The full sl_3 invariant using the Drinfel'd double. For compatibility reasons, we use XX instead of the generator X. This program continues sl2invariant.nb by Dror Bar-Natan and Roland van der Veen.

Profiling

```
(*BeginProfile[];*)
```

External Utilities

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

```
MaxBy[list_, fun_, n_] := list〚Ordering[fun /@ list, -n]〛;
```

Canonical Form:

Program

```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
      Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
   {vs = Cases[ℰ, (XX | Y | Z | A | B | b | s | t | a | x |
          y | z | XX* | Y* | Z* | A* | B* | s* | t* | b* | a* | x* | y* | z*)_, ∞] ⋃
      {XX, Y, Z, A, B, b, s, t, a, x, y, z, XX*, Y*, Z*, A*, B*, s*, t*, b*, a*, x*, y*, z*}},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
  ];
```

Program

The Kronecker δ:

Program

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

```
𝔼3@𝔼sp___[ω_, L_, Q_, P_] := Module[
   {NP = Normal[P]},
   𝔼sp[L, ω^-1 Q, (ω^-1 NP /. ϵ → ω^-4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
𝔼4@𝔼sp___[L_, Q_, P_] := Module[
   {NP = Normal[P], ω},
   ω = (NP /. ϵ → 0)^-1;
   𝔼sp[ω, L, ω Q, (ω NP /. ϵ → ω^4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

```
(u_i_)* := (u*)i;
((u_i_)*)* := ui;
(((u_)*)i_)* := ui;
((u_)*)* := u;
```

Program

Finite Zips:

Program

```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := PPCollect@Collect[ℰ, ζ];
Zip{}[P_] := P; Zip{ζ_,ζs___}[P_] := PPZip[
  (collect[P // Zip{ζs}, ζ] /. f_. ζ^d_. :→ ∂{ζ*,d} f) /. ζ* → 0]
```

Program

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars. $\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a,b’s, Q is a combination of $x_i\, X_j$ (possibly starred and/or mixed with other variables), and P is a perturbation polynomial. It should be interpreted via $\mathbb{O}[\mathbb{E}[\ldots], \{X_1, Y_1, Z_1, A_1, B_1, b_1, a_1, z_1, y_1, x_1\}_i, \ldots]$, with an assumed standard ordering on the generators for an interpretation of the tensor as an expression in $U_q(\mathrm{sl}_3{}^\epsilon)$.

Program

```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := PP_QZip@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

Program

Upper to lower and lower to Upper:

Program

```
U2l = {𝕓_{i_}^{p_·} → 𝕖^{-p b_i}, 𝕓^{p_·} → 𝕖^{-p b}, 𝕒_{i_}^{p_·} → 𝕖^{-p a_i}, 𝕒^{p_·} → 𝕖^{-p a}, T_{i_}^{p_·} → 𝕖^{p t_i}, T^{p_·} → 𝕖^{p t}, S_{i_}^{p_·} → 𝕖^{p s_i},
   S^{p_·} → 𝕖^{p s}, a_{i_}^{p_·} → 𝕖^{p a*_i}, a^{p_·} → 𝕖^{p a*}, b_{i_}^{p_·} → 𝕖^{p b*_i}, b^{p_·} → 𝕖^{p b*}, 𝔹_{i_}^{p_·} → 𝕖^{-p B_i}, 𝔹^{p_·} → 𝕖^{-p B},
   𝔸_{i_}^{p_·} → 𝕖^{-p A_i}, 𝔸^{p_·} → 𝕖^{-p A}, 𝒜_{i_}^{p_·} → 𝕖^{p A*_i}, 𝒜^{p_·} → 𝕖^{p A*}, ℬ_{i_}^{p_·} → 𝕖^{p B*_i}, ℬ^{p_·} → 𝕖^{p B*}};
l2U = {𝕖^{c_· b_{i_} + d_·} ⧴ 𝕓_i^{-c} 𝕖^d, 𝕖^{c_· b + d_·} ⧴ 𝕓^{-c} 𝕖^d, 𝕖^{c_· a_{i_} + d_·} ⧴ 𝕒_i^{-c} 𝕖^d, 𝕖^{c_· a + d_·} ⧴ 𝕒^{-c} 𝕖^d,
   𝕖^{c_· t_{i_} + d_·} ⧴ T_i^c 𝕖^d, 𝕖^{c_· t + d_·} ⧴ T^c 𝕖^d, 𝕖^{c_· s_{i_} + d_·} ⧴ S_i^c 𝕖^d, 𝕖^{c_· s + d_·} ⧴ S^c 𝕖^d,
   𝕖^{c_· a*_{i_} + d_·} ⧴ a_i^c 𝕖^d, 𝕖^{c_· a* + d_·} ⧴ a^c 𝕖^d, 𝕖^{c_· b*_{i_} + d_·} ⧴ b_i^c 𝕖^d, 𝕖^{c_· b* + d_·} ⧴ b^c 𝕖^d,
   𝕖^{c_· B_{i_} + d_·} ⧴ 𝔹_i^{-c} 𝕖^d, 𝕖^{c_· B + d_·} ⧴ 𝔹^{-c} 𝕖^d, 𝕖^{c_· A_{i_} + d_·} ⧴ 𝔸_i^{-c} 𝕖^d, 𝕖^{c_· A + d_·} ⧴ 𝔸^{-c} 𝕖^d,
   𝕖^{c_· A*_{i_} + d_·} ⧴ 𝒜_i^c 𝕖^d, 𝕖^{c_· A* + d_·} ⧴ 𝒜^c 𝕖^d, 𝕖^{c_· B*_{i_} + d_·} ⧴ ℬ_i^c 𝕖^d, 𝕖^{c_· B* + d_·} ⧴ ℬ^c 𝕖^d,
   𝕖^{ℰ_} ⧴ 𝕖^{Expand@ℰ}};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single"P". Here the z's are A, B and a^*, b^* and the ζ's are A^*, B^* and a, b. s and t are not regarded as scalars for zip-technicalities. DB and STB are variations of B with a different choice for ζs in LZip, to speed up the zipping of tensors with s and t instead of A and B.

Program

```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ζ, z, zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. U2l /. (zrule ⋃ ζrule);
    EEQ[ps___] := EEQ[ps] = PP_"EEQ"@
       (CF[e^{-Q1} D[e^{Q1}, Sequence @@ Thread[{zs, {ps}}]]] /. Alternatives @@ zs → 0 //. l2U);
    CF /@ ((*CF/@*)𝔼[
        c + ηs.lt.ys, Q1 /. Alternatives @@ zs → 0,
        Det[lt] (Zip_ζs[(EQ @@ zs) (P /. U2l /. (zrule ⋃ ζrule))] /.
            Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1)
      ] //. l2U)
   ];
```

Program

```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_Bind@Module[{n},
    Times[
       L /. Table[(v : XX | Y | Z | A | 𝔸 | B | 𝔹 | s | t | S | T | b | (*𝕓|*)a(*|𝕒*) | x | y | z)_i →
          v_{n@i}, {i, {is}}],
       R /. Table[(v : XX* | Y* | Z* | A*(*|𝒜*) | B* |
             (*ℬ|*)b* | s* | t* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Flatten@Table[{A*_{n@i},B*_{n@i},(s*)_{n@i},(t*)_{n@i},b_{n@i},a_{n@i}},{i,{is}}]} //
    QZip_{Flatten@Table[{XX_{n@i},Y*_{n@i},y_{n@i},x*_{n@i}},{i,{is}}]} // QZip_{Flatten@Table[{Z*_{n@i},z_{n@i}},{i,{is}}]}];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

```
DB_{}[L_, R_] := L R;
DB_{is__}[L_𝔼, R_𝔼] := PP_DBind@Module[{n},
    Times[
       L /. Table[(v : XX | Y | Z | s | t | b | 𝕓 | a | 𝕒 | x | y | z)_i → v_{n@i}, {i, {is}}],
       R /. Table[(v : XX* | Y* | Z* | s* | t* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Flatten@Table[{(s*)_{n@i},(t*)_{n@i},b_{n@i},a_{n@i}},{i,{is}}]} //
    QZip_{Flatten@Table[{XX_{n@i},Y*_{n@i},y_{n@i},x*_{n@i}},{i,{is}}]} // QZip_{Flatten@Table[{Z*_{n@i},z_{n@i}},{i,{is}}]}];
DB_{is___}[L_, R_] := DB_{{is}}[L, R];
```

```
STB{}[L_, R_] := L R;
STB{is__}[L_𝔼, R_𝔼] := PP_STBind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | b | 𝕓 | a | 𝕒 | x | y | z)_i → v_n@i, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_n@i, {i, {is}}]
    ] // LZip_Flatten@Table[{b_n@i,a_n@i},{i,{is}}] // QZip_Flatten@Table[{XX_n@i,Y*_n@i,y_n@i,x*_n@i},{i,{is}}] //
    QZip_Flatten@Table[{Z*_n@i,z_n@i},{i,{is}}]];
STB_is___[L_, R_] := STB{is}[L, R];
```

Program

𝔼 morphisms with domain and range.

Program

```
B_is_List[𝔼_d1_→r1_[L1_, Q1_, P1_], 𝔼_d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
STB_is_List[𝔼_d1_→r1_[L1_, Q1_, P1_], 𝔼_d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ STB_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] // 𝔼_d2_→r2_[L2_, Q2_, P2_] :=
  B_r1⋂d2[𝔼_d1→r1[L1, Q1, P1], 𝔼_d2→r2[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_d1_→r1_[L1_, Q1_, P1_] 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_d_→r_[L_, Q_, P_]_$k_ := 𝔼_d→r @@ 𝔼[L, Q, P]_$k;
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

Program

“Define” code

Program

Define[lhs = rhs,   ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

Program

Booting Up

Program

```
$k = 1;
```

The multiplication tensors on both halves of the double are defined here. $k indicates the degree $\epsilon^{\$k+1} = 0$ we are working over. The am and bm tensors given here only work for $k=1 and $k=0.

```
Define[am_{i,j→k} =
   𝔼_{{i,j}→{k}}[(a*_i + a*_j) a_k + (b*_i + b*_j) b_k, (ⅇ^((b*)_j+(a*)_j) z*_i + z*_j + ⅇ^(-(b*)_j+2 (a*)_j) x*_i (y*)_j) z_k +
     (ⅇ^(-(a*)_j+2 (b*)_j) y*_i + y*_j) y_k + (ⅇ^(-(b*)_j+2 (a*)_j) x*_i + x*_j) x_k,
    1 + ϵ (ℏ (z*)_j (x*)_i ⅇ^(-(b*)_j+2 (a*)_j) z_k x_k + ℏ (z*)_j (x*)_i (y*)_j ⅇ^(-(b*)_j+2 (a*)_j) z_k z_k -
       ℏ (z*)_j (y*)_i ⅇ^(2 (b*)_j-(a*)_j) z_k y_k - ℏ (y*)_i (x*)_i (y*)_j ⅇ^((a*)_j+(b*)_j) z_k y_k -
       ℏ (x*)_i ⅇ^(-(b*)_j+2 (a*)_j) (y*)_j (y*)_j z_k y_k - ℏ (x*)_i ⅇ^(-(b*)_j+2 (a*)_j) (y*)_j y_k x_k) + O[ϵ]^2]_$k,
  bm_{i,j→k} =
   𝔼_{{i,j}→{k}}[A_k A*_i + A_k A*_j + B_k B*_i + B_k B*_j, XX_k XX*_i + XX_k XX*_j + Y_k Y*_i + Y_k Y*_j + Z_k Z*_i + Z_k Z*_j,
    1 + (-XX_k (A*)_i (XX*)_j + ℏ XX_k Y_k (XX*)_j (Y*)_i - Y_k (B*)_i (Y*)_j + 2 Z_k (XX*)_i (Y*)_j - ℏ XX_k Z_k
         (XX*)_j (Z*)_i + ℏ Y_k Z_k (Y*)_j (Z*)_i - Z_k (A*)_i (Z*)_j - Z_k (B*)_i (Z*)_j) ϵ + O[ϵ]^2]_$k];
```

The R-matrix is defined with the Faddeev-Quesne formula.

Program

$$
\begin{aligned}
&\mathrm{Define}\Big[\\
&\quad R_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar A_i a_j + \hbar B_i b_j,\ \hbar XX_i x_j + \hbar Y_i y_j + \hbar Z_i z_j,\ e\text{^}\left(\sum_{k=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^k (\hbar XX_i x_j)^k}{k\left(1-e^{-2k\epsilon\hbar}\right)}\right)\\
&\qquad e\text{^}\left(\sum_{l=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^l (\hbar Y_i y_j)^l}{l\left(1-e^{-2l\epsilon\hbar}\right)}\right) e\text{^}\left(\sum_{m=2}^{\$k+1} \frac{\left(e^{-2\epsilon\hbar}-1\right)^m (\hbar Z_i z_j)^m}{m\left(1-e^{-2m\epsilon\hbar}\right)}\right)\Big]_{\$k},\\
&\quad \overline{R}_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[-\hbar a_j A_i - \hbar b_j B_i,\ -\hbar x_j XX_i \mathbb{A}_i^{2\hbar} \mathbb{B}_i^{-\hbar} + \hbar^2 XX_i Y_i z_j \mathbb{A}_i^{\hbar} \mathbb{B}_i^{\hbar} -\\
&\qquad \hbar z_j Z_i \mathbb{A}_i^{\hbar} \mathbb{B}_i^{\hbar} - \hbar y_j Y_i \mathbb{A}_i^{-\hbar} \mathbb{B}_i^{2\hbar},\ 1 + \mathrm{If}\big[\$k == 0,\ 0,\ (\overline{R}_{\{i,j\},\$k-1})_{\$k}[3] -\\
&\qquad\quad \big(\big((\overline{R}_{\{i,j\},0})_{\$k}\, R_{1,2}\, (\overline{R}_{\{3,4\},\$k-1})_{\$k}\big) \mathbin{//} (bm_{i,1\to i}\, am_{j,2\to j}) \mathbin{//} (bm_{i,3\to i}\, am_{j,4\to j})\big)[3]\big]\Big],\\
&\quad P_{i,j} = \mathbb{E}_{\{i,j\}\to\{\}}\Big[\frac{1}{\hbar} A^*{}_i\, a^*{}_j + \frac{1}{\hbar} B^*{}_i\, (b^*)_j,\\
&\qquad \frac{1}{\hbar} XX^*{}_i\, x^*{}_j + \frac{1}{\hbar} Y^*{}_i\, y^*{}_j + \frac{1}{\hbar} Z^*{}_i\, z^*{}_j,\ 1 + \mathrm{If}[\$k == 0,\ 0,\ (P_{\{i,j\},\$k-1})_{\$k}[3] -\\
&\qquad\quad (R_{1,2} \mathbin{//} ((P_{\{1,j\},0})_{\$k}\, (P_{\{i,2\},\$k-1})_{\$k}))[3]]\Big]\Big]
\end{aligned}
$$

Program

$$
\begin{aligned}
&\mathrm{Define}\Big[aS_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j},\\
&\quad \overline{aS}_i = \mathbb{E}_{\{i\}\to\{i\}}\Big[-a^*{}_i\, a_i - b^*{}_i\, b_i,\ -e^{-(b^*)_i-(a^*)_i} z^*{}_i\, z_i + e^{-(b^*)_i-(a^*)_i} y^*{}_i\, (x^*)_i\, z_i -\\
&\qquad e^{-2(b^*)_i+(a^*)_i} y^*{}_i\, y_i - e^{(b^*)_i-2(a^*)_i} x^*{}_i\, x_i,\ 1 + \mathrm{If}\big[\$k == 0,\ 0,\ (\overline{aS}_{\{i\},\$k-1})_{\$k}[3] -\\
&\qquad\quad \big((\overline{aS}_{\{i\},0})_{\$k} \sim B_i \sim aS_i \sim B_i \sim (\overline{aS}_{\{i\},\$k-1})_{\$k}\big)[3]\big]\Big]\Big]
\end{aligned}
$$

Program

$$
\begin{aligned}
&\mathrm{Define}\big[bS_i = R_{i,1} \sim B_1 \sim aS_1 \sim B_1 \sim P_{i,1},\\
&\quad \overline{bS}_i = R_{i,1} \sim B_1 \sim \overline{aS}_1 \sim B_1 \sim P_{i,1},\\
&\quad a\Delta_{i\to j,k} = (R_{1,j}\, R_{2,k}) \mathbin{//} bm_{1,2\to 3} \mathbin{//} P_{3,i},\\
&\quad b\Delta_{i\to j,k} = (R_{j,1}\, R_{k,2}) \mathbin{//} am_{1,2\to 3} \mathbin{//} P_{i,3}\big]
\end{aligned}
$$

Program

$$
\begin{aligned}
&\mathrm{Define}\Big[dm_{i,j\to k} = \Big(\mathbb{E}_{\{i,j\}\to\{i,j\}}\big[(A^*)_i\, A_i + (B^*)_i\, B_i + b^*{}_j\, b_j + a^*{}_j\, a_j,\\
&\qquad y^*{}_j\, y_j + x^*{}_j\, x_j + z^*{}_j\, z_j + (XX^*)_i\, XX_i + (Y^*)_i\, Y_i + Z^*{}_i\, Z_i,\ 1\big]\\
&\qquad \big(a\Delta_{i\to 1,2} \mathbin{//} a\Delta_{2\to 2,3} \mathbin{//} \overline{aS}_3\big)\ (b\Delta_{j\to -1,-2} \mathbin{//} b\Delta_{-2\to -2,-3})\Big) \mathbin{//} (P_{-1,3}\, P_{-3,1}\, am_{2,j\to k}\, bm_{i,-2\to k}),\\
&\quad dS_i = \mathbb{E}_{\{i\}\to\{1,2\}}\big[(A^*)_i\, A_1 + (B^*)_i\, B_1 + b^*{}_i\, b_2 + a^*{}_i\, a_2,\\
&\qquad y^*{}_i\, y_2 + x^*{}_i\, x_2 + z^*{}_i\, z_2 + (XX^*)_i\, XX_1 + (Y^*)_i\, Y_1 + Z^*{}_i\, Z_1,\ 1\big] \mathbin{//} \big(\overline{bS}_1\, aS_2\big) \mathbin{//} dm_{2,1\to i},\\
&\quad d\Delta_{i\to j,k} = (b\Delta_{i\to 3,1}\, a\Delta_{i\to 2,4}) \mathbin{//} (dm_{3,4\to k}\, dm_{1,2\to j}),\\
&\quad \overline{dS}_i = \mathbb{E}_{\{i\}\to\{1,2\}}\big[(A^*)_i\, A_1 + (B^*)_i\, B_1 + b^*{}_i\, b_2 + a^*{}_i\, a_2,\\
&\qquad y^*{}_i\, y_2 + x^*{}_i\, x_2 + z^*{}_i\, z_2 + (XX^*)_i\, XX_1 + (Y^*)_i\, Y_1 + Z^*{}_i\, Z_1,\ 1\big] \mathbin{//} \big(bS_1\, \overline{aS}_2\big) \mathbin{//} dm_{2,1\to i}\Big]
\end{aligned}
$$

Program

```
Define[C_i = 𝔼_{{}→{i}}[0, 0, 1/(𝔸_i^ℏ 𝔹_i^ℏ) - ℏ (a_i + b_i) ϵ/(𝔸_i^ℏ 𝔹_i^ℏ) + O[ϵ]^2]_$k,
  C̄_i = 𝔼_{{}→{i}}[0, 0, (𝔸_i^ℏ 𝔹_i^ℏ + (ℏ a_i 𝔸_i^ℏ 𝔹_i^ℏ + ℏ b_i 𝔸_i^ℏ 𝔹_i^ℏ) ϵ) + O[ϵ]^2]_$k,
  Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
  K̄ink_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Program

Note: s=2A-B+ϵa, t=2B-A+ϵb. This substitution is implemented in the following tensors.

Program

```
Define[AB2st_i =
   𝔼_{{i}→{i}}[b*_i b_i + a*_i a_i + (A*)_i (2/3 (s_i + t_i/2)) + (B*)_i (2/3 (t_i + s_i/2)), (XX*)_i XX_i + (Y*)_i Y_i +
      Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i, 1 - ϵ (A*)_i 1/3 (2 a_i + b_i) - ϵ (B*)_i 1/3 (2 b_i + a_i) + O[ϵ]^2]_$k,
  st2AB_i = 𝔼_{{i}→{i}}[b*_i b_i + a*_i a_i + (s*)_i (2 A_i - B_i) + (t*)_i (2 B_i - A_i),
     (XX*)_i XX_i + (Y*)_i Y_i + Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i, 1 + ϵ (s*)_i a_i + ϵ (t*)_i b_i + O[ϵ]^2]_$k]
```

The following definitions are used for a slightly faster implementation of the quantum-double that leaves out the s and the t from the zip. Since s and t are central, this is well defined and yields the same result. These tensors should be zipped using the STB and DB zip function. As such, we also check the axioms for these tensors.

```
Define[stR_{i,j} = (R_{i,j} ~ B_{{i,j}} ~ (AB2st_i AB2st_j)) // Simplify,
  \overline{stR}_{i,j} = (\overline{R}_{i,j} ~ B_{{i,j}} ~ (AB2st_i AB2st_j)) // Simplify,
  \overline{stC}_i = (\overline{C}_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stC_i = (C_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  \overline{stKink}_i = (\overline{Kink}_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stKink_i = (Kink_i ~ B_{{i}} ~ (AB2st_i)) // Simplify,
  stdm_{i,j→k} = ((st2AB_i st2AB_j) ~ B_{i,j} ~ dm_{i,j→k} ~ B_k ~ AB2st_k),
  stdΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)),
  stdS_i = (st2AB_i // dS_i // AB2st_i),
  stP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}),
  ddΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)) /.
    {S_{j|k} → S, T_{j|k} → T, t_{j|k} → t, s_{j|k} → s, (s^*)_i → 0, (t^*)_i → 0},
  ddS_i = (st2AB_i // dS_i // AB2st_i) /. {S_i → S, T_i → T, s_i → s, t_i → t, (s^*)_i → 0, (t^*)_i → 0},
  PP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}) /. {(s^*)_{i|j} → 0, (t^*)_{i|j} → 0},
  RR_{i,j} = (R_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s},
  \overline{RR}_{i,j} = (\overline{R}_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T},
  \overline{CC}_i = (\overline{C}_i ~ B_{{i}} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
  CC_i = (C_i ~ B_{{i}} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
  \overline{KKink}_i = (\overline{Kink}_i ~ B_{{i}} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} // Simplify,
  KKink_i = (Kink_i ~ B_{{i}} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} // Simplify,
  ddm_{i,j→k} = ((st2AB_i st2AB_j) // dm_{i,j→k} // AB2st_k) /.
    {S_k → S, T_k → T, t_k → t, s_k → s, (s^*)_{i|j} → 0, (t^*)_{i|j} → 0}];
```

```
Define[BB_{i, j→k} = ((dΔ_{i→1,r1} dΔ_{j→2,r2}) // dS_{r1} // \overline{dS}_{r2} // dm_{r1,r2→k} // dm_{k,1→k} // dm_{k,2→k})]
```

Testing

```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, \overline{R} → \overline{R}_{i,j}, P → P_{i,j}, aS → aS_i,
   \overline{aS} → \overline{aS}_i, bS → bS_i, \overline{bS} → \overline{bS}_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   C → C_i, \overline{C} → \overline{C}_i, Kink → Kink_i, \overline{Kink} → \overline{Kink}_i, AB2st → AB2st_i, st2AB → st2AB_i
  }] // Column
```

Check that on the generators this agrees with our conventions in the handout:

```
Timing@
 {{"[x,a]" → ((𝔼{}→{1,2}[0, 0, a2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, a1 x2] // am1,2→1)[3]),
    "[A,X]" → ((𝔼{}→{1,2}[0, 0, XX2 A1] // bm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, XX1 A2] // bm1,2→1)[3]),
    "[x,y]" → ((𝔼{}→{1,2}[0, 0, y2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 x2] // am1,2→1)[3]),
    "[Y,X]" → ((𝔼{}→{1,2}[0, 0, XX2 Y1] // bm1,2→1)[3] -
       (𝔼{}→{1,2}[0, 0, XX1 Y2] // bm1,2→1)[3])} /. z_1 → z,
  {"△[X]" → Last[𝔼{}→{1}[0, 0, XX1] ~B1~ bΔ1→1,2],
   "△[A]" → Last[𝔼{}→{1}[0, 0, A1] ~B1~ bΔ1→1,2],
   "△[a]" → Last[𝔼{}→{1}[0, 0, a1] ~B1~ aΔ1→1,2],
   "△[z]" → Last[𝔼{}→{1}[0, 0, z1] ~B1~ aΔ1→1,2],
   "△[x]" → Last[𝔼{}→{1}[0, 0, x1] ~B1~ aΔ1→1,2],
   "△[Z]" → Last[𝔼{}→{1}[0, 0, Z1] ~B1~ bΔ1→1,2]},
  {
    "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~B1~ aS1)[3]),
    "S(z)" → ((𝔼{}→{1}[0, 0, z1] ~B1~ aS1)[3]),
    "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~B1~ aS1)[3]),
    "S(A)" → ((𝔼{}→{1}[0, 0, A1] ~B1~ bS1)[3]),
    "S(X)" → ((𝔼{}→{1}[0, 0, XX1] ~B1~ bS1)[3]),
    "S(Z)" → ((𝔼{}→{1}[0, 0, Z1] ~B1~ bS1)[3])
   } /. z_1 → z}
```

$\{1.265625, \{\{[x,a] \to 2x + O[\epsilon]^2,\ [A,X] \to -XX\,\epsilon + O[\epsilon]^2,\ [x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,$
$[Y,X] \to (-2Z + XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2\},\ \{\triangle[X] \to \left(XX_2 + XX_1\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}\right) + O[\epsilon]^2,\ \triangle[A] \to (A_1 + A_2) + O[\epsilon]^2,$
$\triangle[a] \to (a_1 + a_2) + O[\epsilon]^2,\ \triangle[z] \to (z_1 + z_2) + (2\hbar\,x_1\,y_2 - \hbar\,a_1\,z_2 - \hbar\,b_1\,z_2)\,\epsilon + O[\epsilon]^2,$
$\triangle[x] \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2,\ \triangle[Z] \to \left(Z_2 + Z_1\,\mathbb{A}_2^{-\hbar}\,\mathbb{B}_2^{-\hbar} + \hbar\,XX_1\,Y_2\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}\right) + O[\epsilon]^2\},$
$\{S(a) \to -a + O[\epsilon]^2,\ S(z) \to -z + (2\,x\,y\,\hbar + 2\,z\,\hbar - a\,z\,\hbar - b\,z\,\hbar)\,\epsilon + O[\epsilon]^2,$
$S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2,\ S(A) \to -A + O[\epsilon]^2,\ S(X) \to -XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar} + O[\epsilon]^2,$
$S(Z) \to \left(-Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar} + XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\right) + \left(2\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar - XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar^2\right)\epsilon + O[\epsilon]^2\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
Timing@Block[{$k = 1},
  HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
 ]
```

{0.437500, {True, True}}

R and *P* are inverses:

```
Timing@Block[{$k = 1}, {HL[(Ri,j // Pi,k) ≡ 𝔼{k}→{j}[aj a*k + bj b*k, xj x*k + yj y*k + zj z*k, 1]]}]
```

{0.031250, {True}}

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
Timing[HL /@ {(aS̄1 // aS1) ≡ 𝔼{1}→{1}[a1 a*1 + b1 b*1, x1 x*1 + y1 y*1 + z1 z*1, 1],
   (bS̄1 // bS1) ≡ 𝔼{1}→{1}[A1 A*1 + B1 B*1, XX1 XX*1 + Y1 Y*1 + Z1 Z*1, 1]}]
```

{0.406250, {True, True}}

(co)-associativity on both sides

```
Timing[HL /@ {(aΔ1→1,2 // aΔ2→2,3) ≡ (aΔ1→1,3 // aΔ1→1,2), (bΔ1→1,2 // bΔ2→2,3) ≡ (bΔ1→1,3 // bΔ1→1,2),
   (am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}]
{1.078125, {True, True, True, True}}
```

Δ is an algebra morphism

```
Timing[HL /@ {(am1,2→1 // aΔ1→1,2) ≡ ((aΔ1→1,3 aΔ2→2,4) // (am3,4→2 am1,2→1)),
   (bm1,2→1 // bΔ1→1,2) ≡ ((bΔ1→1,3 bΔ2→2,4) // (bm3,4→2 bm1,2→1))}]
{1.312500, {True, True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@ {
   (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
   (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
{1.015625, {True, True, True, True}}
```

S is an algebra anti-(co)morphism

```
Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
   aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
{2.500000, {True, True, True, True}}
```

R-matrix and antipode

```
R1,2 ~ B1 ~ (bS1) ≡ R̄1,2
True
```

Pairing axioms

```
Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[b*3 b3 + a*3 a3, y*3 y3 + x*3 x3 + z*3 z3, 1]) ~ B1,3 ~ P1,3 ≡
    (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1]
      𝔼{2}→{2}[(A*)2 A2 + (B*)2 B2, (XX*)2 XX2 + (Y*)2 Y2 + Z*2 Z2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
   (bΔ1→1,2 𝔼{3}→{3}[b*3 b3 + a*3 a3, y*3 y3 + x*3 x3 + z*3 z3, 1]
      𝔼{4}→{4}[b*4 b4 + a*4 a4, y*4 y4 + x*4 x4 + z*4 z4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
    (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
{0.34375, {True, True}}
```

```
Timing[HL /@ {((bS1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) // P1,2) ≡
    ((𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] aS2) // P1,2),
   (b̄S1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) ~ B1,2 ~ P1,2 ≡
    (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] āS2) ~ B1,2 ~ P1,2}]
{0.28125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
{
   "[a,y]" →
```

```
  ((𝔼_{{}→{1,2}}[0, 0, y_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, y_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[b,x]"  → ((𝔼_{{}→{1,2}}[0, 0, x_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, x_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[b,y]" →
  ((𝔼_{{}→{1,2}}[0, 0, y_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, y_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[a,x]" → ((𝔼_{{}→{1,2}}[0, 0, x_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, x_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[a,z]" →
  ((𝔼_{{}→{1,2}}[0, 0, z_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, z_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[b,z]" → ((𝔼_{{}→{1,2}}[0, 0, z_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, z_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[x,z]" →
  ((𝔼_{{}→{1,2}}[0, 0, z_2 x_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, z_1 x_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[y,z]" → ((𝔼_{{}→{1,2}}[0, 0, z_2 y_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, z_1 y_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[x,y]" →
  ((𝔼_{{}→{1,2}}[0, 0, y_2 x_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, y_1 x_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[Y,y]" → ((𝔼_{{}→{1,2}}[0, 0, y_2 Y_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, y_1 Y_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify,
 "[Y,x]" → ((𝔼_{{}→{1,2}}[0, 0, x_2 Y_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, x_1 Y_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify,
 "[X,y]" → ((𝔼_{{}→{1,2}}[0, 0, y_2 XX_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, y_1 XX_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify,
 "[XX,x]" →
    ((𝔼_{{}→{1,2}}[0, 0, x_2 XX_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, x_1 XX_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) //
   Expand // Simplify, "[Z,z]" → ((𝔼_{{}→{1,2}}[0, 0, z_2 Z_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, z_1 Z_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify, "[Z,y]" →
    ((𝔼_{{}→{1,2}}[0, 0, y_2 Z_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, y_1 Z_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) //
   Expand // Simplify, "[Z,x]" → ((𝔼_{{}→{1,2}}[0, 0, x_2 Z_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, x_1 Z_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify, "[XX,z]" →
    ((𝔼_{{}→{1,2}}[0, 0, z_2 XX_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, z_1 XX_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) //
   Expand // Simplify, "[Y,z]" → ((𝔼_{{}→{1,2}}[0, 0, z_2 Y_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
      (𝔼_{{}→{1,2}}[0, 0, z_1 Y_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛) // Expand // Simplify, "[a,Z]" →
  ((𝔼_{{}→{1,2}}[0, 0, Z_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, Z_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[a,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, XX_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[a,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 a_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, Y_1 a_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[b,Z]" →
  ((𝔼_{{}→{1,2}}[0, 0, Z_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, Z_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[b,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, XX_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[b,Y]" →
  ((𝔼_{{}→{1,2}}[0, 0, Y_2 b_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, Y_1 b_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[A,Z]" → ((𝔼_{{}→{1,2}}[0, 0, Z_2 A_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, Z_1 A_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[A,XX]" →
  ((𝔼_{{}→{1,2}}[0, 0, XX_2 A_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, XX_1 A_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[A,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 A_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
    (𝔼_{{}→{1,2}}[0, 0, Y_1 A_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛), "[B,Z]" →
  ((𝔼_{{}→{1,2}}[0, 0, Z_2 B_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 - (𝔼_{{}→{1,2}}[0, 0, Z_1 B_2] ~ B_{1,2} ~ dm_{1,2→1})〚3〛),
 "[B,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 B_1] ~ B_{1,2} ~ dm_{1,2→1})〚3〛 -
```

```
    (𝔼{}→{1,2}[0, 0, XX1 B2] ~B1,2~dm1,2→1) ⟦3⟧), "[B,Y]" →
   ((𝔼{}→{1,2}[0, 0, Y2 B1] ~B1,2~dm1,2→1) ⟦3⟧ - (𝔼{}→{1,2}[0, 0, Y1 B2] ~B1,2~dm1,2→1) ⟦3⟧),
  "[XX,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 XX1] ~B1,2~dm1,2→1) ⟦3⟧ -
    (𝔼{}→{1,2}[0, 0, Y1 XX2] ~B1,2~dm1,2→1) ⟦3⟧), "[Z,Y]" →
   ((𝔼{}→{1,2}[0, 0, Y2 Z1] ~B1,2~dm1,2→1) ⟦3⟧ - (𝔼{}→{1,2}[0, 0, Y1 Z2] ~B1,2~dm1,2→1) ⟦3⟧),
  "[Z,XX]" →
   ((𝔼{}→{1,2}[0, 0, XX2 Z1] ~B1,2~dm1,2→1) ⟦3⟧ - (𝔼{}→{1,2}[0, 0, XX1 Z2] ~B1,2~dm1,2→1) ⟦3⟧),
  "[A,x]" → ((𝔼{}→{1,2}[0, 0, x2 A1] ~B1,2~dm1,2→1) ⟦3⟧ -
    (𝔼{}→{1,2}[0, 0, x1 A2] ~B1,2~dm1,2→1) ⟦3⟧), "[A,y]" →
   ((𝔼{}→{1,2}[0, 0, y2 A1] ~B1,2~dm1,2→1) ⟦3⟧ - (𝔼{}→{1,2}[0, 0, y1 A2] ~B1,2~dm1,2→1) ⟦3⟧)
 } /. {ν_1 → v} // Expand // Factor
{
 "Δ(a)" → ((𝔼{1,2}→{}[0, 0, a1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(x)" → ((𝔼{1,2}→{}[0, 0, x1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(b)" → ((𝔼{1,2}→{}[0, 0, b1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(y)" → ((𝔼{1,2}→{}[0, 0, y1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(z)" → ((𝔼{1,2}→{}[0, 0, z1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(XX)" → ((𝔼{1,2}→{}[0, 0, XX1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(Y)" → ((𝔼{1,2}→{}[0, 0, Y1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(Z)" → ((𝔼{1,2}→{}[0, 0, Z1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(A)" → ((𝔼{1,2}→{}[0, 0, A1] ~B1~dΔ1→1,2) ⟦3⟧),
 "Δ(B)" → ((𝔼{1,2}→{}[0, 0, B1] ~B1~dΔ1→1,2) ⟦3⟧)} // Simplify
{
  "S(a)" → ((𝔼{1}→{1}[0, 0, a1] ~B1~dS1) ⟦3⟧),
  "S(x)" → ((𝔼{1}→{1}[0, 0, x1] ~B1~dS1) ⟦3⟧),
  "S(b)" → ((𝔼{1}→{1}[0, 0, b1] ~B1~dS1) ⟦3⟧),
  "S(y)" → ((𝔼{1}→{1}[0, 0, y1] ~B1~dS1) ⟦3⟧), "S(z)" → ((𝔼{1}→{1}[0, 0, z1] ~B1~dS1) ⟦3⟧),
  "S(XX)" → ((𝔼{1}→{1}[0, 0, XX1] ~B1~dS1) ⟦3⟧),
  "S(Y)" → ((𝔼{1}→{1}[0, 0, Y1] ~B1~dS1) ⟦3⟧),
  "S(Z)" → ((𝔼{1}→{1}[0, 0, Z1] ~B1~dS1) ⟦3⟧), "S(A)" → ((𝔼{1}→{1}[0, 0, A1] ~B1~dS1) ⟦3⟧),
  "S(B)" → ((𝔼{1}→{1}[0, 0, B1] ~B1~dS1) ⟦3⟧)
 } /. {ν_1 → v} // Simplify
```

$\{[a,y] \to y + O[\epsilon]^2,\ [b,x] \to x + O[\epsilon]^2,\ [b,y] \to -2y + O[\epsilon]^2,$
$[a,x] \to -2x + O[\epsilon]^2,\ [a,z] \to -z + O[\epsilon]^2,\ [b,z] \to -z + O[\epsilon]^2,$
$[x,z] \to x\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [y,z] \to -y\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,$
$[Y,y] \to \frac{-1 + \mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar}}{\hbar} + \left(-b\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar} + 2\,y\,Y\,\hbar\right)\epsilon + O[\epsilon]^2,\ [Y,x] \to -x\,Y\,\hbar\,\epsilon + O[\epsilon]^2,$
$[X,y] \to -XX\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ [XX,x] \to \frac{-1 + \mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar}}{\hbar} + \left(-a\,\mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar} + 2\,x\,XX\,\hbar\right)\epsilon + O[\epsilon]^2,$
$[Z,z] \to \frac{-1 + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}}{\hbar} + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}\left(-a - b + 2\,z\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,\ [Z,y] \to -XX + y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,$
$[Z,x] \to Y\,\mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar} + \mathbb{A}^{-2\hbar}\left(x\,Z\,\mathbb{A}^{2\hbar} - (-1 + a)\,Y\,\mathbb{B}^{\hbar}\right)\hbar\,\epsilon + O[\epsilon]^2,\ [XX,z] \to (-2\,y + XX\,z\,\hbar)\,\epsilon + O[\epsilon]^2,$
$[Y,z] \to \left(2\,x\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar} + Y\,z\,\hbar\right)\epsilon + O[\epsilon]^2,\ [a,Z] \to Z + O[\epsilon]^2,\ [a,XX] \to 2\,XX + O[\epsilon]^2,$
$[a,Y] \to -Y + O[\epsilon]^2,\ [b,Z] \to Z + O[\epsilon]^2,\ [b,XX] \to -XX + O[\epsilon]^2,\ [b,Y] \to 2\,Y + O[\epsilon]^2,$
$[A,Z] \to -Z\,\epsilon + O[\epsilon]^2,\ [A,XX] \to -XX\,\epsilon + O[\epsilon]^2,\ [A,Y] \to O[\epsilon]^2,\ [B,Z] \to -Z\,\epsilon + O[\epsilon]^2,$
$[B,XX] \to O[\epsilon]^2,\ [B,Y] \to -Y\,\epsilon + O[\epsilon]^2,\ [XX,Y] \to (2\,Z - XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,$
$[Z,Y] \to Y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [Z,XX] \to -XX\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [A,x] \to x\,\epsilon + O[\epsilon]^2,\ [A,y] \to O[\epsilon]^2\}$

$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^2,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2,$
$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^2,\ \Delta(y) \to (y_1 + y_2) - \hbar\,b_1\,y_2\,\epsilon + O[\epsilon]^2,$
$\Delta(z) \to (z_1 + z_2) + \hbar\,(2\,x_1\,y_2 - (a_1 + b_1)\,z_2)\,\epsilon + O[\epsilon]^2,\ \Delta(XX) \to \left(XX_1 + XX_2\,\mathbb{A}_1^{-2\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$
$\Delta(Y) \to \left(Y_1 + Y_2\,\mathbb{A}_1^{\hbar}\,\mathbb{B}_1^{-2\hbar}\right) + O[\epsilon]^2,\ \Delta(Z) \to \left(Z_1 + Z_2\,\mathbb{A}_1^{-\hbar}\,\mathbb{B}_1^{-\hbar} + \hbar\,XX_2\,Y_1\,\mathbb{A}_1^{-2\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$
$\Delta(A) \to (A_1 + A_2) + O[\epsilon]^2,\ \Delta(B) \to (B_1 + B_2) + O[\epsilon]^2\}$

$\{S(a) \to -a + O[\epsilon]^2,\ S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2,\ S(b) \to -b + O[\epsilon]^2,$
$S(y) \to -y - b\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ S(z) \to -z + (2\,x\,y - (-2 + a + b)\,z)\,\hbar\,\epsilon + O[\epsilon]^2,$
$S(XX) \to -XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar} - 2\left(XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,\ S(Y) \to -Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\hbar} - 2\left(Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$
$S(Z) \to \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,(-Z + XX\,Y\,\hbar) + \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\,(-2\,Z + 3\,XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,$
$S(A) \to -A + O[\epsilon]^2,\ S(B) \to -B + O[\epsilon]^2\}$

(co)-associativity

```
Timing[HL /@ {(dΔ_{1→1,2} // dΔ_{2→2,3}) ≡ (dΔ_{1→1,3} // dΔ_{1→1,2}), (dm_{1,2→1} // dm_{1,3→1}) ≡ (dm_{2,3→2} // dm_{1,2→1})}]
```

{14.218750, {True, True}}

```
Timing[HL /@ {(stdΔ_{1→1,2} // stdΔ_{2→2,3}) ≡ (stdΔ_{1→1,3} // stdΔ_{1→1,2}),
   (stdm_{1,2→1} // stdm_{1,3→1}) ≡ (stdm_{2,3→2} // stdm_{1,2→1})}]
```

{7.531250, {True, True}}

```
Timing[HL /@ {(ddm_{1,2→1} ~ STB_1 ~ ddm_{1,3→1}) ≡ (ddm_{2,3→2} ~ STB_2 ~ ddm_{1,2→1})}]
```

{2.125000, {True}}

Δ is an algebra morphism

```
Timing@HL[dm_{1,2→1} ~ B_1 ~ dΔ_{1→1,2} ≡ (dΔ_{1→1,3} dΔ_{2→2,4}) ~ B_{1,2,3,4} ~ (dm_{3,4→2} dm_{1,2→1})]
```

{7.296875, True}

S_2 inverts R, but not S_1:

```
Timing@{HL[R_{1,2} ~ B_2 ~ dS_2 ≡ R̄_{1,2}]}
```

{0.796875, {True}}

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@
  {(dΔ_{1→1,2} ~ B_1 ~ dS_1) ~ B_{1,2} ~ dm_{1,2→1}, (dΔ_{1→1,2} ~ B_2 ~ dS_2) // dm_{1,2→1}}]
```

{4.703125, {True, True}}

S is a (co)-algebra anti-morphism

```
Timing[HL /@
  Expand /@ {dm_{1,2→1} ~ B_1 ~ dS_1 ≡ (dS_1 dS_2) ~ B_{1,2} ~ dm_{2,1→1}, dS_1 ~ B_1 ~ dΔ_{1→1,2} ≡ dΔ_{1→2,1} ~ B_{1,2} ~ (dS_1 dS_2)}]
```

{22.718750, {True, True}}

Quasi-triangular axiom 1:

```
Timing@HL[R_{1,2} ~ B_1 ~ dΔ_{1→1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ dm_{2,4→2}]
```

{0.375000, True}

Quasi-triangular axiom 2:

```
Timing@HL[((dΔ_{1→1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})) ≡ ((dΔ_{1→2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{3,1→1} dm_{4,2→2}))]
```

{2.359375, True}

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)\sim B_{1,2}\sim \mathrm{dm}_{1,2\to 1} \equiv \mathbb{E}[0, 0, 1]$:

```
Timing@HL[((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i} ≡
   𝔼_{{}→{i}}[0, 0, 1]]
```

{2.453125, True}

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy. Taking the square root of z and multiplying it with S(u) yields the ribbon element v.

```
Timing@Block[{$k = 1},
  (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}]
```

$\{8.062500,\ \mathbb{E}_{\{\}\to\{i\}}[0,\ 0,\ \mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar} + (2\,\hbar\,a_i\,\mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar} + 2\,\hbar\,b_i\,\mathbb{A}_i^{2\hbar}\,\mathbb{B}_i^{2\hbar})\,\epsilon + O[\epsilon]^2]\}$

Turaev moves are checked here.

T-4:

```
Timing@Block[{$k = 1},
  HL /@ {(C̄_1 C̄_2 R_{3,4} C_5 C_6) ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ~ B_{2,4} ~ dm_{2,4→2} ~ B_{2,6} ~ dm_{2,6→2} ≡ R_{1,2}}]
```

{3.796875, {True}}

T-5, T-6

$$\text{Timing@Block}\Big[\{\$k=1\},\ \text{HL}\ /@\ \Big\{\big(C_i\,\overline{C}_j\big)\sim B_{i,j}\sim dm_{i,j\to i}\equiv \mathbb{E}_{\{\}\to\{i\}}[0,0,1],\ \big(C_i\,\overline{C}_j\big)\sim B_{i,j}\sim dm_{j,i\to i}\equiv \mathbb{E}_{\{\}\to\{i\}}[0,0,1],\ \big(\overline{C}_i\,\overline{C}_j\big)\sim B_{i,j}\sim dm_{i,j\to i}\equiv (((R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i})\sim B_i\sim dS_i)\ (R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j}))\sim B_{i,j}\sim dm_{i,j\to i}\Big\}\Big]$$

{16.296875, {True, True, True}}

Reidemeister 2 or T-3:

$$\text{Timing}\Big[\text{HL}[\#\equiv \mathbb{E}_{\{\}\to\{1,2\}}[0,0,1]]\ \&\ /@\ \Big\{\big(\overline{R}_{1,2}\,R_{3,4}\big)\sim B_{1,2,3,4}\sim(dm_{1,3\to1}\,dm_{2,4\to2}),\ \big(R_{1,2}\,\overline{R}_{3,4}\big)\sim B_{1,2,3,4}\sim(dm_{1,3\to1}\,dm_{2,4\to2})\Big\}\Big]$$

{2.500000, {True, True}}

$$\text{Timing}\Big[\text{HL}[\#\equiv \mathbb{E}_{\{\}\to\{1,2\}}[0,0,1]]\ \&\ /@\ \Big\{\Big(\Big(\big(\overline{RR}_{1,2}\,RR_{3,4}\big)\Big)\sim B_{1,2,3,4}\sim((ddm_{1,3\to1}\,ddm_{2,4\to2}))\Big),\ \Big(\big(RR_{1,2}\,\overline{RR}_{3,4}\big)\sim STB_{1,2,3,4}\sim(ddm_{1,3\to1}\,ddm_{2,4\to2})\Big)\Big\}\Big]$$

{2.718750, {True, True}}

Cyclic Reidemeister 2 or T-2:

$$\text{Timing@HL}\Big[\big(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3\big)\sim B_{2,4}\sim dm_{2,4\to2}\sim B_{1,3}\sim dm_{1,3\to1}\sim B_{1,5}\sim dm_{1,5\to1}\equiv \overline{C}_1\,\mathbb{E}_{\{\}\to\{2\}}[0,0,1]\Big]$$

{7.765625, True}

Reidemeister 3 or T-1:

$$\text{Timing@HL}[((R_{1,2}\,R_{4,3}\,R_{5,6})\sim B_{1,4}\sim dm_{1,4\to1}\sim B_{2,5}\sim dm_{2,5\to2}\sim B_{3,6}\sim dm_{3,6\to3})\equiv ((R_{1,6}\,R_{2,3}\,R_{4,5})\sim B_{1,4}\sim dm_{1,4\to1}\sim B_{2,5}\sim dm_{2,5\to2}\sim B_{3,6}\sim dm_{3,6\to3})]$$

{5.343750, True}

$$\text{Timing@HL}[((RR_{1,2}\,RR_{4,3}\,RR_{5,6})\sim STB_{1,4}\sim ddm_{1,4\to1}\sim STB_{2,5}\sim ddm_{2,5\to2}\sim STB_{3,6}\sim ddm_{3,6\to3})\equiv ((RR_{1,6}\,RR_{2,3}\,RR_{4,5})\sim STB_{1,4}\sim ddm_{1,4\to1}\sim STB_{2,5}\sim ddm_{2,5\to2}\sim STB_{3,6}\sim ddm_{3,6\to3})]$$

{1.656250, True}

Relations between the four kinks or T-7

$$\text{Timing}\Big[\text{HL}\ /@\ \Big\{\text{Kink}_i\equiv(R_{3,1}\,C_2)\sim B_{1,2}\sim dm_{1,2\to1}\sim B_{1,3}\sim dm_{1,3\to i},\ \overline{\text{Kink}}_j\equiv\big(\overline{R}_{3,1}\,\overline{C}_2\big)\sim B_{1,2}\sim dm_{1,2\to1}\sim B_{1,3}\sim dm_{1,3\to j},\ \big(\text{Kink}_i\,\overline{\text{Kink}}_j\big)\sim B_{i,j}\sim dm_{i,j\to1}\equiv \mathbb{E}_{\{\}\to\{1\}}[0,0,1],\ \big(\text{Kink}_i\,\overline{\text{Kink}}_j\big)\sim B_{i,j}\sim dm_{j,i\to1}\equiv \mathbb{E}_{\{\}\to\{1\}}[0,0,1]\Big\}\Big]$$

{9.187500, {True, True, True, True}}

The Trefoil

$$\text{Timing@Block}\Big[\{\$k=1\},\ ZZ=RR_{1,5}\,RR_{6,2}\,RR_{3,7}\,\overline{CC}_4\,\overline{\text{KKink}}_8\,\overline{\text{KKink}}_9\,\overline{\text{KKink}}_{10};\ \text{Do}[ZZ=(ZZ\ /.\ \{\epsilon\to0\})\sim B_{1,r}\sim(ddm_{1,r\to1}\ /.\ \{\epsilon\to0\}),\ \{r,2,10\}];\ \{\text{Simplify}\ /@\ (ZZ\ /.\ \{\epsilon\to0\})\}\Big]$$

$$\Big\{1.250000,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,0,\frac{S^2\,T^2}{(1-S+S^2)\,(1-T+T^2)\,(1-S\,T+S^2\,T^2)}\Big]\Big\}\Big\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR_{1,5} RR_{6,2} RR_{3,7} \overline{CC}_4 \overline{KKink}_8 \overline{KKink}_9 \overline{KKink}_{10});
  Do[Print["Doing ", r]; ZZ = (ZZ) ~B_{1,r}~ (ddm_{1,r→1}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\{28.890625,$$

$$\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\, 0,\, \frac{S^2 T^2}{(1-S+S^2)\,(1-T+T^2)\,(1-S T+S^2 T^2)} + \Big(2 S^2 T^2 \big(S - 2 S^2 + 3 S^3 - 2 S^4 + T - 2 S T +$$

$$S^2 T + 2 S^3 T - 5 S^4 T + 6 S^5 T - 2 T^2 + S T^2 - S^3 T^2 - 4 S^4 T^2 + 5 S^5 T^2 - 11 S^6 T^2 + 3 T^3 + 2 S T^3 -$$

$$S^2 T^3 + 4 S^3 T^3 + 6 S^4 T^3 - S^5 T^3 + 7 S^6 T^3 + 10 S^7 T^3 - 2 T^4 - 5 S T^4 - 4 S^2 T^4 + 6 S^3 T^4 - 24 S^4 T^4 +$$

$$10 S^5 T^4 - 12 S^6 T^4 - 11 S^7 T^4 - 6 S^8 T^4 + 6 S T^5 + 5 S^2 T^5 - S^3 T^5 + 10 S^4 T^5 + 12 S^5 T^5 - S^6 T^5 +$$

$$10 S^7 T^5 + 13 S^8 T^5 - 11 S^2 T^6 + 7 S^3 T^6 - 12 S^4 T^6 - S^5 T^6 - 16 S^6 T^6 + 11 S^7 T^6 - 20 S^8 T^6 +$$

$$10 S^3 T^7 - 11 S^4 T^7 + 10 S^5 T^7 + 11 S^6 T^7 - 14 S^7 T^7 + 15 S^8 T^7 - 6 S^4 T^8 + 13 S^5 T^8 - 20 S^6 T^8 +$$

$$15 S^7 T^8 - 8 S^8 T^8 + (1 - S + S^2)\,(1 - T + T^2)^2\,\big(-2 + S + 3 S T + 2 S^6 T^4 + S^4 T^2\,(3 + T) -$$

$$S^5 T^3\,(3 + T) - S^2 T\,(1 + 3 T) + S^3 T\,(-1 + T^2)\big)\,a_1 + (1 - S + S^2)^2\,(1 - T + T^2)$$

$$\big(-2 + T + 3 S T - S\,(1 + 3 S)\,T^2 + S\,(-1 + S^2)\,T^3 + S^2\,(3 + S)\,T^4 - S^3\,(3 + S)\,T^5 + 2 S^4 T^6\big)\,b_1 +$$

$$2 x_1 XX_1 + 2 S^3 x_1 XX_1 - 5 T x_1 XX_1 - 2 S T x_1 XX_1 - 3 S^2 T x_1 XX_1 - 2 S^3 T x_1 XX_1 -$$

$$5 S^4 T x_1 XX_1 + 9 T^2 x_1 XX_1 + 5 S T^2 x_1 XX_1 + 9 S^2 T^2 x_1 XX_1 + 9 S^3 T^2 x_1 XX_1 + 5 S^4 T^2 x_1 XX_1 +$$

$$9 S^5 T^2 x_1 XX_1 - 7 T^3 x_1 XX_1 - 12 S T^3 x_1 XX_1 - 12 S^2 T^3 x_1 XX_1 - 14 S^3 T^3 x_1 XX_1 - 12 S^4 T^3 x_1 XX_1 -$$

$$12 S^5 T^3 x_1 XX_1 - 7 S^6 T^3 x_1 XX_1 + 4 T^4 x_1 XX_1 + 10 S T^4 x_1 XX_1 + 18 S^2 T^4 x_1 XX_1 + 14 S^3 T^4 x_1 XX_1 +$$

$$14 S^4 T^4 x_1 XX_1 + 18 S^5 T^4 x_1 XX_1 + 10 S^6 T^4 x_1 XX_1 + 4 S^7 T^4 x_1 XX_1 - 7 S T^5 x_1 XX_1 -$$

$$12 S^2 T^5 x_1 XX_1 - 12 S^3 T^5 x_1 XX_1 - 14 S^4 T^5 x_1 XX_1 - 12 S^5 T^5 x_1 XX_1 - 12 S^6 T^5 x_1 XX_1 -$$

$$7 S^7 T^5 x_1 XX_1 + 9 S^2 T^6 x_1 XX_1 + 5 S^3 T^6 x_1 XX_1 + 9 S^4 T^6 x_1 XX_1 + 9 S^5 T^6 x_1 XX_1 + 5 S^6 T^6 x_1 XX_1 +$$

$$9 S^7 T^6 x_1 XX_1 - 5 S^3 T^7 x_1 XX_1 - 2 S^4 T^7 x_1 XX_1 - 3 S^5 T^7 x_1 XX_1 - 2 S^6 T^7 x_1 XX_1 - 5 S^7 T^7 x_1 XX_1 +$$

$$2 S^4 T^8 x_1 XX_1 + 2 S^7 T^8 x_1 XX_1 + 2 y_1 Y_1 - 5 S y_1 Y_1 + 9 S^2 y_1 Y_1 - 7 S^3 y_1 Y_1 + 4 S^4 y_1 Y_1 -$$

$$2 S T y_1 Y_1 + 5 S^2 T y_1 Y_1 - 12 S^3 T y_1 Y_1 + 10 S^4 T y_1 Y_1 - 7 S^5 T y_1 Y_1 - 3 S T^2 y_1 Y_1 +$$

$$9 S^2 T^2 y_1 Y_1 - 12 S^3 T^2 y_1 Y_1 + 18 S^4 T^2 y_1 Y_1 - 12 S^5 T^2 y_1 Y_1 + 9 S^6 T^2 y_1 Y_1 + 2 T^3 y_1 Y_1 -$$

$$2 S T^3 y_1 Y_1 + 9 S^2 T^3 y_1 Y_1 - 14 S^3 T^3 y_1 Y_1 + 14 S^4 T^3 y_1 Y_1 - 12 S^5 T^3 y_1 Y_1 + 5 S^6 T^3 y_1 Y_1 -$$

$$5 S^7 T^3 y_1 Y_1 - 5 S T^4 y_1 Y_1 + 5 S^2 T^4 y_1 Y_1 - 12 S^3 T^4 y_1 Y_1 + 14 S^4 T^4 y_1 Y_1 - 14 S^5 T^4 y_1 Y_1 +$$

$$9 S^6 T^4 y_1 Y_1 - 2 S^7 T^4 y_1 Y_1 + 2 S^8 T^4 y_1 Y_1 + 9 S^2 T^5 y_1 Y_1 - 12 S^3 T^5 y_1 Y_1 + 18 S^4 T^5 y_1 Y_1 -$$

$$12 S^5 T^5 y_1 Y_1 + 9 S^6 T^5 y_1 Y_1 - 3 S^7 T^5 y_1 Y_1 - 7 S^3 T^6 y_1 Y_1 + 10 S^4 T^6 y_1 Y_1 - 12 S^5 T^6 y_1 Y_1 +$$

$$5 S^6 T^6 y_1 Y_1 - 2 S^7 T^6 y_1 Y_1 + 4 S^4 T^7 y_1 Y_1 - 7 S^5 T^7 y_1 Y_1 + 9 S^6 T^7 y_1 Y_1 - 5 S^7 T^7 y_1 Y_1 +$$

$$2 S^8 T^7 y_1 Y_1 - 2 XX_1 Y_1 z_1 + 7 S XX_1 Y_1 z_1 - 9 S^2 XX_1 Y_1 z_1 + 7 S^3 XX_1 Y_1 z_1 - 2 S^4 XX_1 Y_1 z_1 -$$

$$5 S T XX_1 Y_1 z_1 - 5 S^4 T XX_1 Y_1 z_1 + 12 S T^2 XX_1 Y_1 z_1 - 9 S^2 T^2 XX_1 Y_1 z_1 + 21 S^3 T^2 XX_1 Y_1 z_1 -$$

$$9 S^4 T^2 XX_1 Y_1 z_1 + 12 S^5 T^2 XX_1 Y_1 z_1 - 2 T^3 XX_1 Y_1 z_1 - 5 S T^3 XX_1 Y_1 z_1 - 9 S^2 T^3 XX_1 Y_1 z_1 -$$

$$7 S^3 T^3 XX_1 Y_1 z_1 - 7 S^4 T^3 XX_1 Y_1 z_1 - 9 S^5 T^3 XX_1 Y_1 z_1 - 5 S^6 T^3 XX_1 Y_1 z_1 - 2 S^7 T^3 XX_1 Y_1 z_1 +$$

$$7 S T^4 XX_1 Y_1 z_1 + 21 S^3 T^4 XX_1 Y_1 z_1 - 7 S^4 T^4 XX_1 Y_1 z_1 + 21 S^5 T^4 XX_1 Y_1 z_1 + 7 S^7 T^4 XX_1 Y_1 z_1 -$$

$$9 S^2 T^5 XX_1 Y_1 z_1 - 9 S^4 T^5 XX_1 Y_1 z_1 - 9 S^5 T^5 XX_1 Y_1 z_1 - 9 S^7 T^5 XX_1 Y_1 z_1 + 7 S^3 T^6 XX_1 Y_1 z_1 -$$

$$5 S^4 T^6 XX_1 Y_1 z_1 + 12 S^5 T^6 XX_1 Y_1 z_1 - 5 S^6 T^6 XX_1 Y_1 z_1 + 7 S^7 T^6 XX_1 Y_1 z_1 - 2 S^4 T^7 XX_1 Y_1 z_1 -$$

$$2 S^7 T^7 XX_1 Y_1 z_1 + 4 z_1 Z_1 - 7 S z_1 Z_1 + 9 S^2 z_1 Z_1 - 5 S^3 z_1 Z_1 + 2 S^4 z_1 Z_1 - 7 T z_1 Z_1 +$$

$$10 S T z_1 Z_1 - 12 S^2 T z_1 Z_1 + 5 S^3 T z_1 Z_1 - 2 S^4 T z_1 Z_1 + 9 T^2 z_1 Z_1 - 12 S T^2 z_1 Z_1 +$$

$$18 S^2 T^2 z_1 Z_1 - 12 S^3 T^2 z_1 Z_1 + 9 S^4 T^2 z_1 Z_1 - 3 S^5 T^2 z_1 Z_1 - 5 T^3 z_1 Z_1 + 5 S T^3 z_1 Z_1 -$$

$$12 S^2 T^3 z_1 Z_1 + 14 S^3 T^3 z_1 Z_1 - 14 S^4 T^3 z_1 Z_1 + 9 S^5 T^3 z_1 Z_1 - 2 S^6 T^3 z_1 Z_1 + 2 S^7 T^3 z_1 Z_1 +$$

$$2 T^4 z_1 Z_1 - 2 S T^4 z_1 Z_1 + 9 S^2 T^4 z_1 Z_1 - 14 S^3 T^4 z_1 Z_1 + 14 S^4 T^4 z_1 Z_1 - 12 S^5 T^4 z_1 Z_1 +$$

$$5 S^6 T^4 z_1 Z_1 - 5 S^7 T^4 z_1 Z_1 - 3 S^2 T^5 z_1 Z_1 + 9 S^3 T^5 z_1 Z_1 - 12 S^4 T^5 z_1 Z_1 + 18 S^5 T^5 z_1 Z_1 -$$

$$12 S^6 T^5 z_1 Z_1 + 9 S^7 T^5 z_1 Z_1 - 2 S^3 T^6 z_1 Z_1 + 5 S^4 T^6 z_1 Z_1 - 12 S^5 T^6 z_1 Z_1 + 10 S^6 T^6 z_1 Z_1 -$$

$$7 S^7 T^6 z_1 Z_1 + 2 S^3 T^7 z_1 Z_1 - 5 S^4 T^7 z_1 Z_1 + 9 S^5 T^7 z_1 Z_1 - 7 S^6 T^7 z_1 Z_1 + 4 S^7 T^7 z_1 Z_1\big)\,\epsilon\Big)\,/$$

$$\Big(\big(1 - S + S^2\big)^3\,\big(1 - T + T^2\big)^3\,\big(1 - S T + S^2 T^2\big)^3\Big) + O[\epsilon]^2\Big]\}\}$$

The Figure Eight knot

```
Timing@Block[{$k = 1},
  ZZ = (RR̄8,1 RR̄2,6 RR5,9 RR10,3 CC̄7 CC4);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB1,r~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{2.125000,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\,\left(1-3\,T+T^2\right)\,\left(1-3\,S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR̄8,1 RR̄2,6 RR5,9 RR10,3 CC̄7 CC4);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{532.593750,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\,\left(1-3\,T+T^2\right)\,\left(1-3\,S\,T+S^2\,T^2\right)}-$$
$$\Big(2\,\Big(S^2\,T^2\,\Big(4-9\,S+2\,S^2-9\,T+12\,S\,T+6\,S^2\,T+2\,T^2+6\,S\,T^2-6\,S^3\,T^2-2\,S^4\,T^2-6\,S^2\,T^3-12\,S^3\,T^3+$$
$$9\,S^4\,T^3-2\,S^2\,T^4+9\,S^3\,T^4-4\,S^4\,T^4+\left(1-3\,T+T^2\right)\,\left(-2+2\,S^4\,T^2+3\,S\,(1+T)-3\,S^3\,T\,(1+T)\right)$$
$$a_1+\left(1-3\,S+S^2\right)\,\left(-2+3\,(1+S)\,T-3\,S\,(1+S)\,T^3+2\,S^2\,T^4\right)\,b_1+2\,x_1\,XX_1+$$
$$2\,S\,x_1\,XX_1-9\,T\,x_1\,XX_1-3\,S\,T\,x_1\,XX_1-9\,S^2\,T\,x_1\,XX_1+4\,T^2\,x_1\,XX_1+16\,S\,T^2\,x_1\,XX_1+$$
$$16\,S^2\,T^2\,x_1\,XX_1+4\,S^3\,T^2\,x_1\,XX_1-9\,S\,T^3\,x_1\,XX_1-3\,S^2\,T^3\,x_1\,XX_1-9\,S^3\,T^3\,x_1\,XX_1+$$
$$2\,S^2\,T^4\,x_1\,XX_1+2\,S^3\,T^4\,x_1\,XX_1+2\,y_1\,Y_1-9\,S\,y_1\,Y_1+4\,S^2\,y_1\,Y_1+2\,T\,y_1\,Y_1-3\,S\,T\,y_1\,Y_1+$$
$$16\,S^2\,T\,y_1\,Y_1-9\,S^3\,T\,y_1\,Y_1-9\,S\,T^2\,y_1\,Y_1+16\,S^2\,T^2\,y_1\,Y_1-3\,S^3\,T^2\,y_1\,Y_1+2\,S^4\,T^2\,y_1\,Y_1+$$
$$4\,S^2\,T^3\,y_1\,Y_1-9\,S^3\,T^3\,y_1\,Y_1+2\,S^4\,T^3\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+11\,S\,XX_1\,Y_1\,z_1-2\,S^2\,XX_1\,Y_1\,z_1-$$
$$2\,T\,XX_1\,Y_1\,z_1-8\,S\,T\,XX_1\,Y_1\,z_1-8\,S^2\,T\,XX_1\,Y_1\,z_1-2\,S^3\,T\,XX_1\,Y_1\,z_1+11\,S\,T^2\,XX_1\,Y_1\,z_1-$$
$$8\,S^2\,T^2\,XX_1\,Y_1\,z_1+11\,S^3\,T^2\,XX_1\,Y_1\,z_1-2\,S^2\,T^3\,XX_1\,Y_1\,z_1-2\,S^3\,T^3\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-$$
$$9\,S\,z_1\,Z_1+2\,S^2\,z_1\,Z_1-9\,T\,z_1\,Z_1+16\,S\,T\,z_1\,Z_1-3\,S^2\,T\,z_1\,Z_1+2\,S^3\,T\,z_1\,Z_1+2\,T^2\,z_1\,Z_1-$$
$$3\,S\,T^2\,z_1\,Z_1+16\,S^2\,T^2\,z_1\,Z_1-9\,S^3\,T^2\,z_1\,Z_1+2\,S\,T^3\,z_1\,Z_1-9\,S^2\,T^3\,z_1\,Z_1+4\,S^3\,T^3\,z_1\,Z_1\Big)\Big)\,\epsilon\Big)\Big/$$
$$\left(\left(1-3\,S+S^2\right)^2\,\left(1-3\,T+T^2\right)^2\,\left(1-3\,S\,T+S^2\,T^2\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The 6-3 knot

```
Timing@Block[{$k = 1},
  ZZ = (RR̄6,12 RR̄10,14 RR̄13,7 RR1,5 RR8,2 RR3,9 CC̄4 CC11);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 14}];
  {Simplify@ZZ}]
```

$$\Big\{337.828125,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,$$
$$\left(S^4\,T^4\right)\Big/\left(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)\,\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\,\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)\right)+$$
$$\Big(2\,S^4\,T^4\,\Big(8-21\,S+30\,S^2-15\,S^3+4\,S^4-21\,T+36\,S\,T-30\,S^2\,T-24\,S^3\,T+18\,S^4\,T-6\,S^5\,T+$$
$$30\,T^2-30\,S\,T^2+12\,S^2\,T^2+45\,S^3\,T^2+6\,S^4\,T^2-6\,S^5\,T^2-15\,T^3-24\,S\,T^3+45\,S^2\,T^3-$$
$$48\,S^3\,T^3-24\,S^4\,T^3+6\,S^6\,T^3+6\,S^7\,T^3+4\,T^4+18\,S\,T^4+6\,S^2\,T^4-24\,S^3\,T^4+24\,S^5\,T^4-$$
$$6\,S^6\,T^4-18\,S^7\,T^4-4\,S^8\,T^4-6\,S\,T^5-6\,S^2\,T^5+24\,S^4\,T^5+48\,S^5\,T^5-45\,S^6\,T^5+24\,S^7\,T^5+$$
$$15\,S^8\,T^5+6\,S^3\,T^6-6\,S^4\,T^6-45\,S^5\,T^6-12\,S^6\,T^6+30\,S^7\,T^6-30\,S^8\,T^6+6\,S^3\,T^7-18\,S^4\,T^7+$$
$$24\,S^5\,T^7+30\,S^6\,T^7-36\,S^7\,T^7+21\,S^8\,T^7-4\,S^4\,T^8+15\,S^5\,T^8-30\,S^6\,T^8+21\,S^7\,T^8-8\,S^8\,T^8+$$
$$\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\,\Big(-4+4\,S^8\,T^4+9\,S\,(1+T)-9\,S^7\,T^3\,(1+T)-2\,S^2\,\left(5+9\,T+5\,T^2\right)+$$
$$2\,S^6\,T^2\,\left(5+9\,T+5\,T^2\right)+3\,S^3\,\left(1+5\,T+5\,T^2+T^3\right)-3\,S^5\,T\,\left(1+5\,T+5\,T^2+T^3\right)\Big)\,a_1+$$

$$
\begin{aligned}
&\left(1-3 S+5 S^2-3 S^3+S^4\right) \left(-4+9 (1+S) T-2 \left(5+9 S+5 S^2\right) T^2+3 \left(1+5 S+5 S^2+S^3\right) T^3-\right.\\
&\quad \left.3 S \left(1+5 S+5 S^2+S^3\right) T^5+2 S^2 \left(5+9 S+5 S^2\right) T^6-9 S^3 (1+S) T^7+4 S^4 T^8\right) b_1+\\
&4 x_1 \mathrm{XX}_1-2 S x_1 \mathrm{XX}_1-2 S^2 x_1 \mathrm{XX}_1+4 S^3 x_1 \mathrm{XX}_1-15 T x_1 \mathrm{XX}_1+3 S T x_1 \mathrm{XX}_1-3 S^2 T x_1 \mathrm{XX}_1+\\
&3 S^3 T x_1 \mathrm{XX}_1-15 S^4 T x_1 \mathrm{XX}_1+30 T^2 x_1 \mathrm{XX}_1+6 S T^2 x_1 \mathrm{XX}_1+12 S^2 T^2 x_1 \mathrm{XX}_1+12 S^3 T^2 x_1 \mathrm{XX}_1+\\
&6 S^4 T^2 x_1 \mathrm{XX}_1+30 S^5 T^2 x_1 \mathrm{XX}_1-21 T^3 x_1 \mathrm{XX}_1-51 S T^3 x_1 \mathrm{XX}_1-6 S^2 T^3 x_1 \mathrm{XX}_1-30 S^3 T^3 x_1 \mathrm{XX}_1-\\
&6 S^4 T^3 x_1 \mathrm{XX}_1-51 S^5 T^3 x_1 \mathrm{XX}_1-21 S^6 T^3 x_1 \mathrm{XX}_1+8 T^4 x_1 \mathrm{XX}_1+44 S T^4 x_1 \mathrm{XX}_1+56 S^2 T^4 x_1 \mathrm{XX}_1+\\
&8 S^3 T^4 x_1 \mathrm{XX}_1+8 S^4 T^4 x_1 \mathrm{XX}_1+56 S^5 T^4 x_1 \mathrm{XX}_1+44 S^6 T^4 x_1 \mathrm{XX}_1+8 S^7 T^4 x_1 \mathrm{XX}_1-21 S T^5 x_1 \mathrm{XX}_1-\\
&51 S^2 T^5 x_1 \mathrm{XX}_1-6 S^3 T^5 x_1 \mathrm{XX}_1-30 S^4 T^5 x_1 \mathrm{XX}_1-6 S^5 T^5 x_1 \mathrm{XX}_1-51 S^6 T^5 x_1 \mathrm{XX}_1-21 S^7 T^5 x_1 \mathrm{XX}_1+\\
&30 S^2 T^6 x_1 \mathrm{XX}_1+6 S^3 T^6 x_1 \mathrm{XX}_1+12 S^4 T^6 x_1 \mathrm{XX}_1+12 S^5 T^6 x_1 \mathrm{XX}_1+6 S^6 T^6 x_1 \mathrm{XX}_1+\\
&30 S^7 T^6 x_1 \mathrm{XX}_1-15 S^3 T^7 x_1 \mathrm{XX}_1+3 S^4 T^7 x_1 \mathrm{XX}_1-3 S^5 T^7 x_1 \mathrm{XX}_1+3 S^6 T^7 x_1 \mathrm{XX}_1-15 S^7 T^7 x_1 \mathrm{XX}_1+\\
&4 S^4 T^8 x_1 \mathrm{XX}_1-2 S^5 T^8 x_1 \mathrm{XX}_1-2 S^6 T^8 x_1 \mathrm{XX}_1+4 S^7 T^8 x_1 \mathrm{XX}_1+4 y_1 Y_1-15 S y_1 Y_1+\\
&30 S^2 y_1 Y_1-21 S^3 y_1 Y_1+8 S^4 y_1 Y_1-2 T y_1 Y_1+3 S T y_1 Y_1+6 S^2 T y_1 Y_1-51 S^3 T y_1 Y_1+\\
&44 S^4 T y_1 Y_1-21 S^5 T y_1 Y_1-2 T^2 y_1 Y_1-3 S T^2 y_1 Y_1+12 S^2 T^2 y_1 Y_1-6 S^3 T^2 y_1 Y_1+\\
&56 S^4 T^2 y_1 Y_1-51 S^5 T^2 y_1 Y_1+30 S^6 T^2 y_1 Y_1+4 T^3 y_1 Y_1+3 S T^3 y_1 Y_1+12 S^2 T^3 y_1 Y_1-\\
&30 S^3 T^3 y_1 Y_1+8 S^4 T^3 y_1 Y_1-6 S^5 T^3 y_1 Y_1+6 S^6 T^3 y_1 Y_1-15 S^7 T^3 y_1 Y_1-15 S T^4 y_1 Y_1+\\
&6 S^2 T^4 y_1 Y_1-6 S^3 T^4 y_1 Y_1+8 S^4 T^4 y_1 Y_1-30 S^5 T^4 y_1 Y_1+12 S^6 T^4 y_1 Y_1+3 S^7 T^4 y_1 Y_1+\\
&4 S^8 T^4 y_1 Y_1+30 S^2 T^5 y_1 Y_1-51 S^3 T^5 y_1 Y_1+56 S^4 T^5 y_1 Y_1-6 S^5 T^5 y_1 Y_1+12 S^6 T^5 y_1 Y_1-\\
&3 S^7 T^5 y_1 Y_1-2 S^8 T^5 y_1 Y_1-21 S^3 T^6 y_1 Y_1+44 S^4 T^6 y_1 Y_1-51 S^5 T^6 y_1 Y_1+6 S^6 T^6 y_1 Y_1+\\
&3 S^7 T^6 y_1 Y_1-2 S^8 T^6 y_1 Y_1+8 S^4 T^7 y_1 Y_1-21 S^5 T^7 y_1 Y_1+30 S^6 T^7 y_1 Y_1-15 S^7 T^7 y_1 Y_1+\\
&4 S^8 T^7 y_1 Y_1-4 \mathrm{XX}_1 Y_1 z_1+19 S \mathrm{XX}_1 Y_1 z_1-32 S^2 \mathrm{XX}_1 Y_1 z_1+19 S^3 \mathrm{XX}_1 Y_1 z_1-4 S^4 \mathrm{XX}_1 Y_1 z_1+\\
&2 T \mathrm{XX}_1 Y_1 z_1-22 S T \mathrm{XX}_1 Y_1 z_1+16 S^2 T \mathrm{XX}_1 Y_1 z_1+16 S^3 T \mathrm{XX}_1 Y_1 z_1-22 S^4 T \mathrm{XX}_1 Y_1 z_1+\\
&2 S^5 T \mathrm{XX}_1 Y_1 z_1+2 T^2 \mathrm{XX}_1 Y_1 z_1+35 S T^2 \mathrm{XX}_1 Y_1 z_1-28 S^2 T^2 \mathrm{XX}_1 Y_1 z_1+34 S^3 T^2 \mathrm{XX}_1 Y_1 z_1-\\
&28 S^4 T^2 \mathrm{XX}_1 Y_1 z_1+35 S^5 T^2 \mathrm{XX}_1 Y_1 z_1+2 S^6 T^2 \mathrm{XX}_1 Y_1 z_1-4 T^3 \mathrm{XX}_1 Y_1 z_1-22 S T^3 \mathrm{XX}_1 Y_1 z_1-\\
&28 S^2 T^3 \mathrm{XX}_1 Y_1 z_1-4 S^3 T^3 \mathrm{XX}_1 Y_1 z_1-4 S^4 T^3 \mathrm{XX}_1 Y_1 z_1-28 S^5 T^3 \mathrm{XX}_1 Y_1 z_1-22 S^6 T^3 \mathrm{XX}_1 Y_1 z_1-\\
&4 S^7 T^3 \mathrm{XX}_1 Y_1 z_1+19 S T^4 \mathrm{XX}_1 Y_1 z_1+16 S^2 T^4 \mathrm{XX}_1 Y_1 z_1+34 S^3 T^4 \mathrm{XX}_1 Y_1 z_1-4 S^4 T^4 \mathrm{XX}_1 Y_1 z_1+\\
&34 S^5 T^4 \mathrm{XX}_1 Y_1 z_1+16 S^6 T^4 \mathrm{XX}_1 Y_1 z_1+19 S^7 T^4 \mathrm{XX}_1 Y_1 z_1-32 S^2 T^5 \mathrm{XX}_1 Y_1 z_1+16 S^3 T^5 \mathrm{XX}_1 Y_1 z_1-\\
&28 S^4 T^5 \mathrm{XX}_1 Y_1 z_1-28 S^5 T^5 \mathrm{XX}_1 Y_1 z_1+16 S^6 T^5 \mathrm{XX}_1 Y_1 z_1-32 S^7 T^5 \mathrm{XX}_1 Y_1 z_1+19 S^3 T^6 \mathrm{XX}_1 Y_1 z_1-\\
&22 S^4 T^6 \mathrm{XX}_1 Y_1 z_1+35 S^5 T^6 \mathrm{XX}_1 Y_1 z_1-22 S^6 T^6 \mathrm{XX}_1 Y_1 z_1+19 S^7 T^6 \mathrm{XX}_1 Y_1 z_1-4 S^4 T^7 \mathrm{XX}_1 Y_1 z_1+\\
&2 S^5 T^7 \mathrm{XX}_1 Y_1 z_1+2 S^6 T^7 \mathrm{XX}_1 Y_1 z_1-4 S^7 T^7 \mathrm{XX}_1 Y_1 z_1+8 z_1 Z_1-21 S z_1 Z_1+30 S^2 z_1 Z_1-\\
&15 S^3 z_1 Z_1+4 S^4 z_1 Z_1-21 T z_1 Z_1+44 S T z_1 Z_1-51 S^2 T z_1 Z_1+6 S^3 T z_1 Z_1+3 S^4 T z_1 Z_1-\\
&2 S^5 T z_1 Z_1+30 T^2 z_1 Z_1-51 S T^2 z_1 Z_1+56 S^2 T^2 z_1 Z_1-6 S^3 T^2 z_1 Z_1+12 S^4 T^2 z_1 Z_1-\\
&3 S^5 T^2 z_1 Z_1-2 S^6 T^2 z_1 Z_1-15 T^3 z_1 Z_1+6 S T^3 z_1 Z_1-6 S^2 T^3 z_1 Z_1+8 S^3 T^3 z_1 Z_1-\\
&30 S^4 T^3 z_1 Z_1+12 S^5 T^3 z_1 Z_1+3 S^6 T^3 z_1 Z_1+4 S^7 T^3 z_1 Z_1+4 T^4 z_1 Z_1+3 S T^4 z_1 Z_1+\\
&12 S^2 T^4 z_1 Z_1-30 S^3 T^4 z_1 Z_1+8 S^4 T^4 z_1 Z_1-6 S^5 T^4 z_1 Z_1+6 S^6 T^4 z_1 Z_1-15 S^7 T^4 z_1 Z_1-\\
&2 S T^5 z_1 Z_1-3 S^2 T^5 z_1 Z_1+12 S^3 T^5 z_1 Z_1-6 S^4 T^5 z_1 Z_1+56 S^5 T^5 z_1 Z_1-51 S^6 T^5 z_1 Z_1+\\
&30 S^7 T^5 z_1 Z_1-2 S^2 T^6 z_1 Z_1+3 S^3 T^6 z_1 Z_1+6 S^4 T^6 z_1 Z_1-51 S^5 T^6 z_1 Z_1+44 S^6 T^6 z_1 Z_1-\\
&\left.\left.21 S^7 T^6 z_1 Z_1+4 S^3 T^7 z_1 Z_1-15 S^4 T^7 z_1 Z_1+30 S^5 T^7 z_1 Z_1-21 S^6 T^7 z_1 Z_1+8 S^7 T^7 z_1 Z_1\right) \epsilon\right) \Big/\\
&\left(\left(1-3 S+5 S^2-3 S^3+S^4\right)^2 \left(1-3 T+5 T^2-3 T^3+T^4\right)^2 \left(1-3 S T+5 S^2 T^2-3 S^3 T^3+S^4 T^4\right)^2\right)+\\
&\left.\left.O[\epsilon]^2\right]\right\}\right\}
\end{aligned}
$$

Mirror Trefoil

```
Timing@Block[{$k = 1},
  ZZ = (RR̄_{1,5} RR̄_{6,2} RR̄_{3,7} CC_4 KKink_8 KKink_9 KKink_10);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB_{1,r}~ (ddm_{1,r→1} /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{3.562500, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0, 0, \frac{S^2\, T^2}{\left(1 - S + S^2\right)\,\left(1 - T + T^2\right)\,\left(1 - S\,T + S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR̄_{1,5} RR̄_{6,2} RR̄_{3,7} CC_4 KKink_8 KKink_9 KKink_10);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB_{1,r}~ (ddm_{1,r→1}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\begin{array}{l}
\Big\{49.390625, \\
\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\, 0,\, \dfrac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)} + \Big(2\,S^2\,T^2\,\Big(8-15\,S+20\,S^2-13\,S^3+6\,S^4-15\,T+ \\
\quad 14\,S\,T-11\,S^2\,T-10\,S^3\,T+11\,S^4\,T-10\,S^5\,T+20\,T^2-11\,S\,T^2+16\,S^2\,T^2+S^3\,T^2+12\,S^4\,T^2- \\
\quad 7\,S^5\,T^2+11\,S^6\,T^2-13\,T^3-10\,S\,T^3+S^2\,T^3-12\,S^3\,T^3-10\,S^4\,T^3+S^5\,T^3-5\,S^6\,T^3-6\,S^7\,T^3+ \\
\quad 6\,T^4+11\,S\,T^4+12\,S^2\,T^4-10\,S^3\,T^4+24\,S^4\,T^4-6\,S^5\,T^4+4\,S^6\,T^4+5\,S^7\,T^4+2\,S^8\,T^4-10\,S\,T^5- \\
\quad 7\,S^2\,T^5+S^3\,T^5-6\,S^4\,T^5-4\,S^5\,T^5+S^6\,T^5-2\,S^7\,T^5-3\,S^8\,T^5+11\,S^2\,T^6-5\,S^3\,T^6+4\,S^4\,T^6+ \\
\quad S^5\,T^6-S^7\,T^6+2\,S^8\,T^6-6\,S^3\,T^7+5\,S^4\,T^7-2\,S^5\,T^7-S^6\,T^7+2\,S^7\,T^7-S^8\,T^7+2\,S^4\,T^8-3\,S^5\,T^8+ \\
\quad 2\,S^6\,T^8-S^7\,T^8+\left(1-S+S^2\right)\left(1-T+T^2\right)^2\Big(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)- \\
\qquad S^5\,T^3\,(3+T)-S^2\,T\,(1+3\,T)+S^3\,T\left(-1+T^2\right)\Big)\,a_1+\left(1-S+S^2\right)^2\left(1-T+T^2\right) \\
\qquad \left(-2+T+3\,S\,T-S\,(1+3\,S)\,T^2+S\left(-1+S^2\right)T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\right)b_1+ \\
\quad 2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1- \\
\quad 5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+5\,S^4\,T^2\,x_1\,XX_1+ \\
\quad 9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1- \\
\quad 12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+ \\
\quad 14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1- \\
\quad 12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1- \\
\quad 7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+ \\
\quad 9\,S^7\,T^6\,x_1\,XX_1-5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+ \\
\quad 2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+4\,S^4\,y_1\,Y_1- \\
\quad 2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-3\,S\,T^2\,y_1\,Y_1+ \\
\quad 9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1- \\
\quad 2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1- \\
\quad 5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+ \\
\quad 9\,S^6\,T^4\,y_1\,Y_1-2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1- \\
\quad 12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-12\,S^5\,T^6\,y_1\,Y_1+ \\
\quad 5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+9\,S^6\,T^7\,y_1\,Y_1-5\,S^7\,T^7\,y_1\,Y_1+ \\
\quad 2\,S^8\,T^7\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+7\,S\,XX_1\,Y_1\,z_1-9\,S^2\,XX_1\,Y_1\,z_1+7\,S^3\,XX_1\,Y_1\,z_1-2\,S^4\,XX_1\,Y_1\,z_1- \\
\quad 5\,S\,T\,XX_1\,Y_1\,z_1-5\,S^4\,T\,XX_1\,Y_1\,z_1+12\,S\,T^2\,XX_1\,Y_1\,z_1-9\,S^2\,T^2\,XX_1\,Y_1\,z_1+21\,S^3\,T^2\,XX_1\,Y_1\,z_1- \\
\quad 9\,S^4\,T^2\,XX_1\,Y_1\,z_1+12\,S^5\,T^2\,XX_1\,Y_1\,z_1-2\,T^3\,XX_1\,Y_1\,z_1-5\,S\,T^3\,XX_1\,Y_1\,z_1-9\,S^2\,T^3\,XX_1\,Y_1\,z_1- \\
\quad 7\,S^3\,T^3\,XX_1\,Y_1\,z_1-7\,S^4\,T^3\,XX_1\,Y_1\,z_1-9\,S^5\,T^3\,XX_1\,Y_1\,z_1-5\,S^6\,T^3\,XX_1\,Y_1\,z_1-2\,S^7\,T^3\,XX_1\,Y_1\,z_1+ \\
\quad 7\,S\,T^4\,XX_1\,Y_1\,z_1+21\,S^3\,T^4\,XX_1\,Y_1\,z_1-7\,S^4\,T^4\,XX_1\,Y_1\,z_1+21\,S^5\,T^4\,XX_1\,Y_1\,z_1+7\,S^7\,T^4\,XX_1\,Y_1\,z_1- \\
\quad 9\,S^2\,T^5\,XX_1\,Y_1\,z_1-9\,S^4\,T^5\,XX_1\,Y_1\,z_1-9\,S^5\,T^5\,XX_1\,Y_1\,z_1-9\,S^7\,T^5\,XX_1\,Y_1\,z_1+7\,S^3\,T^6\,XX_1\,Y_1\,z_1- \\
\quad 5\,S^4\,T^6\,XX_1\,Y_1\,z_1+12\,S^5\,T^6\,XX_1\,Y_1\,z_1-5\,S^6\,T^6\,XX_1\,Y_1\,z_1+7\,S^7\,T^6\,XX_1\,Y_1\,z_1-2\,S^4\,T^7\,XX_1\,Y_1\,z_1- \\
\quad 2\,S^7\,T^7\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-7\,S\,z_1\,Z_1+9\,S^2\,z_1\,Z_1-5\,S^3\,z_1\,Z_1+2\,S^4\,z_1\,Z_1-7\,T\,z_1\,Z_1+ \\
\quad 10\,S\,T\,z_1\,Z_1-12\,S^2\,T\,z_1\,Z_1+5\,S^3\,T\,z_1\,Z_1-2\,S^4\,T\,z_1\,Z_1+9\,T^2\,z_1\,Z_1-12\,S\,T^2\,z_1\,Z_1+ \\
\quad 18\,S^2\,T^2\,z_1\,Z_1-12\,S^3\,T^2\,z_1\,Z_1+9\,S^4\,T^2\,z_1\,Z_1-3\,S^5\,T^2\,z_1\,Z_1-5\,T^3\,z_1\,Z_1+5\,S\,T^3\,z_1\,Z_1- \\
\quad 12\,S^2\,T^3\,z_1\,Z_1+14\,S^3\,T^3\,z_1\,Z_1-14\,S^4\,T^3\,z_1\,Z_1+9\,S^5\,T^3\,z_1\,Z_1-2\,S^6\,T^3\,z_1\,Z_1+2\,S^7\,T^3\,z_1\,Z_1+ \\
\quad 2\,T^4\,z_1\,Z_1-2\,S\,T^4\,z_1\,Z_1+9\,S^2\,T^4\,z_1\,Z_1-14\,S^3\,T^4\,z_1\,Z_1+14\,S^4\,T^4\,z_1\,Z_1-12\,S^5\,T^4\,z_1\,Z_1+ \\
\quad 5\,S^6\,T^4\,z_1\,Z_1-5\,S^7\,T^4\,z_1\,Z_1-3\,S^2\,T^5\,z_1\,Z_1+9\,S^3\,T^5\,z_1\,Z_1-12\,S^4\,T^5\,z_1\,Z_1+18\,S^5\,T^5\,z_1\,Z_1- \\
\quad 12\,S^6\,T^5\,z_1\,Z_1+9\,S^7\,T^5\,z_1\,Z_1-2\,S^3\,T^6\,z_1\,Z_1+5\,S^4\,T^6\,z_1\,Z_1-12\,S^5\,T^6\,z_1\,Z_1+10\,S^6\,T^6\,z_1\,Z_1- \\
\quad 7\,S^7\,T^6\,z_1\,Z_1+2\,S^3\,T^7\,z_1\,Z_1-5\,S^4\,T^7\,z_1\,Z_1+9\,S^5\,T^7\,z_1\,Z_1-7\,S^6\,T^7\,z_1\,Z_1+4\,S^7\,T^7\,z_1\,Z_1\Big)\,\epsilon\Big)\Big/ \\
\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}
\end{array}$$

```
(*EndProfile[]; *)
```