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
     z_3 z*_{n$5978[2]} TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
     x_{n$5978[2]} XX_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     x_{n$5978[1]} XX_-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     y_{n$5978[2]} Y_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     y_{n$5978[1]} Y_-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     z_{n$5978[2]} Z_-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     z_{n$5978[1]} Z_-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
     y_3 y*_{n$5978[1]} TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸_-2^ℏ 𝔹_-2^(-2 ℏ)]]] +
     z_3 z*_{n$5978[1]} TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸_-2^(-ℏ) 𝔹_-2^(-ℏ)]]] +
     x_3 x*_{n$5978[1]} TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸_-2^(-2 ℏ) 𝔹_-2^ℏ]]] +
     z_3 x*_{n$5978[1]} y*_{n$5978[2]} TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][𝔸_-2^(-2 ℏ) 𝔹_-2^ℏ]]]],
   TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
     TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
```

```
TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
  IterationLimit][TerminatedEvaluation[IterationLimit][
   TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
     IterationLimit][TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][
         TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
           IterationLimit][TerminatedEvaluation[IterationLimit][
            TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
              IterationLimit][TerminatedEvaluation[IterationLimit][
               TerminatedEvaluation[IterationLimit][
                TerminatedEvaluation[IterationLimit][
                 TerminatedEvaluation[IterationLimit][
                  TerminatedEvaluation[IterationLimit][SeriesData[
                    ϵ, 0, TerminatedEvaluation[IterationLimit][
```

$$\Big\{1,\ \frac{1}{2}\hbar^3\left(x_{n\$5978[2]}^2\,XX_{-2}^2+y_{n\$5978[2]}^2\,Y_{-2}^2+z_{n\$5978[2]}^2\,Z_{-2}^2\right)+\frac{1}{2}\hbar^3\left(x_{n\$5978[1]}^2\,XX_{-1}^2+y_{n\$5978[1]}^2\,Y_{-1}^2+z_{n\$5978[1]}^2\,Z_{-1}^2\right)-\hbar\,\mathbb{A}_{-2}^{\hbar}\,\mathbb{B}_{-2}^{\hbar}\Big(x_3\,y_3\,\mathbb{A}_{-2}^{-3\hbar}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{A}_{-2}^{-2\hbar}\,\mathbb{B}_{-2}^{-2\hbar}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{A}_{-2}^{-3\hbar}\,x^*{}_{n\$5978[1]}\,(y^*)_{n\$5978[2]}^2-x_3\,z_3\,\mathbb{A}_{-2}^{-3\hbar}\,x^*{}_{n\$5978[1]}\,z^*{}_{n\$5978[2]}+y_3\,z_3\,\mathbb{B}_{-2}^{-3\hbar}\,y^*{}_{n\$5978[1]}\,z^*{}_{n\$5978[2]}-z_3^2\,\mathbb{A}_{-2}^{-3\hbar}\,x^*{}_{n\$5978[1]}\,y^*{}_{n\$5978[2]}\,z^*{}_{n\$5978[2]}\Big)\Big\}\Big],\ 0,\ 2,$$

```
1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]
```

TerminatedEvaluation[IterationLimit][$\mathbb{E}_{\{3\}\to\{3\}}$[$-a_3\,a^*{}_3-b_3\,b^*{}_3,\ -e^{-2a^*{}_3+b^*{}_3}$

x_3

$x^*{}_3-$

$e^{a^*{}_3-2b^*{}_3}$

y_3

$y^*{}_3+$

$e^{-a^*{}_3-b^*{}_3}$

z_3

$x^*{}_3$

$y^*{}_3-$

$e^{-a^*{}_3-b^*{}_3}$

```
      z3
      z*3,
    1 + TerminatedEvaluation[IterationLimit][𝔼{3}→{3}[-a3 a*3 - b3 b*3,
          -ⅇ^(-2 a*3+b*3) x3 x*3 - ⅇ^(a*3-2 b*3) y3 y*3 + ⅇ^(-a*3-b*3) z3 x*3 y*3 - ⅇ^(-a*3-b*3) z3 z*3, 1]]1[
      3] - TerminatedEvaluation[IterationLimit][
      3]]]]
TerminatedEvaluation[IterationLimit][
  {a*2 A*3/ℏ +
     b*2 B*3/ℏ, x*2 XX*3/ℏ +
     y*2 Y*3/ℏ +
     z*2 Z*3/ℏ,
    1 - {a*2 A*1/ℏ + a*2 A*3/ℏ + b*2 B*1/ℏ + b*2 B*3/ℏ,
       x*2 XX*1/ℏ + x*2 XX*3/ℏ + y*2 Y*1/ℏ + y*2 Y*3/ℏ + z*2 Z*1/ℏ + z*2 Z*3/ℏ, 1}[[
      TerminatedEvaluation[IterationLimit][𝔼{}→{1,2}[ℏ a2 A1 + ℏ b2 B1,
         ℏ x2 XX1 + ℏ y2 Y1 + ℏ z2 Z1, 1 + 1/2 ℏ^3 (x2^2 XX1^2 + y2^2 Y1^2 + z2^2 Z1^2) ϵ + O[ϵ]^2]]]][3] +
     TerminatedEvaluation[IterationLimit][𝔼{3,2}→{}[a*2 A*3/ℏ + b*2 B*3/ℏ,
       x*2 XX*3/ℏ + y*2 Y*3/ℏ + z*2 Z*3/ℏ, 1]]1[3]}[[
  𝔼{}→{3,2,3}[QZip{Z*n$5933[1],zn$5933[1],Z*n$5933[2],zn$5933[2]}[
    QZip{XXn$5933[1],Y*n$5933[1],yn$5933[1],x*n$5933[1],XXn$5933[2],Y*n$5933[2],yn$5933[2],x*n$5933[2]}[
     TerminatedEvaluation[IterationLimit][
      𝔼[TerminatedEvaluation[IterationLimit][
        a2 A3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
             IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
         a3 A3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
             IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
         b2 B3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
             IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
         b3 B3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
             IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
       TerminatedEvaluation[IterationLimit][XX3 XX*n$5933[1] TerminatedEvaluation[
           IterationLimit][TerminatedEvaluation[IterationLimit][
           TerminatedEvaluation[IterationLimit][1]]] +
         XX3 XX*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
```

```
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Y3 Y*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Y3 Y*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Z3 Z*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Z3 Z*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    x2 XXn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    x3 XXn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    y2 Yn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    y3 Yn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z2 Zn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z3 Zn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
  TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
    TerminatedEvaluation[IterationLimit]⟦
     TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[
        IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
        TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[
           IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
           TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[
              IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
              TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[
                 IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
                 TerminatedEvaluation[IterationLimit]⟦TerminatedEvaluation[
                    IterationLimit]⟦TerminatedEvaluation[IterationLimit]⟦
                    TerminatedEvaluation[IterationLimit]⟦
                     TerminatedEvaluation[IterationLimit]⟦
                      TerminatedEvaluation[IterationLimit]⟦
```

$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\text{SeriesData}\Big[$$
$$\epsilon,\ 0,\ \text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\Big\{1,\ \frac{1}{2}\hbar^3\left(x_2^2\,XX^2_{n\$5933[1]}+y_2^2\,Y^2_{n\$5933[1]}+z_2^2\,Z^2_{n\$5933[1]}\right)+$$
$$\frac{1}{2}\hbar^3\left(x_3^2\,XX^2_{n\$5933[2]}+y_3^2\,Y^2_{n\$5933[2]}+z_3^2\,Z^2_{n\$5933[2]}\right)-$$
$$\hbar\,a_2\,XX_3\,XX^*{}_{n\$5933[2]}+\hbar\,XX_3\,Y_3\,XX^*{}_{n\$5933[2]}\,Y^*{}_{n\$5933[1]}-$$
$$\hbar\,b_2\,Y_3\,Y^*{}_{n\$5933[2]}+2\,Z_3\,XX^*{}_{n\$5933[1]}\,Y^*{}_{n\$5933[2]}-$$
$$\hbar\,XX_3\,Z_3\,XX^*{}_{n\$5933[2]}\,Z^*{}_{n\$5933[1]}+\hbar\,Y_3\,Z_3\,Y^*{}_{n\$5933[2]}$$
$$Z^*{}_{n\$5933[1]}-\hbar\,a_2\,Z_3\,Z^*{}_{n\$5933[2]}-\hbar\,b_2\,Z_3\,Z^*{}_{n\$5933[2]}\Big\}\Big],$$
$$0,\ 2,\ 1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]\rrbracket]][$$

$$\Big\{\frac{a^*{}_i\,A^*{}_3}{\hbar}+\frac{b^*{}_i\,B^*{}_3}{\hbar},\ \frac{x^*{}_i\,XX^*{}_3}{\hbar}+\frac{y^*{}_i\,Y^*{}_3}{\hbar}+\frac{z^*{}_i\,Z^*{}_3}{\hbar},\ 1-\Big\{\frac{a^*{}_i\,A^*{}_1}{\hbar}+\frac{a^*{}_2\,A^*{}_3}{\hbar}+\frac{b^*{}_i\,B^*{}_1}{\hbar}+\frac{b^*{}_2\,B^*{}_3}{\hbar},$$
$$\frac{x^*{}_i\,XX^*{}_1}{\hbar}+\frac{x^*{}_2\,XX^*{}_3}{\hbar}+\frac{y^*{}_i\,Y^*{}_1}{\hbar}+\frac{y^*{}_2\,Y^*{}_3}{\hbar}+\frac{z^*{}_i\,Z^*{}_1}{\hbar}+\frac{z^*{}_2\,Z^*{}_3}{\hbar},\ 1\Big\}\llbracket$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\,a_2\,A_1+\hbar\,b_2\,B_1,\ \hbar\,x_2\,XX_1+\hbar\,y_2\,Y_1+\hbar\,z_2\,Z_1,$$
$$1+\frac{1}{2}\hbar^3\left(x_2^2\,XX_1^2+y_2^2\,Y_1^2+z_2^2\,Z_1^2\right)\epsilon+O[\epsilon]^2\Big]\Big]\rrbracket[3]+$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{3,i\}\to\{\}}\Big[\frac{a^*{}_i\,A^*{}_3}{\hbar}+\frac{b^*{}_i\,B^*{}_3}{\hbar},$$
$$\frac{x^*{}_i\,XX^*{}_3}{\hbar}+\frac{y^*{}_i\,Y^*{}_3}{\hbar}+\frac{z^*{}_i\,Z^*{}_3}{\hbar},\ 1\Big]\Big]_1[3]\Big\}\llbracket$$
$$\mathbb{E}_{\{\}\to\{3,1,2\}}\Big[\text{QZip}_{\{Z^*{}_{n\$5933[1]},z_{n\$5933[1]},Z^*{}_{n\$5933[2]},z_{n\$5933[2]}\}}\Big[$$
$$\text{QZip}_{\{XX_{n\$5933[1]},Y^*{}_{n\$5933[1]},y_{n\$5933[1]},x^*{}_{n\$5933[1]},XX_{n\$5933[2]},Y^*{}_{n\$5933[2]},y_{n\$5933[2]},x^*{}_{n\$5933[2]}\}}\Big[$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[$$
$$\mathbb{E}\Big[\text{TerminatedEvaluation}[\text{IterationLimit}][$$
$$a_1\,A_3\,\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\hbar]]]+$$
$$a_2\,A_3\,\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\hbar]]]+$$
$$b_1\,B_3\,\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\hbar]]]+$$
$$b_2\,B_3\,\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\hbar]]]],$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}][XX_3\,XX^*{}_{n\$5933[1]}$$
$$\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][\text{TerminatedEvaluation}[\text{IterationLimit}][1]]]+$$

```
    XX3 XX*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Y3 Y*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Y3 Y*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Z3 Z*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    Z3 Z*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
    x1 XXn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    x2 XXn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    y1 Yn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    y2 Yn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z1 Zn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z2 Zn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
  TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][
          TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
              IterationLimit][TerminatedEvaluation[IterationLimit][
              TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                  IterationLimit][TerminatedEvaluation[IterationLimit][
                  TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                      IterationLimit][TerminatedEvaluation[IterationLimit][
                      TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                          IterationLimit][TerminatedEvaluation[IterationLimit][
                          TerminatedEvaluation[IterationLimit][
                           TerminatedEvaluation[IterationLimit][
                            TerminatedEvaluation[IterationLimit][
```

```
TerminatedEvaluation[IterationLimit][
 TerminatedEvaluation[IterationLimit][SeriesData[
   ϵ, 0, TerminatedEvaluation[IterationLimit][
    {1, 1/2 ℏ^3 (x_1^2 XX^2_n$5933[1] + y_1^2 Y^2_n$5933[1] + z_1^2 Z^2_n$5933[1]) +
      1/2 ℏ^3 (x_2^2 XX^2_n$5933[2] + y_2^2 Y^2_n$5933[2] + z_2^2 Z^2_n$5933[2]) -
      ℏ a_1 XX_3 XX*_n$5933[2] + ℏ XX_3 Y_3 XX*_n$5933[2] Y*_n$5933[1] -
      ℏ b_1 Y_3 Y*_n$5933[2] + 2 Z_3 XX*_n$5933[1] Y*_n$5933[2] -
      ℏ XX_3 Z_3 XX*_n$5933[2] Z*_n$5933[1] + ℏ Y_3 Z_3 Y*_n$5933[2]
      Z*_n$5933[1] - ℏ a_1 Z_3 Z*_n$5933[2] - ℏ b_1 Z_3 Z*_n$5933[2]}],
   0, 2, 1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]
```

In[•]:= **stR$_{i,j}$**

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

General: Further output of $IterationLimit::itlim will be suppressed during this calculation.

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$6203[, a*_n$6203[j]], EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_≪1≫, B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$6203[j] , a*_n$6203[j]] (≪1≫)}] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$62 , a*_n$6203[j]], EQ$6214[A_n$6203[≪1≫], B_n$6203[≪1≫], s_n$6203[≪1≫], t_n$6203[≪1≫], ≪5≫, t_n$6203[≪1≫], ≪1≫*_n$6203[≪1≫], ≪1≫*_n$6203[≪

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$6203[, 0], EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$6203[j] , 0] (≪1≫)}] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], b*_n$62 , 0], EQ$6214[A_n$6203[≪1≫], B_n$6203[≪1≫], s_n$6203[≪1≫], t_n$6203[≪1≫], ≪1≫*_n$6203[≪1≫], ≪1≫*_≪1≫, A_≪1≫, B_≪1≫, s_n$6203[≪1≫], t_n$6203[≪1≫], ≪1≫*_n$6203[≪1≫], 0] (≪1≫)}], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], 0, 0] , EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], 0, 0] (≪1≫)}] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][{EQ$6214[A_n$6203[i], B_n$6203[i], s_n$6203[i], t_n$6203[i], b*_n$6203[i], a*_n$6203[i], A_n$6203[j], B_n$6203[j], s_n$6203[j], t_n$6203[j], 0, 0], EQ$6214[A_n$6203[≪1≫], B_n$6203[≪1≫], s_n$6203[≪1≫], t_n$6203[≪1≫], ≪1≫*_n$6203[≪1≫], ≪1≫*_n$6203[≪1≫], A_≪1≫, B_n$6203[≪1≫], s_n$6203[≪1≫], t_n$6203[≪1≫], 0, 0] (≪1≫)}], 0, 2, 1] is not a list.

General: Further output of SeriesData::sdatc will be suppressed during this calculation.

In[ ]:= $\mathbf{RR_{i,j}}$

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

General: Further output of $IterationLimit::itlim will be suppressed during this calculation.

SeriesData: Coefficient specification TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$6100[}$
, $a^*_{n\$6100[j]}$], EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{\ll 1\gg}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$6100[j]}$
, $a^*_{n\$6100[j]}$] (≪1≫)}] in SeriesData[$\epsilon$, 0, TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$61}$
, $a^*_{n\$6100[j]}$], EQ$6105[$A_{n\$6100[\ll 1\gg]}$, $B_{n\$6100[\ll 1\gg]}$, $s_{n\$6100[\ll 1\gg]}$, $t_{n\$6100[\ll 1\gg]}$, ≪5≫, $t_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll}$

SeriesData: Coefficient specification TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$6100[}$
, 0], EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$6100[j]}$
, 0] (≪1≫)}] in SeriesData[ϵ, 0, TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, $b^*_{n\$61}$
, 0], EQ$6105[$A_{n\$6100[\ll 1\gg]}$, $B_{n\$6100[\ll 1\gg]}$, $s_{n\$6100[\ll 1\gg]}$, $t_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{\ll 1\gg}$, $A_{\ll 1\gg}$, $B_{\ll 1\gg}$,
$s_{n\$6100[\ll 1\gg]}$, $t_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll 1\gg]}$, 0] (≪1≫)}], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, 0, 0]
, EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, 0, 0] (≪1≫)}]
in SeriesData[ϵ, 0, TerminatedEvaluation[
IterationLimit][{EQ$6105[$A_{n\$6100[i]}$, $B_{n\$6100[i]}$, $s_{n\$6100[i]}$, $t_{n\$6100[i]}$, $b^*_{n\$6100[i]}$, $a^*_{n\$6100[i]}$, $A_{n\$6100[j]}$, $B_{n\$6100[j]}$, $s_{n\$6100[j]}$, $t_{n\$6100[j]}$, 0,
0], EQ$6105[$A_{n\$6100[\ll 1\gg]}$, $B_{n\$6100[\ll 1\gg]}$, $s_{n\$6100[\ll 1\gg]}$, $t_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll 1\gg]}$, $\ll 1\gg^*_{n\$6100[\ll 1\gg]}$, $A_{\ll 1\gg}$, $B_{n\$6100[\ll 1\gg]}$,
$s_{n\$6100[\ll 1\gg]}$, $t_{n\$6100[\ll 1\gg]}$, 0, 0] (≪1≫)}], 0, 2, 1] is not a list.

General: Further output of SeriesData::sdatc will be suppressed during this calculation.

In[ ]:= $\mathbf{ddm_{i,j\to k}}$

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

$IterationLimit: Iteration limit of 4096 exceeded.

General: Further output of $IterationLimit::itlim will be suppressed during this calculation.

Part: The expression TerminatedEvaluation[
IterationLimit]$\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, A_1 + \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg])\,\epsilon + O[\epsilon]^2\right]\right]$ cannot be used as a part specification.

Part: The expression TerminatedEvaluation[
IterationLimit]$\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, A_1 + \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg])\,\epsilon + O[\epsilon]^2\right]\right]$ cannot be used as a part specification.

Part: The expression TerminatedEvaluation[IterationLimit][$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, A_1 + \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\,(\text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg] + \text{Times}[\ll 2\gg])\,\epsilon + O[\epsilon]^2\right]$] cannot be used as a part specification.

General: Further output of Part::pkspec1 will be suppressed during this calculation.

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][{EQ$5824[$A_{n\$5823[1]}$, $B_{n\$5823[1]}$, $s_{n\$5823[1]}$, $t_{n\$5823[1]}$, $b^*_{n\$5823[1]}$, $a^*_{n\$5823[1]}$, $A_{n\$5823[2]}$, $B_{n\$5823[2]}$, $s_{n\$5823[2]}$, $t_{n\$5823[2]}$, $b^*_{n\$}$, $a^*_{n\$5823[2]}$, $A_{\ll 1\gg}$, $B_{n\$5823[i\$]}$, $s_{n\$5823[i\$]}$, $t_{n\$5823[i\$]}$, $b^*_{n\$5823[i\$]}$, $a^*_{n\$5823[i\$]}$, $A_{n\$5823[j\$]}$, $B_{n\$5823[j\$]}$, $s_{n\$5823[j\$]}$, $t_{n\$5823[j\$]}$, $b^*_{n\$5823[j\$]}$, $a^*_{n\$5823[j\$]}$], EQ$5824[≪1≫] (≪1≫)}] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][≪1≫], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][{EQ$5824[$A_{n\$5823[1]}$, $B_{n\$5823[1]}$, $s_{n\$5823[1]}$, $t_{n\$5823[1]}$, $b^*_{n\$5823[1]}$, $a^*_{n\$5823[1]}$, $A_{n\$5823[2]}$, $B_{n\$5823[2]}$, $s_{n\$5823[2]}$, $t_{n\$5823[2]}$, $b^*_{n\$}$, $a^*_{n\$5823[2]}$, $A_{n\$5823[i\$]}$, $B_{n\$5823[i\$]}$, $s_{n\$5823[i\$]}$, $t_{n\$5823[i\$]}$, $b^*_{n\$5823[i\$]}$, $a^*_{n\$5823[i\$]}$, $A_{n\$5823[j\$]}$, $B_{n\$5823[j\$]}$, $s_{n\$5823[j\$]}$, $t_{n\$5823[j\$]}$, $b^*_{n\$5823[j\$]}$, 0], EQ$5824[$A_{\ll 1\gg}$, ≪22≫, 0] ≪1≫}] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][{EQ$5824[$A_{n\$5823[1]}$, $B_{n\$5823[1]}$, $s_{n\$5823[1]}$, $t_{n\$5823[1]}$, $b^*_{n\$5823[1]}$, $a^*_{n\$5823[1]}$, $A_{n\$5823[2]}$, $B_{n\$5823[2]}$, $s_{n\$5823[2]}$, $t_{n\$5823[2]}$, $b^*_{n\$5823[2]}$, $a^*_{\ll 6\gg[2]}$, $A_{\ll 1\gg}$, $B_{n\$5823[i\$]}$, $s_{n\$5823[i\$]}$, $t_{n\$5823[i\$]}$, $b^*_{n\$5823[i\$]}$, $a^*_{n\$5823[i\$]}$, $A_{n\$5823[j\$]}$, $B_{n\$5823[j\$]}$, $s_{n\$5823[j\$]}$, $t_{n\$5823[j\$]}$, $b^*_{n\$5823[j\$]}$, 0], ≪7≫[≪1≫] ≪1≫}], 0, 2, 1] is not a list.

SeriesData: Coefficient specification TerminatedEvaluation[IterationLimit][≪1≫] in SeriesData[ϵ, 0, TerminatedEvaluation[IterationLimit][{EQ$5824[$A_{n\$5823[1]}$, $B_{n\$5823[1]}$, $s_{n\$5823[1]}$, $t_{n\$5823[1]}$, $b^*_{n\$5823[1]}$, $a^*_{n\$5823[1]}$, $A_{n\$5823[2]}$, $B_{n\$5823[2]}$, $s_{n\$5823[2]}$, $t_{n\$5823[2]}$, $b^*_{n\$5823[2]}$, $a^*_{n\$5823[2]}$, $A_{\ll 6\gg[i\$]}$, $B_{n\$5823[i\$]}$, $s_{n\$5823[i\$]}$, $t_{n\$5823[i\$]}$, $b^*_{n\$5823[i\$]}$, $a^*_{n\$5823[i\$]}$, $A_{n\$5823[j\$]}$, $B_{n\$5823[j\$]}$, $s_{n\$5823[j\$]}$, $t_{n\$5823[j\$]}$, 0, 0], EQ$5824[≪1≫] ≪1≫}], 0, 2, 1] is not a list.

General: Further output of SeriesData::sdatc will be suppressed during this calculation.

Out[•]=

$$
\begin{aligned}
&\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\\
&\quad\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{k\}\to\{k\}}\Big[a_k\, a^*_k + \frac{2}{3}\left(s + \frac{t}{2}\right) A^*_k + b_k\, b^*_k + \frac{2}{3}\left(\frac{s}{2} + t\right) B^*_k,\\
&\qquad x_k\, x^*_k + XX_k\, XX^*_k + y_k\, y^*_k + Y_k\, Y^*_k + z_k\, z^*_k + Z_k\, Z^*_k,\\
&\qquad 1 + \left(-\frac{1}{3}\,(2\, a_k + b_k)\, A^*_k - \frac{1}{3}\,(a_k + 2\, b_k)\, B^*_k\right)\epsilon + O[\epsilon]^2\Big]\Big]\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\\
&\qquad\Big(\mathbb{E}_{\{-3,1\}\to\{\}}\Big[\frac{a^*_1\, A^*_{-3}}{\hbar} + \frac{b^*_1\, B^*_{-3}}{\hbar},\ \frac{x^*_1\, XX^*_{-3}}{\hbar} + \frac{y^*_1\, Y^*_{-3}}{\hbar} + \frac{z^*_1\, Z^*_{-3}}{\hbar},\\
&\qquad\quad 1 - \Big\{\frac{a^*_2\, A^*_{-3}}{\hbar} + \frac{a^*_1\, A^*_1}{\hbar} + \frac{b^*_2\, B^*_{-3}}{\hbar} + \frac{b^*_1\, B^*_1}{\hbar},\ \frac{x^*_2\, XX^*_{-3}}{\hbar} + \frac{x^*_1\, XX^*_1}{\hbar} + \frac{y^*_2\, Y^*_{-3}}{\hbar} + \frac{y^*_1\, Y^*_1}{\hbar} +\\
&\qquad\qquad \frac{z^*_2\, Z^*_{-3}}{\hbar} + \frac{z^*_1\, Z^*_1}{\hbar},\ 1\Big\}\Big[\!\Big[\text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\, a_2\, A_1 +\\
&\qquad\qquad \hbar\, b_2\, B_1,\ \hbar\, x_2\, XX_1 + \hbar\, y_2\, Y_1 + \hbar\, z_2\, Z_1,\ 1 + \frac{1}{2}\hbar^3\left(x_2^2\, XX_1^2 + y_2^2\, Y_1^2 + z_2^2\, Z_1^2\right)\epsilon + O[\epsilon]^2\Big]\Big]\Big]\!\Big][3] +\\
&\qquad\quad \text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\mathbb{E}_{\{-3,1\}\to\{\}}\Big[\frac{a^*_1\, A^*_{-3}}{\hbar} + \frac{b^*_1\, B^*_{-3}}{\hbar},\\
&\qquad\qquad \frac{x^*_1\, XX^*_{-3}}{\hbar} + \frac{y^*_1\, Y^*_{-3}}{\hbar} + \frac{z^*_1\, Z^*_{-3}}{\hbar},\ 1\Big]\Big]_1[3]\Big]\ \text{TerminatedEvaluation}[\text{IterationLimit}]\Big[\\
&\qquad\mathbb{E}_{\{-1,3\}\to\{\}}\Big[\frac{a^*_3\, A^*_{-1}}{\hbar} + \frac{b^*_3\, B^*_{-1}}{\hbar},\ \frac{x^*_3\, XX^*_{-1}}{\hbar} + \frac{y^*_3\, Y^*_{-1}}{\hbar} + \frac{z^*_3\, Z^*_{-1}}{\hbar},
\end{aligned}
$$

$$1-\left\{\frac{a^*{}_2 A^*{}_{-1}}{\hbar}+\frac{a^*{}_3 A^*{}_1}{\hbar}+\frac{b^*{}_2 B^*{}_{-1}}{\hbar}+\frac{b^*{}_3 B^*{}_1}{\hbar},\ \frac{x^*{}_2 XX^*{}_{-1}}{\hbar}+\frac{x^*{}_3 XX^*{}_1}{\hbar}+\frac{y^*{}_2 Y^*{}_{-1}}{\hbar}+\frac{y^*{}_3 Y^*{}_1}{\hbar}+\frac{z^*{}_2 Z^*{}_{-1}}{\hbar}+\frac{z^*{}_3 Z^*{}_1}{\hbar},\ 1\right\}\Big[\!\Big[\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\ \hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,\ 1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\Big]\Big]\Big]\!\Big][3]+$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{-1,3\}\to\{\}}\Big[\frac{a^*{}_3 A^*{}_{-1}}{\hbar}+\frac{b^*{}_3 B^*{}_{-1}}{\hbar},\ \frac{x^*{}_3 XX^*{}_{-1}}{\hbar}+\frac{y^*{}_3 Y^*{}_{-1}}{\hbar}+\frac{z^*{}_3 Z^*{}_{-1}}{\hbar},\ 1\Big]\Big]_1[3]\Big]\Big]$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{2,j\}\to\{k\}}\Big[a_k\left(a^*{}_2+a^*{}_j\right)+b_k\left(b^*{}_2+b^*{}_j\right),$$
$$x_k\left(e^{2a^*{}_j-b^*{}_j}x^*{}_2+x^*{}_j\right)+y_k\left(e^{-a^*{}_j+2b^*{}_j}y^*{}_2+y^*{}_j\right)+z_k\left(e^{2a^*{}_j-b^*{}_j}x^*{}_2 y^*{}_j+e^{a^*{}_j+b^*{}_j}z^*{}_2+z^*{}_j\right),$$
$$1-e^{-a^*{}_j-b^*{}_j}\hbar\left(e^{3a^*{}_j}x_k y_k x^*{}_2 y^*{}_j+e^{2a^*{}_j+2b^*{}_j}y_k z_k x^*{}_2 y^*{}_2 y^*{}_j+e^{3a^*{}_j}y_k z_k x^*{}_2\left(y^*\right)_j^2-e^{3a^*{}_j}x_k z_k x^*{}_2 z^*{}_j+e^{3b^*{}_j}y_k z_k y^*{}_2 z^*{}_j-e^{3a^*{}_j}z_k^2 x^*{}_2 y^*{}_j z^*{}_j\right)\epsilon+O[\epsilon]^2\Big]\Big]$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{i,-2\}\to\{k\}}\Big[A_k A^*{}_{-2}+A_k A^*{}_i+B_k B^*{}_{-2}+B_k B^*{}_i,$$
$$XX_k XX^*{}_{-2}+XX_k XX^*{}_i+Y_k Y^*{}_{-2}+Y_k Y^*{}_i+Z_k Z^*{}_{-2}+Z_k Z^*{}_i,$$
$$1+\left(-XX_k A^*{}_i XX^*{}_{-2}-Y_k B^*{}_i Y^*{}_{-2}+2 Z_k XX^*{}_i Y^*{}_{-2}+\hbar\, XX_k Y_k XX^*{}_{-2} Y^*{}_i-Z_k A^*{}_i Z^*{}_{-2}-Z_k B^*{}_i Z^*{}_{-2}-\hbar\, XX_k Z_k XX^*{}_{-2} Z^*{}_i+\hbar\, Y_k Z_k Y^*{}_{-2} Z^*{}_i\right)\epsilon+O[\epsilon]^2\Big]\Big]\Bigg)\Bigg[$$

$$\mathbb{E}_{\{i,j\}\to\{i,j\}}\Big[a_j a^*{}_j+A_i A^*{}_i+b_j b^*{}_j+B_i B^*{}_i,\ x_j x^*{}_j+XX_i XX^*{}_i+y_j y^*{}_j+Y_i Y^*{}_i+z_j z^*{}_j+Z_i Z^*{}_i,\ 1\Big]$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\Big\{\frac{a^*{}_3 A^*{}_{-2}}{\hbar}+\frac{b^*{}_3 B^*{}_{-2}}{\hbar},\ \frac{x^*{}_3 XX^*{}_{-2}}{\hbar}+\frac{y^*{}_3 Y^*{}_{-2}}{\hbar}+\frac{z^*{}_3 Z^*{}_{-2}}{\hbar},$$
$$1-\left\{\frac{a^*{}_2 A^*{}_{-2}}{\hbar}+\frac{a^*{}_3 A^*{}_1}{\hbar}+\frac{b^*{}_2 B^*{}_{-2}}{\hbar}+\frac{b^*{}_3 B^*{}_1}{\hbar},\ \frac{x^*{}_2 XX^*{}_{-2}}{\hbar}+\frac{x^*{}_3 XX^*{}_1}{\hbar}+\frac{y^*{}_2 Y^*{}_{-2}}{\hbar}+\frac{y^*{}_3 Y^*{}_1}{\hbar}+\frac{z^*{}_2 Z^*{}_{-2}}{\hbar}+\frac{z^*{}_3 Z^*{}_1}{\hbar},\ 1\right\}\Big[\!\Big[\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{\}\to\{1,2\}}\Big[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\ \hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,\ 1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\Big]\Big]\Big]\!\Big][3]+$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}_{\{-2,3\}\to\{\}}\Big[\frac{a^*{}_3 A^*{}_{-2}}{\hbar}+\frac{b^*{}_3 B^*{}_{-2}}{\hbar},\ \frac{x^*{}_3 XX^*{}_{-2}}{\hbar}+\frac{y^*{}_3 Y^*{}_{-2}}{\hbar}+\frac{z^*{}_3 Z^*{}_{-2}}{\hbar},\ 1\Big]\Big]_1[3]\Big\}\Big[\!\Big[\mathbb{E}_{\{\}\to\{3,-2,-3\}}\Big[\mathrm{QZip}_{\{Z^*{}_{n\$5978[1]},z_{n\$5978[1]},Z^*{}_{n\$5978[2]},z_{n\$5978[2]}\}}\Big[$$

$$\mathrm{QZip}_{\{XX_{n\$5978[1]},Y^*{}_{n\$5978[1]},y_{n\$5978[1]},x^*{}_{n\$5978[1]},XX_{n\$5978[2]},Y^*{}_{n\$5978[2]},y_{n\$5978[2]},x^*{}_{n\$5978[2]}\}}\Big[$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[\mathbb{E}\Big[\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][$$
$$a_3 A_{-3}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$
$$a_3 A_{-2}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$
$$b_3 B_{-3}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$
$$b_3 B_{-2}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]]],$$
$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[x_3 x^*{}_{n\$5978[2]}\,\mathrm{TerminatedEvaluation}[$$

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      IterationLimit][TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][1]]] + y3 y*n$5978[2]
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
  z3 z*n$5978[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
  xn$5978[2] XX-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  xn$5978[1] XX-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  yn$5978[2] Y-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  yn$5978[1] Y-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  zn$5978[2] Z-3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  zn$5978[1] Z-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
  y3 y*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-3^ℏ 𝔹-3^(-2 ℏ)]]] +
  z3 z*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-3^(-ℏ) 𝔹-3^(-ℏ)]]] +
  x3 x*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-3^(-2 ℏ) 𝔹-3^ℏ]]] +
  z3 x*n$5978[1] y*n$5978[2] TerminatedEvaluation[IterationLimit][
     TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][𝔸-3^(-2 ℏ) 𝔹-3^ℏ]]]],
TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
    IterationLimit][TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
       IterationLimit][TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][
          TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
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                IterationLimit][TerminatedEvaluation[IterationLimit][
                TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                   IterationLimit][TerminatedEvaluation[IterationLimit][
                   TerminatedEvaluation[IterationLimit]
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  TerminatedEvaluation[IterationLimit][
   TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][
     TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[
        IterationLimit][SeriesData[ϵ, 0,
        TerminatedEvaluation[IterationLimit][{1,
```

$$\frac{1}{2}\hbar^3\left(x_{n\$5978[2]}^2 XX_{-3}^2+y_{n\$5978[2]}^2 Y_{-3}^2+z_{n\$5978[2]}^2 Z_{-3}^2\right)+\frac{1}{2}\hbar^3\left(x_{n\$5978[1]}^2 XX_{-2}^2+y_{n\$5978[1]}^2 Y_{-2}^2+z_{n\$5978[1]}^2 Z_{-2}^2\right)-\hbar\,\mathbb{A}_{-3}^{\hbar}\mathbb{B}_{-3}^{\hbar}\left(x_3 y_3 \mathbb{A}_{-3}^{-3\hbar} x^*{}_{n\$5978[1]} y^*{}_{n\$5978[2]}+y_3 z_3 \mathbb{A}_{-3}^{-2\hbar}\mathbb{B}_{-3}^{-2\hbar} x^*{}_{n\$5978[1]} y^*{}_{n\$5978[1]} y^*{}_{n\$5978[2]}+y_3 z_3 \mathbb{A}_{-3}^{-3\hbar} x^*{}_{n\$5978[1]} (y^*)^2_{n\$5978[2]}-x_3 z_3 \mathbb{A}_{-3}^{-3\hbar} x^*{}_{n\$5978[1]} z^*{}_{n\$5978[2]}+y_3 z_3 \mathbb{B}_{-3}^{-3\hbar} y^*{}_{n\$5978[1]} z^*{}_{n\$5978[2]}-z_3^2 \mathbb{A}_{-3}^{-3\hbar} x^*{}_{n\$5978[1]} y^*{}_{n\$5978[2]} z^*{}_{n\$5978[2]}\right)\Big\}\Big], 0, 2,$$

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1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]][
```

$$\left\{\frac{a^*{}_3 A^*{}_j}{\hbar}+\frac{b^*{}_3 B^*{}_j}{\hbar},\ \frac{x^*{}_3 XX^*{}_j}{\hbar}+\frac{y^*{}_3 Y^*{}_j}{\hbar}+\frac{z^*{}_3 Z^*{}_j}{\hbar},\ 1-\left\{\frac{a^*{}_3 A^*{}_1}{\hbar}+\frac{a^*{}_2 A^*{}_j}{\hbar}+\frac{b^*{}_3 B^*{}_1}{\hbar}+\frac{b^*{}_2 B^*{}_j}{\hbar},\ \frac{x^*{}_3 XX^*{}_1}{\hbar}+\frac{x^*{}_2 XX^*{}_j}{\hbar}+\frac{y^*{}_3 Y^*{}_1}{\hbar}+\frac{y^*{}_2 Y^*{}_j}{\hbar}+\frac{z^*{}_3 Z^*{}_1}{\hbar}+\frac{z^*{}_2 Z^*{}_j}{\hbar},\ 1\right\}[\![\right.$$

$$\text{TerminatedEvaluation[IterationLimit]}\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\ \hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,\ 1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\right]\right]]\!][3]+$$

$$\text{TerminatedEvaluation[IterationLimit]}\left[\mathbb{E}_{\{j,3\}\to\{\}}\left[\frac{a^*{}_3 A^*{}_j}{\hbar}+\frac{b^*{}_3 B^*{}_j}{\hbar},\ \frac{x^*{}_3 XX^*{}_j}{\hbar}+\frac{y^*{}_3 Y^*{}_j}{\hbar}+\frac{z^*{}_3 Z^*{}_j}{\hbar},\ 1\right]\right]_1[3]\Big\}[\![$$

$$\mathbb{E}_{\{\}\to\{3,-1,-2\}}\Big[\text{QZip}_{\{Z^*{}_{n\$5978[1]},z_{n\$5978[1]},Z^*{}_{n\$5978[2]},z_{n\$5978[2]}\}}\Big[$$

$$\text{QZip}_{\{XX_{n\$5978[1]},Y^*{}_{n\$5978[1]},y_{n\$5978[1]},x^*{}_{n\$5978[1]},XX_{n\$5978[2]},Y^*{}_{n\$5978[2]},y_{n\$5978[2]},x^*{}_{n\$5978[2]}\}}\Big[$$

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  TerminatedEvaluation[IterationLimit][
   𝔼[TerminatedEvaluation[IterationLimit][
     a₃ A₋₂ TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
         IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + a₃
      A₋₁ TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
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        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + b3
    B-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + b3
    B-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
TerminatedEvaluation[IterationLimit][x3 x*n$5978[2] TerminatedEvaluation[
      IterationLimit][TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][1]]] + y3 y*n$5978[2]
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] + z3
    z*n$5978[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][1]]] + xn$5978[2]
    XX-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + xn$5978[1]
    XX-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + yn$5978[2]
    Y-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + yn$5978[1]
    Y-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + zn$5978[2]
    Z-2 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + zn$5978[1]
    Z-1 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] + y3
    y*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-2^ℏ 𝔹-2^(-2 ℏ)]]] + z3
    z*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-2^(-ℏ) 𝔹-2^(-ℏ)]]] + x3
    x*n$5978[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][𝔸-2^(-2 ℏ) 𝔹-2^ℏ]]] +
  z3 x*n$5978[1] y*n$5978[2] TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[IterationLimit][𝔸-2^(-2 ℏ) 𝔹-2^ℏ]]]],
TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
    IterationLimit][TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][
     TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
            IterationLimit][TerminatedEvaluation[IterationLimit][
            TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
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IterationLimit] [TerminatedEvaluation[IterationLimit] [
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TerminatedEvaluation[IterationLimit] [
TerminatedEvaluation[IterationLimit] [
SeriesData[ϵ, 0, TerminatedEvaluation[
IterationLimit] [{1, $\frac{1}{2}\hbar^3 \left(x^2_{n\$5978[2]}\, XX^2_{-2} + y^2_{n\$5978[2]}\, Y^2_{-2} + z^2_{n\$5978[2]}\, Z^2_{-2}\right) + \frac{1}{2}\hbar^3 \left(x^2_{n\$5978[1]}\, XX^2_{-1} + y^2_{n\$5978[1]}\, Y^2_{-1} + z^2_{n\$5978[1]}\, Z^2_{-1}\right) - \hbar\, \mathbb{A}_{-2}^{\hbar}\, \mathbb{B}_{-2}^{\hbar} \left(x_3\, y_3\, \mathbb{A}_{-2}^{-3\hbar}\, x^*{}_{n\$5978[1]}\, y^*{}_{n\$5978[2]} + y_3\, z_3\, \mathbb{A}_{-2}^{-2\hbar}\, \mathbb{B}_{-2}^{-2\hbar}\, x^*{}_{n\$5978[1]}\, y^*{}_{n\$5978[1]}\, y^*{}_{n\$5978[2]} + y_3\, z_3\, \mathbb{A}_{-2}^{-3\hbar}\, x^*{}_{n\$5978[1]}\, (y^*)^2_{n\$5978[2]} - x_3\, z_3\, \mathbb{A}_{-2}^{-3\hbar}\, x^*{}_{n\$5978[1]}\, z^*{}_{n\$5978[2]} + y_3\, z_3\, \mathbb{B}_{-2}^{-3\hbar}\, y^*{}_{n\$5978[1]}\, z^*{}_{n\$5978[2]} - z_3^2\, \mathbb{A}_{-2}^{-3\hbar}\, x^*{}_{n\$5978[1]}\, y^*{}_{n\$5978[2]}\, z^*{}_{n\$5978[2]}\right)$}], 0, 2,
1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]
TerminatedEvaluation[IterationLimit] [$\mathbb{E}_{\{3\}\to\{3\}}$[$-a_3\, a^*{}_3 - b_3\, b^*{}_3$, $-e^{-2 a^*{}_3 + b^*{}_3}$
$x_3\, x^*{}_3 - e^{a^*{}_3 - 2 b^*{}_3}$
$y_3\, y^*{}_3 + e^{-a^*{}_3 - b^*{}_3}\, z_3$
$x^*{}_3\, y^*{}_3 - e^{-a^*{}_3 - b^*{}_3}$
$z_3\, z^*{}_3$,
1 + TerminatedEvaluation[IterationLimit] [$\mathbb{E}_{\{3\}\to\{3\}}$[$-a_3\, a^*{}_3 - b_3\, b^*{}_3$,
$-e^{-2 a^*{}_3 + b^*{}_3}\, x_3\, x^*{}_3 - e^{a^*{}_3 - 2 b^*{}_3}\, y_3\, y^*{}_3 + e^{-a^*{}_3 - b^*{}_3}\, z_3\, x^*{}_3\, y^*{}_3 - e^{-a^*{}_3 - b^*{}_3}\, z_3\, z^*{}_3$, 1]]$_1$[3] -
TerminatedEvaluation[IterationLimit][3]]] [TerminatedEvaluation[
IterationLimit] [
{$\frac{a^*{}_2\, A^*{}_3}{\hbar} + \frac{b^*{}_2\, B^*{}_3}{\hbar}$, $\frac{x^*{}_2\, XX^*{}_3}{\hbar} + \frac{y^*{}_2\, Y^*{}_3}{\hbar} + \frac{z^*{}_2\, Z^*{}_3}{\hbar}$,

$$1-\left\{\frac{a^*_2 A^*_1}{\hbar}+\frac{a^*_2 A^*_3}{\hbar}+\frac{b^*_2 B^*_1}{\hbar}+\frac{b^*_2 B^*_3}{\hbar},\ \frac{x^*_2 XX^*_1}{\hbar}+\frac{x^*_2 XX^*_3}{\hbar}+\frac{y^*_2 Y^*_1}{\hbar}+\frac{y^*_2 Y^*_3}{\hbar}+\frac{z^*_2 Z^*_1}{\hbar}+\frac{z^*_2 Z^*_3}{\hbar},\ 1\right\}\Big[\!\Big[$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\left[\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2 A_1+\hbar\, b_2 B_1,\ \hbar\, x_2 XX_1+\hbar\, y_2 Y_1+\hbar\, z_2 Z_1,\ 1+\frac{1}{2}\hbar^3\left(x_2^2 XX_1^2+y_2^2 Y_1^2+z_2^2 Z_1^2\right)\epsilon+O[\epsilon]^2\right]\right]\Big]\!\Big][3]+$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\left[\mathbb{E}_{\{3,2\}\to\{\}}\left[\frac{a^*_2 A^*_3}{\hbar}+\frac{b^*_2 B^*_3}{\hbar},\ \frac{x^*_2 XX^*_3}{\hbar}+\frac{y^*_2 Y^*_3}{\hbar}+\frac{z^*_2 Z^*_3}{\hbar},\ 1\right]\right]_1[3]\Big\}\Big[\!\Big[$$

$$\mathbb{E}_{\{\}\to\{3,2,3\}}\Big[\mathrm{QZip}_{\{Z^*_{n\$5933[1]},z_{n\$5933[1]},Z^*_{n\$5933[2]},z_{n\$5933[2]}\}}\Big[$$

$$\mathrm{QZip}_{\{XX_{n\$5933[1]},Y^*_{n\$5933[1]},y_{n\$5933[1]},x^*_{n\$5933[1]},XX_{n\$5933[2]},Y^*_{n\$5933[2]},y_{n\$5933[2]},x^*_{n\$5933[2]}\}}\Big[$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}]\Big[$$

$$\mathbb{E}\Big[\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][$$

$$a_2 A_3\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$a_3 A_3\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$b_2 B_3\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$b_3 B_3\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]],$$

$$\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][XX_3\, XX^*_{n\$5933[1]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$XX_3\, XX^*_{n\$5933[2]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$Y_3\, Y^*_{n\$5933[1]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$Y_3\, Y^*_{n\$5933[2]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$Z_3\, Z^*_{n\$5933[1]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$Z_3\, Z^*_{n\$5933[2]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][1]]]+$$

$$x_2\, XX_{n\$5933[1]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$x_3\, XX_{n\$5933[2]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$y_2\, Y_{n\$5933[1]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\hbar]]]+$$

$$y_3\, Y_{n\$5933[2]}\,\mathrm{TerminatedEvaluation}[\mathrm{IterationLimit}][\mathrm{TerminatedEvaluation}[$$

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        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z2 Zn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z3 Zn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
  TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
     IterationLimit][TerminatedEvaluation[IterationLimit][
     TerminatedEvaluation[IterationLimit][
      TerminatedEvaluation[IterationLimit][
       TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][
          TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
             IterationLimit][TerminatedEvaluation[IterationLimit][
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                IterationLimit][TerminatedEvaluation[IterationLimit][
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                 TerminatedEvaluation[IterationLimit][
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                     TerminatedEvaluation[IterationLimit][
                      TerminatedEvaluation[IterationLimit][
                       TerminatedEvaluation[IterationLimit][
                        TerminatedEvaluation[IterationLimit][
                         SeriesData[ϵ, 0, TerminatedEvaluation[
                         IterationLimit][{1, 1/2 ℏ^3 (x2^2 XX^2_n$5933[1] +
                         y2^2 Y^2_n$5933[1] + z2^2 Z^2_n$5933[1]) + 1/2 ℏ^3 (x3^2 XX^2_n$5933[2] +
                         y3^2 Y^2_n$5933[2] + z3^2 Z^2_n$5933[2]) - ℏ a2 XX3 XX*_n$5933[2] +
                         ℏ XX3 Y3 XX*_n$5933[2] Y*_n$5933[1] - ℏ b2 Y3
                         Y*_n$5933[2] + 2 Z3 XX*_n$5933[1] Y*_n$5933[2] - ℏ XX3 Z3
                         XX*_n$5933[2] Z*_n$5933[1] + ℏ Y3 Z3 Y*_n$5933[2] Z*_n$5933[1] -
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                         ℏ a2 Z3 Z*n$5933[2] - ℏ b2 Z3 Z*n$5933[2]}], 0, 2,
                         1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]][
{a*i A*3/ℏ + b*i B*3/ℏ, x*i XX*3/ℏ + y*i Y*3/ℏ + z*i Z*3/ℏ, 1 - {a*i A*1/ℏ + a*2 A*3/ℏ + b*i B*1/ℏ +
    b*2 B*3/ℏ, x*i XX*1/ℏ + x*2 XX*3/ℏ + y*i Y*1/ℏ + y*2 Y*3/ℏ + z*i Z*1/ℏ + z*2 Z*3/ℏ, 1}⟦
  TerminatedEvaluation[IterationLimit][𝔼{}→{1,2}[ℏ a2 A1 + ℏ b2 B1, ℏ x2 XX1 +
    ℏ y2 Y1 + ℏ z2 Z1, 1 + 1/2 ℏ^3 (x2^2 XX1^2 + y2^2 Y1^2 + z2^2 Z1^2) ϵ + O[ϵ]^2]]⟧[3] +
  TerminatedEvaluation[IterationLimit][𝔼{3,i}→{}[a*i A*3/ℏ + b*i B*3/ℏ,
    x*i XX*3/ℏ + y*i Y*3/ℏ + z*i Z*3/ℏ, 1]]_1[3]}⟦
𝔼{}→{3,1,2}[QZip{Z*n$5933[1],zn$5933[1],Z*n$5933[2],zn$5933[2]}[
  QZip{XXn$5933[1],Y*n$5933[1],yn$5933[1],x*n$5933[1],XXn$5933[2],Y*n$5933[2],yn$5933[2],x*n$5933[2]}[
   TerminatedEvaluation[IterationLimit][
    𝔼[TerminatedEvaluation[IterationLimit][
      a1 A3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
       a2 A3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
       b1 B3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
       b2 B3 TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
     TerminatedEvaluation[IterationLimit][XX3 XX*n$5933[1]
        TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       XX3 XX*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       Y3 Y*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       Y3 Y*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       Z3 Z*n$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       Z3 Z*n$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][1]]] +
       x1 XXn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
       x2 XXn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
          IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
       y1 Yn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
```

```
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    y2 Yn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z1 Zn$5933[1] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]] +
    z2 Zn$5933[2] TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][ℏ]]]],
  TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
    IterationLimit][TerminatedEvaluation[IterationLimit][
    TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
        IterationLimit][TerminatedEvaluation[IterationLimit][
        TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
            IterationLimit][TerminatedEvaluation[IterationLimit][
            TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                IterationLimit][TerminatedEvaluation[IterationLimit][
                TerminatedEvaluation[IterationLimit][TerminatedEvaluation[
                    IterationLimit][TerminatedEvaluation[IterationLimit][
                    TerminatedEvaluation[IterationLimit][
                     TerminatedEvaluation[IterationLimit][
                      TerminatedEvaluation[IterationLimit][
                       TerminatedEvaluation[IterationLimit][
                        TerminatedEvaluation[IterationLimit][
                         TerminatedEvaluation[IterationLimit][
                          TerminatedEvaluation[IterationLimit][
                           TerminatedEvaluation[IterationLimit][
                            SeriesData[ϵ, 0, TerminatedEvaluation[
                            IterationLimit][{1, 1/2 ℏ^3 (x1^2 XX^2_n$5933[1] +
                            y1^2 Y^2_n$5933[1] + z1^2 Z^2_n$5933[1]) + 1/2 ℏ^3 (x2^2 XX^2_n$5933[2] +
                            y2^2 Y^2_n$5933[2] + z2^2 Z^2_n$5933[2]) - ℏ a1 XX3 XX*_n$5933[2] +
                            ℏ XX3 Y3 XX*_n$5933[2] Y*_n$5933[1] - ℏ b1 Y3
                            Y*_n$5933[2] + 2 Z3 XX*_n$5933[1] Y*_n$5933[2] - ℏ XX3 Z3
```

```
XX*n$5933[2] Z*n$5933[1] + ℏ Y3 Z3 Y*n$5933[2] Z*n$5933[1] −
ℏ a1 Z3 Z*n$5933[2] − ℏ b1 Z3 Z*n$5933[2]}], 0, 2,
1]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]][
𝔼{j}→{j}[aj a*j + bj b*j, xj x*j + XXj XX*j + yj y*j + Yj Y*j +
   zj
    z*j +
   Zj
    Z*j, 1 +
   O[
     ∈]2]
 TerminatedEvaluation[IterationLimit][
  𝔼{i}→{i}[
   ai
     a*i +
    bi
     b*i, xi
     x*i +
    XXi
     XX*i +
    yi
     y*i +
    Yi
     Y*i +
    zi
     z*i +
    Zi
     Z*i, 1 +
    O[
      ∈]2]]]]]
```

Testing

```
Block[{$k = 1}, {
   am → ami,j→k, bm → bmi,j→k, dm → dmi,j→k, R → Ri,j, R̄ → R̄i,j, P → Pi,j, aS → aSi,
   aS̄ → aS̄i, bS → bSi, bS̄ → bS̄i, dS → dSi, aΔ → aΔi→j,k, bΔ → bΔi→j,k, dΔ → dΔi→j,k,
   C → Ci, C̄ → C̄i, Kink → Kinki, Kink̄ → Kink̄i, AB2st → AB2sti, st2AB → st2ABi
  }] // Column
```

Check that on the generators this agrees with our conventions in the handout:

```
Timing@
 {{"[x,a]" → ((𝔼{}→{1,2}[0, 0, a2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, a1 x2] // am1,2→1)[3]),
    "[A,X]" → ((𝔼{}→{1,2}[0, 0, XX2 A1] // bm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, XX1 A2] // bm1,2→1)[3]),
    "[x,y]" → ((𝔼{}→{1,2}[0, 0, y2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 x2] // am1,2→1)[3]),
    "[Y,X]" → ((𝔼{}→{1,2}[0, 0, XX2 Y1] // bm1,2→1)[3] -
       (𝔼{}→{1,2}[0, 0, XX1 Y2] // bm1,2→1)[3])} /. z_1 → z,
  {"Δ[X]" → Last[𝔼{}→{1}[0, 0, XX1] ~ B1 ~ bΔ1→1,2],
   "Δ[A]" → Last[𝔼{}→{1}[0, 0, A1] ~ B1 ~ bΔ1→1,2],
   "Δ[a]" → Last[𝔼{}→{1}[0, 0, a1] ~ B1 ~ aΔ1→1,2],
   "Δ[z]" → Last[𝔼{}→{1}[0, 0, z1] ~ B1 ~ aΔ1→1,2],
   "Δ[x]" → Last[𝔼{}→{1}[0, 0, x1] ~ B1 ~ aΔ1→1,2],
   "Δ[Z]" → Last[𝔼{}→{1}[0, 0, Z1] ~ B1 ~ bΔ1→1,2]},
  {
    "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ aS1)[3]),
    "S(z)" → ((𝔼{}→{1}[0, 0, z1] ~ B1 ~ aS1)[3]),
    "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ aS1)[3]),
    "S(A)" → ((𝔼{}→{1}[0, 0, A1] ~ B1 ~ bS1)[3]),
    "S(X)" → ((𝔼{}→{1}[0, 0, XX1] ~ B1 ~ bS1)[3]),
    "S(Z)" → ((𝔼{}→{1}[0, 0, Z1] ~ B1 ~ bS1)[3])
   } /. z_1 → z}
```

$$\{1.265625, \{\{[x,a] \to 2x + O[\epsilon]^2, [A,X] \to -XX\,\epsilon + O[\epsilon]^2, [x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[Y,X] \to (-2Z + XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2\}, \{\Delta[X] \to (XX_2 + XX_1\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}) + O[\epsilon]^2, \Delta[A] \to (A_1 + A_2) + O[\epsilon]^2,$$
$$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^2, \Delta[z] \to (z_1 + z_2) + (2\hbar\,x_1\,y_2 - \hbar\,a_1\,z_2 - \hbar\,b_1\,z_2)\,\epsilon + O[\epsilon]^2,$$
$$\Delta[x] \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2, \Delta[Z] \to (Z_2 + Z_1\,\mathbb{A}_2^{-\hbar}\,\mathbb{B}_2^{-\hbar} + \hbar\,XX_1\,Y_2\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}) + O[\epsilon]^2\},$$
$$\{S(a) \to -a + O[\epsilon]^2, S(z) \to -z + (2\,x\,y\,\hbar + 2\,z\,\hbar - a\,z\,\hbar - b\,z\,\hbar)\,\epsilon + O[\epsilon]^2,$$
$$S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2, S(A) \to -A + O[\epsilon]^2, S(X) \to -XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar} + O[\epsilon]^2,$$
$$S(Z) \to (-Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar} + XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar) + (2\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar - XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar^2)\,\epsilon + O[\epsilon]^2\}\}\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
Timing@Block[{$k = 1},
  HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
 ]
```

{0.437500, {True, True}}

R and *P* are inverses:

```
Timing@Block[{$k = 1}, {HL[(Ri,j // Pi,k) ≡ 𝔼{k}→{j}[aj a*k + bj b*k, xj x*k + yj y*k + zj z*k, 1]]}]
```

{0.031250, {True}}

as and $\overline{aS}$ are inverses, bs and $\overline{bS}$ are inverses:

```
Timing[HL /@ {(aS̄1 // aS1) ≡ 𝔼{1}→{1}[a1 a*1 + b1 b*1, x1 x*1 + y1 y*1 + z1 z*1, 1],
    (bS̄1 // bS1) ≡ 𝔼{1}→{1}[A1 A*1 + B1 B*1, XX1 XX*1 + Y1 Y*1 + Z1 Z*1, 1]}]
```

{0.406250, {True, True}}

(co)-associativity on both sides

```
Timing[HL /@ {(aΔ1→1,2 // aΔ2→2,3) ≡ (aΔ1→1,3 // aΔ1→1,2), (bΔ1→1,2 // bΔ2→2,3) ≡ (bΔ1→1,3 // bΔ1→1,2),
    (am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}]
{1.078125, {True, True, True, True}}
```

Δ is an algebra morphism

```
Timing[HL /@ {(am1,2→1 // aΔ1→1,2) ≡ ((aΔ1→1,3 aΔ2→2,4) // (am3,4→2 am1,2→1)),
    (bm1,2→1 // bΔ1→1,2) ≡ ((bΔ1→1,3 bΔ2→2,4) // (bm3,4→2 bm1,2→1))}]
{1.312500, {True, True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
{1.015625, {True, True, True, True}}
```

S is an algebra anti-(co)morphism

```
Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
    aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
{2.500000, {True, True, True, True}}
```

R-matrix and antipode

```
R1,2 ~ B1 ~ (bS1) ≡ R̄1,2
True
```

Pairing axioms

```
Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[b*3 b3 + a*3 a3, y*3 y3 + x*3 x3 + z*3 z3, 1]) ~ B1,3 ~ P1,3 ≡
     (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1]
        𝔼{2}→{2}[(A*)2 A2 + (B*)2 B2, (XX*)2 XX2 + (Y*)2 Y2 + Z*2 Z2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
    (bΔ1→1,2 𝔼{3}→{3}[b*3 b3 + a*3 a3, y*3 y3 + x*3 x3 + z*3 z3, 1]
        𝔼{4}→{4}[b*4 b4 + a*4 a4, y*4 y4 + x*4 x4 + z*4 z4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
     (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] am3,4→3) ~ B1,3 ~ P1,3}]
{0.34375, {True, True}}
```

```
Timing[HL /@ {((bS1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) // P1,2) ≡
     ((𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] aS2) // P1,2),
    (b̄S1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) ~ B1,2 ~ P1,2 ≡
     (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] āS2) ~ B1,2 ~ P1,2}]
{0.28125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
{
    "[a,y]" →
```

```
 ((𝔼{}→{1,2}[0, 0, y2 a1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, y1 a2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[b,x]"  → ((𝔼{}→{1,2}[0, 0, x2 b1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, x1 b2] ~ B1,2 ~ dm1,2→1) [[3]]), "[b,y]" →
 ((𝔼{}→{1,2}[0, 0, y2 b1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, y1 b2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[a,x]" → ((𝔼{}→{1,2}[0, 0, x2 a1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, x1 a2] ~ B1,2 ~ dm1,2→1) [[3]]), "[a,z]" →
 ((𝔼{}→{1,2}[0, 0, z2 a1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, z1 a2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[b,z]" → ((𝔼{}→{1,2}[0, 0, z2 b1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, z1 b2] ~ B1,2 ~ dm1,2→1) [[3]]), "[x,z]" →
 ((𝔼{}→{1,2}[0, 0, z2 x1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, z1 x2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[y,z]" → ((𝔼{}→{1,2}[0, 0, z2 y1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, z1 y2] ~ B1,2 ~ dm1,2→1) [[3]]), "[x,y]" →
 ((𝔼{}→{1,2}[0, 0, y2 x1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, y1 x2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[Y,y]" → ((𝔼{}→{1,2}[0, 0, y2 Y1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, y1 Y2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify,
"[Y,x]" → ((𝔼{}→{1,2}[0, 0, x2 Y1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, x1 Y2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify,
"[X,y]" → ((𝔼{}→{1,2}[0, 0, y2 XX1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, y1 XX2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify,
"[XX,x]" →
   ((𝔼{}→{1,2}[0, 0, x2 XX1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, x1 XX2] ~ B1,2 ~ dm1,2→1) [[3]]) //
  Expand // Simplify, "[Z,z]" → ((𝔼{}→{1,2}[0, 0, z2 Z1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, z1 Z2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify, "[Z,y]" →
   ((𝔼{}→{1,2}[0, 0, y2 Z1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, y1 Z2] ~ B1,2 ~ dm1,2→1) [[3]]) //
  Expand // Simplify, "[Z,x]" → ((𝔼{}→{1,2}[0, 0, x2 Z1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, x1 Z2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify, "[XX,z]" →
   ((𝔼{}→{1,2}[0, 0, z2 XX1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, z1 XX2] ~ B1,2 ~ dm1,2→1) [[3]]) //
  Expand // Simplify, "[Y,z]" → ((𝔼{}→{1,2}[0, 0, z2 Y1] ~ B1,2 ~ dm1,2→1) [[3]] -
     (𝔼{}→{1,2}[0, 0, z1 Y2] ~ B1,2 ~ dm1,2→1) [[3]]) // Expand // Simplify, "[a,Z]" →
 ((𝔼{}→{1,2}[0, 0, Z2 a1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, Z1 a2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[a,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 a1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, XX1 a2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[a,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 a1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, Y1 a2] ~ B1,2 ~ dm1,2→1) [[3]]), "[b,Z]" →
 ((𝔼{}→{1,2}[0, 0, Z2 b1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, Z1 b2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[b,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 b1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, XX1 b2] ~ B1,2 ~ dm1,2→1) [[3]]), "[b,Y]" →
 ((𝔼{}→{1,2}[0, 0, Y2 b1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, Y1 b2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[A,Z]" → ((𝔼{}→{1,2}[0, 0, Z2 A1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, Z1 A2] ~ B1,2 ~ dm1,2→1) [[3]]), "[A,XX]" →
 ((𝔼{}→{1,2}[0, 0, XX2 A1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, XX1 A2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[A,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 A1] ~ B1,2 ~ dm1,2→1) [[3]] -
   (𝔼{}→{1,2}[0, 0, Y1 A2] ~ B1,2 ~ dm1,2→1) [[3]]), "[B,Z]" →
 ((𝔼{}→{1,2}[0, 0, Z2 B1] ~ B1,2 ~ dm1,2→1) [[3]] - (𝔼{}→{1,2}[0, 0, Z1 B2] ~ B1,2 ~ dm1,2→1) [[3]]),
"[B,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 B1] ~ B1,2 ~ dm1,2→1) [[3]] -
```

```
        (𝔼{}→{1,2}[0, 0, XX1 B2] ~ B1,2 ~ dm1,2→1)[[3]]), "[B,Y]" →
      ((𝔼{}→{1,2}[0, 0, Y2 B1] ~ B1,2 ~ dm1,2→1)[[3]] - (𝔼{}→{1,2}[0, 0, Y1 B2] ~ B1,2 ~ dm1,2→1)[[3]]),
    "[XX,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 XX1] ~ B1,2 ~ dm1,2→1)[[3]] -
        (𝔼{}→{1,2}[0, 0, Y1 XX2] ~ B1,2 ~ dm1,2→1)[[3]]), "[Z,Y]" →
      ((𝔼{}→{1,2}[0, 0, Y2 Z1] ~ B1,2 ~ dm1,2→1)[[3]] - (𝔼{}→{1,2}[0, 0, Y1 Z2] ~ B1,2 ~ dm1,2→1)[[3]]),
    "[Z,XX]" →
      ((𝔼{}→{1,2}[0, 0, XX2 Z1] ~ B1,2 ~ dm1,2→1)[[3]] - (𝔼{}→{1,2}[0, 0, XX1 Z2] ~ B1,2 ~ dm1,2→1)[[3]]),
    "[A,x]" → ((𝔼{}→{1,2}[0, 0, x2 A1] ~ B1,2 ~ dm1,2→1)[[3]] -
        (𝔼{}→{1,2}[0, 0, x1 A2] ~ B1,2 ~ dm1,2→1)[[3]]), "[A,y]" →
      ((𝔼{}→{1,2}[0, 0, y2 A1] ~ B1,2 ~ dm1,2→1)[[3]] - (𝔼{}→{1,2}[0, 0, y1 A2] ~ B1,2 ~ dm1,2→1)[[3]])
   } /. {v_1 → v} // Expand // Factor
{
  "Δ(a)" → ((𝔼{1,2}→{}[0, 0, a1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(x)" → ((𝔼{1,2}→{}[0, 0, x1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(b)" → ((𝔼{1,2}→{}[0, 0, b1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(y)" → ((𝔼{1,2}→{}[0, 0, y1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(z)" → ((𝔼{1,2}→{}[0, 0, z1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(XX)" → ((𝔼{1,2}→{}[0, 0, XX1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(Y)" → ((𝔼{1,2}→{}[0, 0, Y1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(Z)" → ((𝔼{1,2}→{}[0, 0, Z1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(A)" → ((𝔼{1,2}→{}[0, 0, A1] ~ B1 ~ dΔ1→1,2)[[3]]),
  "Δ(B)" → ((𝔼{1,2}→{}[0, 0, B1] ~ B1 ~ dΔ1→1,2)[[3]])} // Simplify
{
   "S(a)" → ((𝔼{1}→{1}[0, 0, a1] ~ B1 ~ dS1)[[3]]),
   "S(x)" → ((𝔼{1}→{1}[0, 0, x1] ~ B1 ~ dS1)[[3]]),
   "S(b)" → ((𝔼{1}→{1}[0, 0, b1] ~ B1 ~ dS1)[[3]]),
   "S(y)" → ((𝔼{1}→{1}[0, 0, y1] ~ B1 ~ dS1)[[3]]), "S(z)" → ((𝔼{1}→{1}[0, 0, z1] ~ B1 ~ dS1)[[3]]),
   "S(XX)" → ((𝔼{1}→{1}[0, 0, XX1] ~ B1 ~ dS1)[[3]]),
   "S(Y)" → ((𝔼{1}→{1}[0, 0, Y1] ~ B1 ~ dS1)[[3]]),
   "S(Z)" → ((𝔼{1}→{1}[0, 0, Z1] ~ B1 ~ dS1)[[3]]), "S(A)" → ((𝔼{1}→{1}[0, 0, A1] ~ B1 ~ dS1)[[3]]),
   "S(B)" → ((𝔼{1}→{1}[0, 0, B1] ~ B1 ~ dS1)[[3]])
  } /. {v_1 → v} // Simplify
```

$$\Big\{[a,y]\to y+O[\epsilon]^2,\ [b,x]\to x+O[\epsilon]^2,\ [b,y]\to -2y+O[\epsilon]^2,$$
$$[a,x]\to -2x+O[\epsilon]^2,\ [a,z]\to -z+O[\epsilon]^2,\ [b,z]\to -z+O[\epsilon]^2,$$
$$[x,z]\to x z\hbar\epsilon+O[\epsilon]^2,\ [y,z]\to -y z\hbar\epsilon+O[\epsilon]^2,\ [x,y]\to z-x y\hbar\epsilon+O[\epsilon]^2,$$
$$[Y,y]\to \frac{-1+\mathbb{A}^{\hbar}\mathbb{B}^{-2\hbar}}{\hbar}+\left(-b\,\mathbb{A}^{\hbar}\mathbb{B}^{-2\hbar}+2 y Y\hbar\right)\epsilon+O[\epsilon]^2,\ [Y,x]\to -x Y\hbar\epsilon+O[\epsilon]^2,$$
$$[X,y]\to -XX\, y\hbar\epsilon+O[\epsilon]^2,\ [XX,x]\to \frac{-1+\mathbb{A}^{-2\hbar}\mathbb{B}^{\hbar}}{\hbar}+\left(-a\,\mathbb{A}^{-2\hbar}\mathbb{B}^{\hbar}+2 x\, XX\hbar\right)\epsilon+O[\epsilon]^2,$$
$$[Z,z]\to \frac{-1+\mathbb{A}^{-\hbar}\mathbb{B}^{-\hbar}}{\hbar}+\mathbb{A}^{-\hbar}\mathbb{B}^{-\hbar}\left(-a-b+2 z Z\,\mathbb{A}^{\hbar}\mathbb{B}^{\hbar}\hbar\right)\epsilon+O[\epsilon]^2,\ [Z,y]\to -XX+y Z\hbar\epsilon+O[\epsilon]^2,$$
$$[Z,x]\to Y\,\mathbb{A}^{-2\hbar}\mathbb{B}^{\hbar}+\mathbb{A}^{-2\hbar}\left(x Z\,\mathbb{A}^{2\hbar}-(-1+a)\,Y\,\mathbb{B}^{\hbar}\right)\hbar\epsilon+O[\epsilon]^2,\ [XX,z]\to(-2y+XX\, z\hbar)\epsilon+O[\epsilon]^2,$$
$$[Y,z]\to\left(2x\,\mathbb{A}^{\hbar}\mathbb{B}^{-2\hbar}+Y z\hbar\right)\epsilon+O[\epsilon]^2,\ [a,Z]\to Z+O[\epsilon]^2,\ [a,XX]\to 2\,XX+O[\epsilon]^2,$$
$$[a,Y]\to -Y+O[\epsilon]^2,\ [b,Z]\to Z+O[\epsilon]^2,\ [b,XX]\to -XX+O[\epsilon]^2,\ [b,Y]\to 2Y+O[\epsilon]^2,$$
$$[A,Z]\to -Z\epsilon+O[\epsilon]^2,\ [A,XX]\to -XX\epsilon+O[\epsilon]^2,\ [A,Y]\to O[\epsilon]^2,\ [B,Z]\to -Z\epsilon+O[\epsilon]^2,$$
$$[B,XX]\to O[\epsilon]^2,\ [B,Y]\to -Y\epsilon+O[\epsilon]^2,\ [XX,Y]\to(2Z-XX\,Y\hbar)\epsilon+O[\epsilon]^2,$$
$$[Z,Y]\to Y Z\hbar\epsilon+O[\epsilon]^2,\ [Z,XX]\to -XX\,Z\hbar\epsilon+O[\epsilon]^2,\ [A,x]\to x\epsilon+O[\epsilon]^2,\ [A,y]\to O[\epsilon]^2\Big\}$$

$$\Big\{\Delta(a)\to(a_1+a_2)+O[\epsilon]^2,\ \Delta(x)\to(x_1+x_2)-\hbar\, a_1 x_2\epsilon+O[\epsilon]^2,$$
$$\Delta(b)\to(b_1+b_2)+O[\epsilon]^2,\ \Delta(y)\to(y_1+y_2)-\hbar\, b_1 y_2\epsilon+O[\epsilon]^2,$$
$$\Delta(z)\to(z_1+z_2)+\hbar\,(2x_1 y_2-(a_1+b_1)\,z_2)\epsilon+O[\epsilon]^2,\ \Delta(XX)\to\left(XX_1+XX_2\,\mathbb{A}_1^{-2\hbar}\mathbb{B}_1^{\hbar}\right)+O[\epsilon]^2,$$
$$\Delta(Y)\to\left(Y_1+Y_2\,\mathbb{A}_1^{\hbar}\mathbb{B}_1^{-2\hbar}\right)+O[\epsilon]^2,\ \Delta(Z)\to\left(Z_1+Z_2\,\mathbb{A}_1^{-\hbar}\mathbb{B}_1^{-\hbar}+\hbar\, XX_2 Y_1\,\mathbb{A}_1^{-2\hbar}\mathbb{B}_1^{\hbar}\right)+O[\epsilon]^2,$$
$$\Delta(A)\to(A_1+A_2)+O[\epsilon]^2,\ \Delta(B)\to(B_1+B_2)+O[\epsilon]^2\Big\}$$

$$\Big\{S(a)\to -a+O[\epsilon]^2,\ S(x)\to -x-a x\hbar\epsilon+O[\epsilon]^2,\ S(b)\to -b+O[\epsilon]^2,$$
$$S(y)\to -y-b y\hbar\epsilon+O[\epsilon]^2,\ S(z)\to -z+(2xy-(-2+a+b)\,z)\hbar\epsilon+O[\epsilon]^2,$$
$$S(XX)\to -XX\,\mathbb{A}^{2\hbar}\mathbb{B}^{-\hbar}-2\left(XX\,\mathbb{A}^{2\hbar}\mathbb{B}^{-\hbar}\hbar\right)\epsilon+O[\epsilon]^2,\ S(Y)\to -Y\,\mathbb{A}^{-\hbar}\mathbb{B}^{2\hbar}-2\left(Y\,\mathbb{A}^{-\hbar}\mathbb{B}^{2\hbar}\hbar\right)\epsilon+O[\epsilon]^2,$$
$$S(Z)\to\mathbb{A}^{\hbar}\mathbb{B}^{\hbar}(-Z+XX\,Y\hbar)+\mathbb{A}^{\hbar}\mathbb{B}^{\hbar}\hbar\,(-2Z+3\,XX\,Y\hbar)\epsilon+O[\epsilon]^2,$$
$$S(A)\to -A+O[\epsilon]^2,\ S(B)\to -B+O[\epsilon]^2\Big\}$$

(co)-associativity

```
Timing[HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
```

{14.218750, {True, True}}

```
Timing[HL /@ {(stdΔ1→1,2 // stdΔ2→2,3) ≡ (stdΔ1→1,3 // stdΔ1→1,2),
   (stdm1,2→1 // stdm1,3→1) ≡ (stdm2,3→2 // stdm1,2→1)}]
```

{7.531250, {True, True}}

```
Timing[HL /@ {(ddm1,2→1 ~ STB1 ~ ddm1,3→1) ≡ (ddm2,3→2 ~ STB2 ~ ddm1,2→1)}]
```

{2.125000, {True}}

Δ is an algebra morphism

```
Timing@HL[dm1,2→1 ~ B1 ~ dΔ1→1,2 ≡ (dΔ1→1,3 dΔ2→2,4) ~ B1,2,3,4 ~ (dm3,4→2 dm1,2→1)]
```

{7.296875, True}

S_2 inverts R, but not S_1:

```
Timing@{HL[R_{1,2} ~ B_2 ~ dS_2 ≡ R̄_{1,2}]}
```

{0.796875, {True}}

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@
  {(dΔ_{1→1,2} ~ B_1 ~ dS_1) ~ B_{1,2} ~ dm_{1,2→1}, (dΔ_{1→1,2} ~ B_2 ~ dS_2) // dm_{1,2→1}}]
```

{4.703125, {True, True}}

S is a (co)-algebra anti-morphism

```
Timing[HL /@
  Expand /@ {dm_{1,2→1} ~ B_1 ~ dS_1 ≡ (dS_1 dS_2) ~ B_{1,2} ~ dm_{2,1→1}, dS_1 ~ B_1 ~ dΔ_{1→1,2} ≡ dΔ_{1→2,1} ~ B_{1,2} ~ (dS_1 dS_2)}]
```

{22.718750, {True, True}}

Quasi-triangular axiom 1:

```
Timing@HL[R_{1,2} ~ B_1 ~ dΔ_{1→1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ dm_{2,4→2}]
```

{0.375000, True}

Quasi-triangular axiom 2:

```
Timing@HL[((dΔ_{1→1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})) ≡ ((dΔ_{1→2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{3,1→1} dm_{4,2→2}))]
```

{2.359375, True}

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)\sim B_{1,2}\sim \mathrm{dm}_{1,2\to1} \equiv \mathbb{E}[0, 0, 1]$:

```
Timing@HL[(((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i} ≡
   𝔼_{{}→{i}}[0, 0, 1]]
```

{2.453125, True}

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner $C=uv^{-1}$. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy. Taking the square root of z and multiplying it with S(u) yields the ribbon element v.

```
Timing@Block[{$k = 1},
  (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}]
```

$\{8.062500,\ \mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar} + \left(2\,\hbar\,a_i\,\mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar} + 2\,\hbar\,b_i\,\mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar}\right)\epsilon + O[\epsilon]^2\right]\}$

Turaev moves are checked here.

T-4:

```
Timing@Block[{$k = 1},
  HL /@ {(C̄_1 C̄_2 R_{3,4} C_5 C_6) ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ~ B_{2,4} ~ dm_{2,4→2} ~ B_{2,6} ~ dm_{2,6→2} ≡ R_{1,2}}]
```

{3.796875, {True}}

T-5, T-6

```
Timing@Block[{$k = 1}, HL /@ {(Ci C̄j) ~ Bi,j ~ dmi,j→i ≡ 𝔼{}→{i}[0, 0, 1],
    (Ci C̄j) ~ Bi,j ~ dmj,i→i ≡ 𝔼{}→{i}[0, 0, 1], (C̄i C̄j) ~ Bi,j ~ dmi,j→i ≡
     (((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) ~ Bi ~ dSi) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~ Bi,j ~ dmi,j→i}]
{16.296875, {True, True, True}}
```

Reidemeister 2 or T-3:

```
Timing[HL[# ≡ 𝔼{}→{1,2}[0, 0, 1]] & /@
  {(R̄1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3→1 dm2,4→2), (R1,2 R̄3,4) ~ B1,2,3,4 ~ (dm1,3→1 dm2,4→2)}]
{2.500000, {True, True}}
```

```
Timing[HL[# ≡ 𝔼{}→{1,2}[0, 0, 1]] & /@ {(((R̄R̄1,2 RR3,4)) ~ B1,2,3,4 ~ ((ddm1,3→1 ddm2,4→2))),
    ((RR1,2 R̄R̄3,4) ~ STB1,2,3,4 ~ (ddm1,3→1 ddm2,4→2))}]
{2.718750, {True, True}}
```

Cyclic Reidemeister 2 or T-2:

```
Timing@HL[(R1,4 R̄5,2 C̄3) ~ B2,4 ~ dm2,4→2 ~ B1,3 ~ dm1,3→1 ~ B1,5 ~ dm1,5→1 ≡ C̄1 𝔼{}→{2}[0, 0, 1]]
{7.765625, True}
```

Reidemeister 3 or T-1:

```
Timing@HL[((R1,2 R4,3 R5,6) ~ B1,4 ~ dm1,4→1 ~ B2,5 ~ dm2,5→2 ~ B3,6 ~ dm3,6→3 ) ≡
    ((R1,6 R2,3 R4,5) ~ B1,4 ~ dm1,4→1 ~ B2,5 ~ dm2,5→2 ~ B3,6 ~ dm3,6→3)]
{5.343750, True}
```

```
Timing@HL[((RR1,2 RR4,3 RR5,6) ~ STB1,4 ~ ddm1,4→1 ~ STB2,5 ~ ddm2,5→2 ~ STB3,6 ~ ddm3,6→3 ) ≡
    ((RR1,6 RR2,3 RR4,5) ~ STB1,4 ~ ddm1,4→1 ~ STB2,5 ~ ddm2,5→2 ~ STB3,6 ~ ddm3,6→3)]
{1.656250, True}
```

Relations between the four kinks or T-7

```
Timing[
 HL /@ {Kinki ≡ (R3,1 C2) ~ B1,2 ~ dm1,2→1 ~ B1,3 ~ dm1,3→i, K̄ink̄j ≡ (R̄3,1 C̄2) ~ B1,2 ~ dm1,2→1 ~ B1,3 ~ dm1,3→j,
    (Kinki K̄ink̄j) ~ Bi,j ~ dmi,j→1 ≡ 𝔼{}→{1}[0, 0, 1], (Kinki K̄ink̄j) ~ Bi,j ~ dmj,i→1 ≡ 𝔼{}→{1}[0, 0, 1]}]
{9.187500, {True, True, True, True}}
```

The Trefoil

```
Timing@Block[{$k = 1}, ZZ = RR1,5 RR6,2 RR3,7 C̄C̄4 K̄K̄ink̄8 K̄K̄ink̄9 K̄K̄ink̄10;
  Do[ZZ = (ZZ /. {ϵ → 0}) ~ B1,r ~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify /@ (ZZ /. {ϵ → 0})}]
```

$$\left\{1.250000, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{S^2 T^2}{(1 - S + S^2)\,(1 - T + T^2)\,(1 - S\,T + S^2\,T^2)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR_{1,5} RR_{6,2} RR_{3,7} CC̄_4 KKink̄_8 KKink̄_9 KKink̄_{10});
  Do[Print["Doing ", r]; ZZ = (ZZ) ~B_{1,r}~ (ddm_{1,r→1}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{28.890625,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}+\Big(2\,S^2\,T^2\,\Big(S-2\,S^2+3\,S^3-2\,S^4+T-2\,S\,T+$$

$$S^2\,T+2\,S^3\,T-5\,S^4\,T+6\,S^5\,T-2\,T^2+S\,T^2-S^3\,T^2-4\,S^4\,T^2+5\,S^5\,T^2-11\,S^6\,T^2+3\,T^3+2\,S\,T^3-$$

$$S^2\,T^3+4\,S^3\,T^3+6\,S^4\,T^3-S^5\,T^3+7\,S^6\,T^3+10\,S^7\,T^3-2\,T^4-5\,S\,T^4-4\,S^2\,T^4+6\,S^3\,T^4-24\,S^4\,T^4+$$

$$10\,S^5\,T^4-12\,S^6\,T^4-11\,S^7\,T^4-6\,S^8\,T^4+6\,S\,T^5+5\,S^2\,T^5-S^3\,T^5+10\,S^4\,T^5+12\,S^5\,T^5-S^6\,T^5+$$

$$10\,S^7\,T^5+13\,S^8\,T^5-11\,S^2\,T^6+7\,S^3\,T^6-12\,S^4\,T^6-S^5\,T^6-16\,S^6\,T^6+11\,S^7\,T^6-20\,S^8\,T^6+$$

$$10\,S^3\,T^7-11\,S^4\,T^7+10\,S^5\,T^7+11\,S^6\,T^7-14\,S^7\,T^7+15\,S^8\,T^7-6\,S^4\,T^8+13\,S^5\,T^8-20\,S^6\,T^8+$$

$$15\,S^7\,T^8-8\,S^8\,T^8+\left(1-S+S^2\right)\left(1-T+T^2\right)^2\Big(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)-$$

$$S^5\,T^3\,(3+T)-S^2\,T\,(1+3\,T)+S^3\,T\left(-1+T^2\right)\Big)\,a_1+\left(1-S+S^2\right)^2\left(1-T+T^2\right)$$

$$\Big(-2+T+3\,S\,T-S\,(1+3\,S)\,T^2+S\left(-1+S^2\right)T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\Big)\,b_1+$$

$$2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1-$$

$$5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+5\,S^4\,T^2\,x_1\,XX_1+$$

$$9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1-$$

$$12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+$$

$$14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1-$$

$$12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1-$$

$$7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+$$

$$9\,S^7\,T^6\,x_1\,XX_1-5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+$$

$$2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+4\,S^4\,y_1\,Y_1-$$

$$2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-3\,S\,T^2\,y_1\,Y_1+$$

$$9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1-$$

$$2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1-$$

$$5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+$$

$$9\,S^6\,T^4\,y_1\,Y_1-2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1-$$

$$12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-12\,S^5\,T^6\,y_1\,Y_1+$$

$$5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+9\,S^6\,T^7\,y_1\,Y_1-5\,S^7\,T^7\,y_1\,Y_1+$$

$$2\,S^8\,T^7\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+7\,S\,XX_1\,Y_1\,z_1-9\,S^2\,XX_1\,Y_1\,z_1+7\,S^3\,XX_1\,Y_1\,z_1-2\,S^4\,XX_1\,Y_1\,z_1-$$

$$5\,S\,T\,XX_1\,Y_1\,z_1-5\,S^4\,T\,XX_1\,Y_1\,z_1+12\,S\,T^2\,XX_1\,Y_1\,z_1-9\,S^2\,T^2\,XX_1\,Y_1\,z_1+21\,S^3\,T^2\,XX_1\,Y_1\,z_1-$$

$$9\,S^4\,T^2\,XX_1\,Y_1\,z_1+12\,S^5\,T^2\,XX_1\,Y_1\,z_1-2\,T^3\,XX_1\,Y_1\,z_1-5\,S\,T^3\,XX_1\,Y_1\,z_1-9\,S^2\,T^3\,XX_1\,Y_1\,z_1-$$

$$7\,S^3\,T^3\,XX_1\,Y_1\,z_1-7\,S^4\,T^3\,XX_1\,Y_1\,z_1-9\,S^5\,T^3\,XX_1\,Y_1\,z_1-5\,S^6\,T^3\,XX_1\,Y_1\,z_1-2\,S^7\,T^3\,XX_1\,Y_1\,z_1+$$

$$7\,S\,T^4\,XX_1\,Y_1\,z_1+21\,S^3\,T^4\,XX_1\,Y_1\,z_1-7\,S^4\,T^4\,XX_1\,Y_1\,z_1+21\,S^5\,T^4\,XX_1\,Y_1\,z_1+7\,S^7\,T^4\,XX_1\,Y_1\,z_1-$$

$$9\,S^2\,T^5\,XX_1\,Y_1\,z_1-9\,S^4\,T^5\,XX_1\,Y_1\,z_1-9\,S^5\,T^5\,XX_1\,Y_1\,z_1-9\,S^7\,T^5\,XX_1\,Y_1\,z_1+7\,S^3\,T^6\,XX_1\,Y_1\,z_1-$$

$$5\,S^4\,T^6\,XX_1\,Y_1\,z_1+12\,S^5\,T^6\,XX_1\,Y_1\,z_1-5\,S^6\,T^6\,XX_1\,Y_1\,z_1+7\,S^7\,T^6\,XX_1\,Y_1\,z_1-2\,S^4\,T^7\,XX_1\,Y_1\,z_1-$$

$$2\,S^7\,T^7\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-7\,S\,z_1\,Z_1+9\,S^2\,z_1\,Z_1-5\,S^3\,z_1\,Z_1+2\,S^4\,z_1\,Z_1-7\,T\,z_1\,Z_1+$$

$$10\,S\,T\,z_1\,Z_1-12\,S^2\,T\,z_1\,Z_1+5\,S^3\,T\,z_1\,Z_1-2\,S^4\,T\,z_1\,Z_1+9\,T^2\,z_1\,Z_1-12\,S\,T^2\,z_1\,Z_1+$$

$$18\,S^2\,T^2\,z_1\,Z_1-12\,S^3\,T^2\,z_1\,Z_1+9\,S^4\,T^2\,z_1\,Z_1-3\,S^5\,T^2\,z_1\,Z_1-5\,T^3\,z_1\,Z_1+5\,S\,T^3\,z_1\,Z_1-$$

$$12\,S^2\,T^3\,z_1\,Z_1+14\,S^3\,T^3\,z_1\,Z_1-14\,S^4\,T^3\,z_1\,Z_1+9\,S^5\,T^3\,z_1\,Z_1-2\,S^6\,T^3\,z_1\,Z_1+2\,S^7\,T^3\,z_1\,Z_1+$$

$$2\,T^4\,z_1\,Z_1-2\,S\,T^4\,z_1\,Z_1+9\,S^2\,T^4\,z_1\,Z_1-14\,S^3\,T^4\,z_1\,Z_1+14\,S^4\,T^4\,z_1\,Z_1-12\,S^5\,T^4\,z_1\,Z_1+$$

$$5\,S^6\,T^4\,z_1\,Z_1-5\,S^7\,T^4\,z_1\,Z_1-3\,S^2\,T^5\,z_1\,Z_1+9\,S^3\,T^5\,z_1\,Z_1-12\,S^4\,T^5\,z_1\,Z_1+18\,S^5\,T^5\,z_1\,Z_1-$$

$$12\,S^6\,T^5\,z_1\,Z_1+9\,S^7\,T^5\,z_1\,Z_1-2\,S^3\,T^6\,z_1\,Z_1+5\,S^4\,T^6\,z_1\,Z_1-12\,S^5\,T^6\,z_1\,Z_1+10\,S^6\,T^6\,z_1\,Z_1-$$

$$7\,S^7\,T^6\,z_1\,Z_1+2\,S^3\,T^7\,z_1\,Z_1-5\,S^4\,T^7\,z_1\,Z_1+9\,S^5\,T^7\,z_1\,Z_1-7\,S^6\,T^7\,z_1\,Z_1+4\,S^7\,T^7\,z_1\,Z_1\Big)\,\epsilon\Big)\Big/$$

$$\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The Figure Eight knot

```
Timing@Block[{$k = 1},
  ZZ = (RR̄8,1 RR̄2,6 RR5,9 RR10,3 CC̄7 CC4);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB1,r~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{2.125000,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR̄8,1 RR̄2,6 RR5,9 RR10,3 CC̄7 CC4);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{532.593750,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}-\right.\right.\right.$$
$$\Big(2\,\Big(S^2\,T^2\,\Big(4-9\,S+2\,S^2-9\,T+12\,S\,T+6\,S^2\,T+2\,T^2+6\,S\,T^2-6\,S^3\,T^2-2\,S^4\,T^2-6\,S^2\,T^3-12\,S^3\,T^3+$$
$$9\,S^4\,T^3-2\,S^2\,T^4+9\,S^3\,T^4-4\,S^4\,T^4+\left(1-3\,T+T^2\right)\left(-2+2\,S^4\,T^2+3\,S\,(1+T)-3\,S^3\,T\,(1+T)\right)$$
$$a_1+\left(1-3\,S+S^2\right)\left(-2+3\,(1+S)\,T-3\,S\,(1+S)\,T^3+2\,S^2\,T^4\right)b_1+2\,x_1\,XX_1+$$
$$2\,S\,x_1\,XX_1-9\,T\,x_1\,XX_1-3\,S\,T\,x_1\,XX_1-9\,S^2\,T\,x_1\,XX_1+4\,T^2\,x_1\,XX_1+16\,S\,T^2\,x_1\,XX_1+$$
$$16\,S^2\,T^2\,x_1\,XX_1+4\,S^3\,T^2\,x_1\,XX_1-9\,S\,T^3\,x_1\,XX_1-3\,S^2\,T^3\,x_1\,XX_1-9\,S^3\,T^3\,x_1\,XX_1+$$
$$2\,S^2\,T^4\,x_1\,XX_1+2\,S^3\,T^4\,x_1\,XX_1+2\,y_1\,Y_1-9\,S\,y_1\,Y_1+4\,S^2\,y_1\,Y_1+2\,T\,y_1\,Y_1-3\,S\,T\,y_1\,Y_1+$$
$$16\,S^2\,T\,y_1\,Y_1-9\,S^3\,T\,y_1\,Y_1-9\,S\,T^2\,y_1\,Y_1+16\,S^2\,T^2\,y_1\,Y_1-3\,S^3\,T^2\,y_1\,Y_1+2\,S^4\,T^2\,y_1\,Y_1+$$
$$4\,S^2\,T^3\,y_1\,Y_1-9\,S^3\,T^3\,y_1\,Y_1+2\,S^4\,T^3\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+11\,S\,XX_1\,Y_1\,z_1-2\,S^2\,XX_1\,Y_1\,z_1-$$
$$2\,T\,XX_1\,Y_1\,z_1-8\,S\,T\,XX_1\,Y_1\,z_1-8\,S^2\,T\,XX_1\,Y_1\,z_1-2\,S^3\,T\,XX_1\,Y_1\,z_1+11\,S\,T^2\,XX_1\,Y_1\,z_1-$$
$$8\,S^2\,T^2\,XX_1\,Y_1\,z_1+11\,S^3\,T^2\,XX_1\,Y_1\,z_1-2\,S^2\,T^3\,XX_1\,Y_1\,z_1-2\,S^3\,T^3\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-$$
$$9\,S\,z_1\,Z_1+2\,S^2\,z_1\,Z_1-9\,T\,z_1\,Z_1+16\,S\,T\,z_1\,Z_1-3\,S^2\,T\,z_1\,Z_1+2\,S^3\,T\,z_1\,Z_1+2\,T^2\,z_1\,Z_1-$$
$$\left.\left.\left.3\,S\,T^2\,z_1\,Z_1+16\,S^2\,T^2\,z_1\,Z_1-9\,S^3\,T^2\,z_1\,Z_1+2\,S\,T^3\,z_1\,Z_1-9\,S^2\,T^3\,z_1\,Z_1+4\,S^3\,T^3\,z_1\,Z_1\right)\right)\epsilon\right)\Big/$$
$$\left.\left.\left.\left(\left(1-3\,S+S^2\right)^2\left(1-3\,T+T^2\right)^2\left(1-3\,S\,T+S^2\,T^2\right)^2\right)+O[\epsilon]^2\right]\right\}\right\}$$

The 6-3 knot

```
Timing@Block[{$k = 1},
  ZZ = (RR̄6,12 RR̄10,14 RR̄13,7 RR1,5 RR8,2 RR3,9 CC̄4 CC11);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 14}];
  {Simplify@ZZ}]
```

$$\left\{337.828125,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\right.\right.\right.$$
$$\left(S^4\,T^4\right)\Big/\left(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)\right)+$$
$$\Big(2\,S^4\,T^4\,\Big(8-21\,S+30\,S^2-15\,S^3+4\,S^4-21\,T+36\,S\,T-30\,S^2\,T-24\,S^3\,T+18\,S^4\,T-6\,S^5\,T+$$
$$30\,T^2-30\,S\,T^2+12\,S^2\,T^2+45\,S^3\,T^2+6\,S^4\,T^2-6\,S^5\,T^2-15\,T^3-24\,S\,T^3+45\,S^2\,T^3-$$
$$48\,S^3\,T^3-24\,S^4\,T^3+6\,S^6\,T^3+6\,S^7\,T^3+4\,T^4+18\,S\,T^4+6\,S^2\,T^4-24\,S^3\,T^4+24\,S^5\,T^4-$$
$$6\,S^6\,T^4-18\,S^7\,T^4-4\,S^8\,T^4-6\,S\,T^5-6\,S^2\,T^5+24\,S^4\,T^5+48\,S^5\,T^5-45\,S^6\,T^5+24\,S^7\,T^5+$$
$$15\,S^8\,T^5+6\,S^3\,T^6-6\,S^4\,T^6-45\,S^5\,T^6-12\,S^6\,T^6+30\,S^7\,T^6-30\,S^8\,T^6+6\,S^3\,T^7-18\,S^4\,T^7+$$
$$24\,S^5\,T^7+30\,S^6\,T^7-36\,S^7\,T^7+21\,S^8\,T^7-4\,S^4\,T^8+15\,S^5\,T^8-30\,S^6\,T^8+21\,S^7\,T^8-8\,S^8\,T^8+$$
$$\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\Big(-4+4\,S^8\,T^4+9\,S\,(1+T)-9\,S^7\,T^3\,(1+T)-2\,S^2\left(5+9\,T+5\,T^2\right)+$$
$$2\,S^6\,T^2\left(5+9\,T+5\,T^2\right)+3\,S^3\left(1+5\,T+5\,T^2+T^3\right)-3\,S^5\,T\left(1+5\,T+5\,T^2+T^3\right)\Big)\,a_1+$$

$$\left(1-3S+5S^2-3S^3+S^4\right)\left(-4+9(1+S)T-2\left(5+9S+5S^2\right)T^2+3\left(1+5S+5S^2+S^3\right)T^3-3S\left(1+5S+5S^2+S^3\right)T^5+2S^2\left(5+9S+5S^2\right)T^6-9S^3(1+S)T^7+4S^4T^8\right)b_1+$$
$$4x_1XX_1-2Sx_1XX_1-2S^2x_1XX_1+4S^3x_1XX_1-15Tx_1XX_1+3STx_1XX_1-3S^2Tx_1XX_1+$$
$$3S^3Tx_1XX_1-15S^4Tx_1XX_1+30T^2x_1XX_1+6ST^2x_1XX_1+12S^2T^2x_1XX_1+12S^3T^2x_1XX_1+$$
$$6S^4T^2x_1XX_1+30S^5T^2x_1XX_1-21T^3x_1XX_1-51ST^3x_1XX_1-6S^2T^3x_1XX_1-30S^3T^3x_1XX_1-$$
$$6S^4T^3x_1XX_1-51S^5T^3x_1XX_1-21S^6T^3x_1XX_1+8T^4x_1XX_1+44ST^4x_1XX_1+56S^2T^4x_1XX_1+$$
$$8S^3T^4x_1XX_1+8S^4T^4x_1XX_1+56S^5T^4x_1XX_1+44S^6T^4x_1XX_1+8S^7T^4x_1XX_1-21ST^5x_1XX_1-$$
$$51S^2T^5x_1XX_1-6S^3T^5x_1XX_1-30S^4T^5x_1XX_1-6S^5T^5x_1XX_1-51S^6T^5x_1XX_1-21S^7T^5x_1XX_1+$$
$$30S^2T^6x_1XX_1+6S^3T^6x_1XX_1+12S^4T^6x_1XX_1+12S^5T^6x_1XX_1+6S^6T^6x_1XX_1+$$
$$30S^7T^6x_1XX_1-15S^3T^7x_1XX_1+3S^4T^7x_1XX_1-3S^5T^7x_1XX_1+3S^6T^7x_1XX_1-15S^7T^7x_1XX_1+$$
$$4S^4T^8x_1XX_1-2S^5T^8x_1XX_1-2S^6T^8x_1XX_1+4S^7T^8x_1XX_1+4y_1Y_1-15Sy_1Y_1+$$
$$30S^2y_1Y_1-21S^3y_1Y_1+8S^4y_1Y_1-2Ty_1Y_1+3STy_1Y_1+6S^2Ty_1Y_1-51S^3Ty_1Y_1+$$
$$44S^4Ty_1Y_1-21S^5Ty_1Y_1-2T^2y_1Y_1-3ST^2y_1Y_1+12S^2T^2y_1Y_1-6S^3T^2y_1Y_1+$$
$$56S^4T^2y_1Y_1-51S^5T^2y_1Y_1+30S^6T^2y_1Y_1+4T^3y_1Y_1+3ST^3y_1Y_1+12S^2T^3y_1Y_1-$$
$$30S^3T^3y_1Y_1+8S^4T^3y_1Y_1-6S^5T^3y_1Y_1+6S^6T^3y_1Y_1-15S^7T^3y_1Y_1-15ST^4y_1Y_1+$$
$$6S^2T^4y_1Y_1-6S^3T^4y_1Y_1+8S^4T^4y_1Y_1-30S^5T^4y_1Y_1+12S^6T^4y_1Y_1+3S^7T^4y_1Y_1+$$
$$4S^8T^4y_1Y_1+30S^2T^5y_1Y_1-51S^3T^5y_1Y_1+56S^4T^5y_1Y_1-6S^5T^5y_1Y_1+12S^6T^5y_1Y_1-$$
$$3S^7T^5y_1Y_1-2S^8T^5y_1Y_1-21S^3T^6y_1Y_1+44S^4T^6y_1Y_1-51S^5T^6y_1Y_1+6S^6T^6y_1Y_1+$$
$$3S^7T^6y_1Y_1-2S^8T^6y_1Y_1+8S^4T^7y_1Y_1-21S^5T^7y_1Y_1+30S^6T^7y_1Y_1-15S^7T^7y_1Y_1+$$
$$4S^8T^7y_1Y_1-4XX_1Y_1z_1+19SXX_1Y_1z_1-32S^2XX_1Y_1z_1+19S^3XX_1Y_1z_1-4S^4XX_1Y_1z_1+$$
$$2TXX_1Y_1z_1-22STXX_1Y_1z_1+16S^2TXX_1Y_1z_1+16S^3TXX_1Y_1z_1-22S^4TXX_1Y_1z_1+$$
$$2S^5TXX_1Y_1z_1+2T^2XX_1Y_1z_1+35ST^2XX_1Y_1z_1-28S^2T^2XX_1Y_1z_1+34S^3T^2XX_1Y_1z_1-$$
$$28S^4T^2XX_1Y_1z_1+35S^5T^2XX_1Y_1z_1+2S^6T^2XX_1Y_1z_1-4T^3XX_1Y_1z_1-22ST^3XX_1Y_1z_1-$$
$$28S^2T^3XX_1Y_1z_1-4S^3T^3XX_1Y_1z_1-4S^4T^3XX_1Y_1z_1-28S^5T^3XX_1Y_1z_1-22S^6T^3XX_1Y_1z_1-$$
$$4S^7T^3XX_1Y_1z_1+19ST^4XX_1Y_1z_1+16S^2T^4XX_1Y_1z_1+34S^3T^4XX_1Y_1z_1-4S^4T^4XX_1Y_1z_1+$$
$$34S^5T^4XX_1Y_1z_1+16S^6T^4XX_1Y_1z_1+19S^7T^4XX_1Y_1z_1-32S^2T^5XX_1Y_1z_1+16S^3T^5XX_1Y_1z_1-$$
$$28S^4T^5XX_1Y_1z_1-28S^5T^5XX_1Y_1z_1+16S^6T^5XX_1Y_1z_1-32S^7T^5XX_1Y_1z_1+19S^3T^6XX_1Y_1z_1-$$
$$22S^4T^6XX_1Y_1z_1+35S^5T^6XX_1Y_1z_1-22S^6T^6XX_1Y_1z_1+19S^7T^6XX_1Y_1z_1-4S^4T^7XX_1Y_1z_1+$$
$$2S^5T^7XX_1Y_1z_1+2S^6T^7XX_1Y_1z_1-4S^7T^7XX_1Y_1z_1+8z_1Z_1-21Sz_1Z_1+30S^2z_1Z_1-$$
$$15S^3z_1Z_1+4S^4z_1Z_1-21Tz_1Z_1+44STz_1Z_1-51S^2Tz_1Z_1+6S^3Tz_1Z_1+3S^4Tz_1Z_1-$$
$$2S^5Tz_1Z_1+30T^2z_1Z_1-51ST^2z_1Z_1+56S^2T^2z_1Z_1-6S^3T^2z_1Z_1+12S^4T^2z_1Z_1-$$
$$3S^5T^2z_1Z_1-2S^6T^2z_1Z_1-15T^3z_1Z_1+6ST^3z_1Z_1-6S^2T^3z_1Z_1+8S^3T^3z_1Z_1-$$
$$30S^4T^3z_1Z_1+12S^5T^3z_1Z_1+3S^6T^3z_1Z_1+4S^7T^3z_1Z_1+4T^4z_1Z_1+3ST^4z_1Z_1+$$
$$12S^2T^4z_1Z_1-30S^3T^4z_1Z_1+8S^4T^4z_1Z_1-6S^5T^4z_1Z_1+6S^6T^4z_1Z_1-15S^7T^4z_1Z_1-$$
$$2ST^5z_1Z_1-3S^2T^5z_1Z_1+12S^3T^5z_1Z_1-6S^4T^5z_1Z_1+56S^5T^5z_1Z_1-51S^6T^5z_1Z_1+$$
$$30S^7T^5z_1Z_1-2S^2T^6z_1Z_1+3S^3T^6z_1Z_1+6S^4T^6z_1Z_1-51S^5T^6z_1Z_1+44S^6T^6z_1Z_1-$$
$$21S^7T^6z_1Z_1+4S^3T^7z_1Z_1-15S^4T^7z_1Z_1+30S^5T^7z_1Z_1-21S^6T^7z_1Z_1+8S^7T^7z_1Z_1)\epsilon\Big)\Big/$$
$$\left(\left(1-3S+5S^2-3S^3+S^4\right)^2\left(1-3T+5T^2-3T^3+T^4\right)^2\left(1-3ST+5S^2T^2-3S^3T^3+S^4T^4\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

Mirror Trefoil

```
Timing@Block[{$k = 1},
  ZZ = (RR̄1,5 RR̄6,2 RR̄3,7 CC4 KKink8 KKink9 KKink10);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~ STB1,r ~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{3.562500, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{S^2 T^2}{\left(1 - S + S^2\right)\left(1 - T + T^2\right)\left(1 - S T + S^2 T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR̄1,5 RR̄6,2 RR̄3,7 CC4 KKink8 KKink9 KKink10);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~ STB1,r ~ (ddm1,r→1), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{49.390625,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}+\Big(2\,S^2\,T^2\,\Big(8-15\,S+20\,S^2-13\,S^3+6\,S^4-15\,T+$$

$$14\,S\,T-11\,S^2\,T-10\,S^3\,T+11\,S^4\,T-10\,S^5\,T+20\,T^2-11\,S\,T^2+16\,S^2\,T^2+S^3\,T^2+12\,S^4\,T^2-$$

$$7\,S^5\,T^2+11\,S^6\,T^2-13\,T^3-10\,S\,T^3+S^2\,T^3-12\,S^3\,T^3-10\,S^4\,T^3+S^5\,T^3-5\,S^6\,T^3-6\,S^7\,T^3+$$

$$6\,T^4+11\,S\,T^4+12\,S^2\,T^4-10\,S^3\,T^4+24\,S^4\,T^4-6\,S^5\,T^4+4\,S^6\,T^4+5\,S^7\,T^4+2\,S^8\,T^4-10\,S\,T^5-$$

$$7\,S^2\,T^5+S^3\,T^5-6\,S^4\,T^5-4\,S^5\,T^5+S^6\,T^5-2\,S^7\,T^5-3\,S^8\,T^5+11\,S^2\,T^6-5\,S^3\,T^6+4\,S^4\,T^6+$$

$$S^5\,T^6-S^7\,T^6+2\,S^8\,T^6-6\,S^3\,T^7+5\,S^4\,T^7-2\,S^5\,T^7-S^6\,T^7+2\,S^7\,T^7-S^8\,T^7+2\,S^4\,T^8-3\,S^5\,T^8+$$

$$2\,S^6\,T^8-S^7\,T^8+\left(1-S+S^2\right)\left(1-T+T^2\right)^2\Big(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)-$$

$$S^5\,T^3\,(3+T)-S^2\,T\,(1+3\,T)+S^3\,T\left(-1+T^2\right)\Big)\,a_1+\left(1-S+S^2\right)^2\left(1-T+T^2\right)$$

$$\Big(-2+T+3\,S\,T-S\,(1+3\,S)\,T^2+S\left(-1+S^2\right)T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\Big)\,b_1+$$

$$2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1-$$

$$5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+5\,S^4\,T^2\,x_1\,XX_1+$$

$$9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1-$$

$$12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+$$

$$14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1-$$

$$12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1-$$

$$7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+$$

$$9\,S^7\,T^6\,x_1\,XX_1-5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+$$

$$2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+4\,S^4\,y_1\,Y_1-$$

$$2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-3\,S\,T^2\,y_1\,Y_1+$$

$$9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1-$$

$$2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1-$$

$$5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+$$

$$9\,S^6\,T^4\,y_1\,Y_1-2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1-$$

$$12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-12\,S^5\,T^6\,y_1\,Y_1+$$

$$5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+9\,S^6\,T^7\,y_1\,Y_1-5\,S^7\,T^7\,y_1\,Y_1+$$

$$2\,S^8\,T^7\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+7\,S\,XX_1\,Y_1\,z_1-9\,S^2\,XX_1\,Y_1\,z_1+7\,S^3\,XX_1\,Y_1\,z_1-2\,S^4\,XX_1\,Y_1\,z_1-$$

$$5\,S\,T\,XX_1\,Y_1\,z_1-5\,S^4\,T\,XX_1\,Y_1\,z_1+12\,S\,T^2\,XX_1\,Y_1\,z_1-9\,S^2\,T^2\,XX_1\,Y_1\,z_1+21\,S^3\,T^2\,XX_1\,Y_1\,z_1-$$

$$9\,S^4\,T^2\,XX_1\,Y_1\,z_1+12\,S^5\,T^2\,XX_1\,Y_1\,z_1-2\,T^3\,XX_1\,Y_1\,z_1-5\,S\,T^3\,XX_1\,Y_1\,z_1-9\,S^2\,T^3\,XX_1\,Y_1\,z_1-$$

$$7\,S^3\,T^3\,XX_1\,Y_1\,z_1-7\,S^4\,T^3\,XX_1\,Y_1\,z_1-9\,S^5\,T^3\,XX_1\,Y_1\,z_1-5\,S^6\,T^3\,XX_1\,Y_1\,z_1-2\,S^7\,T^3\,XX_1\,Y_1\,z_1+$$

$$7\,S\,T^4\,XX_1\,Y_1\,z_1+21\,S^3\,T^4\,XX_1\,Y_1\,z_1-7\,S^4\,T^4\,XX_1\,Y_1\,z_1+21\,S^5\,T^4\,XX_1\,Y_1\,z_1+7\,S^7\,T^4\,XX_1\,Y_1\,z_1-$$

$$9\,S^2\,T^5\,XX_1\,Y_1\,z_1-9\,S^4\,T^5\,XX_1\,Y_1\,z_1-9\,S^5\,T^5\,XX_1\,Y_1\,z_1-9\,S^7\,T^5\,XX_1\,Y_1\,z_1+7\,S^3\,T^6\,XX_1\,Y_1\,z_1-$$

$$5\,S^4\,T^6\,XX_1\,Y_1\,z_1+12\,S^5\,T^6\,XX_1\,Y_1\,z_1-5\,S^6\,T^6\,XX_1\,Y_1\,z_1+7\,S^7\,T^6\,XX_1\,Y_1\,z_1-2\,S^4\,T^7\,XX_1\,Y_1\,z_1-$$

$$2\,S^7\,T^7\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-7\,S\,z_1\,Z_1+9\,S^2\,z_1\,Z_1-5\,S^3\,z_1\,Z_1+2\,S^4\,z_1\,Z_1-7\,T\,z_1\,Z_1+$$

$$10\,S\,T\,z_1\,Z_1-12\,S^2\,T\,z_1\,Z_1+5\,S^3\,T\,z_1\,Z_1-2\,S^4\,T\,z_1\,Z_1+9\,T^2\,z_1\,Z_1-12\,S\,T^2\,z_1\,Z_1+$$

$$18\,S^2\,T^2\,z_1\,Z_1-12\,S^3\,T^2\,z_1\,Z_1+9\,S^4\,T^2\,z_1\,Z_1-3\,S^5\,T^2\,z_1\,Z_1-5\,T^3\,z_1\,Z_1+5\,S\,T^3\,z_1\,Z_1-$$

$$12\,S^2\,T^3\,z_1\,Z_1+14\,S^3\,T^3\,z_1\,Z_1-14\,S^4\,T^3\,z_1\,Z_1+9\,S^5\,T^3\,z_1\,Z_1-2\,S^6\,T^3\,z_1\,Z_1+2\,S^7\,T^3\,z_1\,Z_1+$$

$$2\,T^4\,z_1\,Z_1-2\,S\,T^4\,z_1\,Z_1+9\,S^2\,T^4\,z_1\,Z_1-14\,S^3\,T^4\,z_1\,Z_1+14\,S^4\,T^4\,z_1\,Z_1-12\,S^5\,T^4\,z_1\,Z_1+$$

$$5\,S^6\,T^4\,z_1\,Z_1-5\,S^7\,T^4\,z_1\,Z_1-3\,S^2\,T^5\,z_1\,Z_1+9\,S^3\,T^5\,z_1\,Z_1-12\,S^4\,T^5\,z_1\,Z_1+18\,S^5\,T^5\,z_1\,Z_1-$$

$$12\,S^6\,T^5\,z_1\,Z_1+9\,S^7\,T^5\,z_1\,Z_1-2\,S^3\,T^6\,z_1\,Z_1+5\,S^4\,T^6\,z_1\,Z_1-12\,S^5\,T^6\,z_1\,Z_1+10\,S^6\,T^6\,z_1\,Z_1-$$

$$7\,S^7\,T^6\,z_1\,Z_1+2\,S^3\,T^7\,z_1\,Z_1-5\,S^4\,T^7\,z_1\,Z_1+9\,S^5\,T^7\,z_1\,Z_1-7\,S^6\,T^7\,z_1\,Z_1+4\,S^7\,T^7\,z_1\,Z_1\Big)\,\epsilon\Big)\Big/$$

$$\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

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(*EndProfile[]; *)
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