In[ ]:=
```
Clear[initialcrossing, initialmatrix, braidword, Quad, A, F,
  HigherOrder, kerbeforesub, keraftersub, exp2, poly, poly2, tr, knot, n, p1, p2]
```

In[ ]:=
```
knot = {3, 1};
f[{i_, n_}] := If[n > 0, ConstantArray[{i, i + 1}, {n}], ConstantArray[{i + 1, i}, {-n}]]
list = Flatten[f /@ KnotData[knot, "BraidWord"]];
initialcrossing = firstcrossing[Take[list, {1, 2}]];
initialmatrix = firstmatrix[Take[list, {1, 2}]];
braidword = Sequence[Partition[Take[list, 2 - Length[list]], 2]];
n = KnotData[knot, "BraidIndex"];
firstcrossing[{x_, y_}] := If[x < y, HO_{x,y}, INV_{y,x}] ;
firstmatrix[{x_, y_}] := If[x < y, Matrix_{x,y}, Imatrix_{y,x}];
alex = KnotData[knot, "AlexanderPolynomial"][t^2];
amp = KnotData[knot, "Amphichiral"];
rep2[i_, j_] :=
 Which[i == j == 0, 1, True, ((i ! j ! Coefficient[exp2, d_2^i z_2^j]) /. {z_2 → 0, d_2 → 0})]
rep3[i_, j_, k_, l_] := Which[i == j == k == l == 0, 1, True,
  ((i ! j ! k ! l ! Coefficient[exp2, d_2^i z_2^j d_3 ^k z_3^l]) /. {z_2 → 0, d_2 → 0, z_3 → 0, d_3 → 0})]
rep4[i_, j_, k_, l_, m_, n_] := Which[i == j == k == l == m == n == 0, 1, True,
  ((i ! j ! k ! l ! m ! n ! Coefficient[exp2, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n]) /.
    {z_2 → 0, d_2 → 0, z_3 → 0, d_3 → 0, z_4 → 0, d_4 → 0})]
rep5[i_, j_, k_, l_, m_, n_, o_, p_] := Which[i == j == k == l == m == n == o == p == 0, 1, True,
  ((i ! j ! k ! l ! m ! n ! o ! p ! Coefficient[exp2, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n d_5 ^o z_5^p]) /.
    {z_2 → 0, d_2 → 0, z_3 → 0, d_3 → 0, z_4 → 0, d_4 → 0, z_5 → 0, d_5 → 0})]
rep6[i_, j_, k_, l_, m_, n_, o_, p_, q_, r_] :=
 Which[i == j == k == l == m == n == o == p == q == r == 0, 1, True, ((i ! j ! k ! l ! m ! n ! o ! p ! q !
      r ! Coefficient[exp2, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n d_5 ^o z_5^p d_6^q z_6^r]) /.
    {z_2 → 0, d_2 → 0, z_3 → 0, d_3 → 0, z_4 → 0, d_4 → 0, z_5 → 0, d_5 → 0, z_6 → 0, d_6 → 0})]
trace_2[s_] := Expand[Plus @@ (s /. Rule[{a_, b_}, c_] :> rep2[a, b] c)]
trace_3[s_] := Expand[Plus @@ (s /. Rule[{a_, b_, e_, f_}, c_] :> rep3[a, b, e, f] c)]
trace_4[s_] :=
 Expand[Plus @@ (s /. Rule[{a_, b_, e_, f_, g_, h_}, c_] :> rep4[a, b, e, f, g, h] c)]
trace_5[s_] := Expand[
  Plus @@ (s /. Rule[{a_, b_, e_, f_, g_, h_, i_, j_}, c_] :> rep5[a, b, e, f, g, h, i, j] c)]
trace_6[s_] := Expand[Plus @@ (s /. Rule[{a_, b_, e_, f_, g_, h_, i_, j_, k_, l_}, c_] :>
      rep6[a, b, e, f, g, h, i, j, k, l] c)]
qh[n_] := (1 + h Coefficient[Product[1 + 2 h z_i d_i, {i, 2, n}], h, 1] +
   h^2 Coefficient[Product[1 + 2 h z_i d_i + 2 h^2 (z_i d_i + z_i^2 d_i^2), {i, 2, n}], h, 2] +
   h^3 Coefficient[Product[1 + 2 h z_i d_i + 2 h^2 (z_i d_i + z_i^2 d_i^2) +
        (4 / 3) h^3 (z_i d_i + 3 z_i^2 d_i^2 + z_i^3 d_i^3), {i, 2, n}], h, 3] + h^4 Coefficient[
      Product[1 + 2 h z_i d_i + 2 h^2 (z_i d_i + z_i^2 d_i^2) + (4 / 3) h^3 (z_i d_i + 3 z_i^2 d_i^2 + z_i^3 d_i^3) +
        (2 / 3) h^4 (z_i d_i + 7 z_i^2 d_i^2 + 6 z_i^3 d_i^3 + z_i^4 d_i^4), {i, 2, n}], h, 4])
p_{l_,k_} [i_, j_] := Which[i == k && j == l, t, i == l && j == k, t, i == j == l,
   0, i == j == k, 1 - t^2, i == j, 1, True, 0];
Matrix_{l_,k_} := Array[p_{l,k}, {n, n}];
Imatrix_{l_,k_} := Inverse[Matrix_{l,k}];
Sz_{l_,k_} [z_{j_}] := Expand[Sum[z_i Imatrix_{l,k}[[i, j]], {i, 1, n}]];
Sder_{l_,k_} [d_{j_}] := Expand[Sum[Matrix_{l,k}[[i, j]] d_i, {i, 1, n}]];
```

```
ISz_{l_,k_}[z_{j_}] := Expand[Sum[z_i Matrix_{l,k}[[i, j]], {i, 1, n}]];
ISder_{l_,k_}[d_{j_}] := Expand[Sum[Imatrix_{l,k}[[i, j]] d_i, {i, 1, n}]];
Higher[a_, b_, w_, x_] :=
 1 + h (2 a b w x - (2 a^2 w x)/t + 2 a^2 t w x + 3 a^2 x^2 - (a^2 x^2)/t^2 + (a b x^2)/t - 3 a b t x^2 - 2 a^2 t^2 x^2) +
  h^2 (2 a b w x - (2 a^2 w x)/t + 2 a^2 t w x + 2 a^2 b w^2 x - (2 a^3 w^2 x)/t + 2 a^3 t w^2 x + 5 a^2 x^2 - (a^2 x^2)/t^2 + (a b x^2)/t -
    5 a b t x^2 - 4 a^2 t^2 x^2 + 4 a^3 w x^2 + 2 a b^2 w x^2 - (2 a^2 b w x^2)/t - 2 a^2 b t w x^2 - 4 a^3 t^2 w x^2 -
    4 a^4 w^2 x^2 + 2 a^2 b^2 w^2 x^2 + (2 a^4 w^2 x^2)/t^2 - (4 a^3 b w^2 x^2)/t + 4 a^3 b t w^2 x^2 + 2 a^4 t^2 w^2 x^2 +
    10/3 a^2 b x^3 - (2 a^2 b x^3)/(3 t^2) + (4 a^3 x^3)/(3 t) + (2 a b^2 x^3)/(3 t) - 16/3 a^3 t x^3 - 14/3 a b^2 t x^3 + 4/3 a^2 b t^2 x^3 +
    4 a^3 t^3 x^3 + 14 a^3 b w x^3 + (2 a^4 w x^3)/t^3 - (4 a^3 b w x^3)/t^2 - (8 a^4 w x^3)/t + (2 a^2 b^2 w x^3)/t + 10 a^4 t w x^3 -
    6 a^2 b^2 t w x^3 - 10 a^3 b t^2 w x^3 - 4 a^4 t^3 w x^3 + (13 a^4 x^4)/2 - 3 a^2 b^2 x^4 + (a^4 x^4)/(2 t^4) - (a^3 b x^4)/t^3 - (3 a^4 x^4)/t^2 +
    (a^2 b^2 x^4)/(2 t^2) + (6 a^3 b x^4)/t - 11 a^3 b t x^4 - 6 a^4 t^2 x^4 + 9/2 a^2 b^2 t^2 x^4 + 6 a^3 b t^3 x^4 + 2 a^4 t^4 x^4)
Inv[a_, b_, w_, x_] := 1 + h (b^2 w^2 + (a b w^2)/t + a b t w^2 - b^2 t^2 w^2 - 2 a b w x) +
  h^2 (b^2 w^2 - (a b w^2)/t + a b t w^2 - b^2 t^2 w^2 + 10/3 a b^2 w^3 + (4 a b^2 w^3)/(3 t^2) - (2 a^2 b w^3)/(3 t) + (8 b^3 w^3)/(3 t) + 2/3 a^2 b t w^3 -
    8/3 b^3 t w^3 - 2/3 a b^2 t^2 w^3 + a^2 b^2 w^4 + (b^4 w^4)/2 + (a^2 b^2 w^4)/(2 t^2) + (a b^3 w^4)/t + 1/2 a^2 b^2 t^2 w^4 - b^4 t^2 w^4 -
    a b^3 t^3 w^4 + 1/2 b^4 t^4 w^4 + 2 a b w x + 2 a^2 b w^2 x - 4 b^3 w^2 x - (4 a b^2 w^2 x)/t - 4 a b^2 t w^2 x +
    4 b^3 t^2 w^2 x - 2 a b^3 w^3 x - (2 a^2 b^2 w^3 x)/t - 2 a^2 b^2 t w^3 x + 2 a b^3 t^2 w^3 x + 2 a b^2 w x^2 + 2 a^2 b^2 w^2 x^2)
HO_{i_,j_} := Higher[z_i, z_j, d_i, d_j]
INV_{i_,j_} := Inv[z_i, z_j, d_i, d_j]
Der_{i_}[f_] := f + Sum[(1/k!) D[f, {w_i, k}, {x_i, k}], {k, 1, 8}]
NO3[l_, {i_, j_}] /; i < j :=
 (Der_j[Der_i[Expand[1 + h Coefficient[((l /. Flatten[Table[{z_k → Sz_{i,j}[z_k], d_k → Sder_{i,j}[d_k]},
             {k, 1, n}]]) /. {d_i → w_i, d_j → w_j}) * Higher[x_i, x_j, d_i, d_j], h, 1] +
       h^2 Coefficient[((l /. Flatten[Table[{z_k → Sz_{i,j}[z_k], d_k → Sder_{i,j}[d_k]},
             {k, 1, n}]]) /. {d_i → w_i, d_j → w_j}) *
          Higher[x_i, x_j, d_i, d_j], h, 2]]]]) /. {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}
NO3[l_, {i_, j_}] /; i > j := (Der_j[Der_i[Expand[1 + h Coefficient[
         ((l /. Flatten[Table[{z_k → ISz_{j,i}[z_k], d_k → ISder_{j,i}[d_k]}, {k, 1, n}]]) /.
            {d_i → w_i, d_j → w_j}) * Inv[x_j, x_i, d_j, d_i], h, 1] + h^2 Coefficient[
         ((l /. Flatten[Table[{z_k → ISz_{j,i}[z_k], d_k → ISder_{j,i}[d_k]}, {k, 1, n}]]) /. {d_i → w_i,
             d_j → w_j}) * Inv[x_j, x_i, d_j, d_i], h, 2]]]]) /. {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}

subd = Flatten[Table[d_i → w_i, {i, 2, n}]];
```

```
subz = Flatten[Table[z_i → x_i, {i, 2, n}]];

g[l_, {x_ , y_}] := If[x < y, l.(Matrix_{x,y}), l.(Imatrix_{y,x})]

qhend3[f_] := Expand[
  (f /. subd) + h (Coefficient[qh[n], h, 1] /. subz) + h^2 (Coefficient[qh[n], h, 2] /. subz) +
   h^2 ((Coefficient[f, h, 1] /. subd) * (Coefficient[qh[n], h, 1] /. subz))]

Quad = Expand[Fold[g, initialmatrix, braidword]];
a_{i_,j_} := If[i == j, Quad[[i, j]] - 1, Quad[[i, j]]];
A = Table[a_{i,j}, {i, 2, n}, {j, 2, n}];
F = Inverse[-A];
```

In[∘]:= `AbsoluteTiming[HigherOrder3 = Fold[NO3, initialcrossing, braidword];]`

Out[∘]= {0.937114, Null}

In[∘]:=
```
AbsoluteTiming[kerbeforesub3 = ((l = qhend3[HigherOrder3];
      Do[l = Der_i[l], {i, 2, n, 1}];
      l) /. Flatten[Table[{x_i → z_i, w_i → d_i}, {i, 2, n}]]);]
```

Out[∘]= {0.172053, Null}

In[∘]:= `sub = Flatten[Table[z_k → Expand[Sum[Quad[[i, k]] z_i, {i, 1, n}]], {k, 1, n}]]`

Out[∘]= $\{z_1 \to t^2 z_1 - t^4 z_1 + t\, z_2 - t^3 z_2 + t^5 z_2,\ z_2 \to t\, z_1 - t^3 z_1 + t^5 z_1 + z_2 - t^2 z_2 + t^4 z_2 - t^6 z_2\}$

In[∘]:=
```
AbsoluteTiming[keraftersub = 1 + h Expand[Coefficient[kerbeforesub3 /. sub, h, 1]] +
    h^2 Expand[Coefficient[kerbeforesub3 /. sub, h, 2]];]
```

Out[∘]= {0.399409, Null}

In[∘]:=
```
sub3 = Flatten[Table[{z_i → z_i + Expand[Sum[F[[k, i - 1]] Quad[[1, k + 1]] z_1, {k, 1, n - 1}]],
    d_i -> d_i + Expand[Sum[F[[i - 1, l]] Quad[[l + 1, 1]] d_1, {l, 1, n - 1}]]}, {i, 2, n}]]
```

Out[∘]= $$\left\{z_2 \to \frac{t\, z_1}{t^2 - t^4 + t^6} - \frac{t^3 z_1}{t^2 - t^4 + t^6} + \frac{t^5 z_1}{t^2 - t^4 + t^6} + z_2,\ d_2 \to \frac{t\, d_1}{t^2 - t^4 + t^6} - \frac{t^3 d_1}{t^2 - t^4 + t^6} + \frac{t^5 d_1}{t^2 - t^4 + t^6} + d_2\right\}$$

In[∘]:=
```
AbsoluteTiming[twopoly = 1 + h Expand[Coefficient[keraftersub /. sub3, h, 1]] +
    h^2 Expand[Coefficient[keraftersub /. sub3, h, 2]];]
```

Out[∘]= {0.252061, Null}

In[∘]:= `sub4 = Sum[(Expand[Sum[F[[i - 1, j - 1]] d_i z_j, {j, 2, n}, {i, 2, n}]])^i / i!, {i, 1, 10}]`

Out[∘]=
$$\frac{d_2 z_2}{t^2 - t^4 + t^6} + \frac{d_2^2 z_2^2}{2\left(t^2 - t^4 + t^6\right)^2} + \frac{d_2^3 z_2^3}{6\left(t^2 - t^4 + t^6\right)^3} + \frac{d_2^4 z_2^4}{24\left(t^2 - t^4 + t^6\right)^4} + \frac{d_2^5 z_2^5}{120\left(t^2 - t^4 + t^6\right)^5} + \frac{d_2^6 z_2^6}{720\left(t^2 - t^4 + t^6\right)^6} + \frac{d_2^7 z_2^7}{5040\left(t^2 - t^4 + t^6\right)^7} + \frac{d_2^8 z_2^8}{40\,320\left(t^2 - t^4 + t^6\right)^8} + \frac{d_2^9 z_2^9}{362\,880\left(t^2 - t^4 + t^6\right)^9} + \frac{d_2^{10} z_2^{10}}{3\,628\,800\left(t^2 - t^4 + t^6\right)^{10}}$$

In[∘]:=
```
AbsoluteTiming[exp2 = Expand[1 + sub4];]
```
Out[∘]= {0.0001075, Null}

In[∘]:=
```
sub5 = Flatten[Table[{d_i, z_i}, {i, 2, n}]]
```
Out[∘]= $\{d_2, z_2\}$

In[∘]:=
```
AbsoluteTiming[poly2 = CoefficientRules[twopoly, sub5]]
```
Out[∘]=

$$\Big\{0.318247, \Big\{\{4,4\} \to \frac{h^2 t^4}{2} - 2 h^2 t^6 + 6 h^2 t^8 - 13 h^2 t^{10} + 23 h^2 t^{12} - 33 h^2 t^{14} + \frac{77 h^2 t^{16}}{2} - 37 h^2 t^{18} + \frac{55 h^2 t^{20}}{2} - 15 h^2 t^{22} + \frac{9 h^2 t^{24}}{2},$$

$$\{4,3\} \to -h^2 t^3 z_1 + 6 h^2 t^5 z_1 - 19 h^2 t^7 z_1 + \cdots 27 \cdots + \frac{188 h^2 t^{25} z_1}{t^2-t^4+t^6} - \frac{78 h^2 t^{27} z_1}{t^2-t^4+t^6} + \frac{18 h^2 t^{29} z_1}{t^2-t^4+t^6},$$

$$\cdots 21 \cdots, \{0,1\} \to \frac{2 h t d_1}{t^2-t^4+t^6} + \cdots 416 \cdots + \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$$

$$\{0,0\} \to 1 + \frac{2 h t^2 d_1 z_1}{(t^2-t^4+t^6)^2} + \frac{2 h^2 t^2 d_1 z_1}{(t^2-t^4+t^6)^2} - \frac{6 h t^4 d_1 z_1}{(t^2-t^4+t^6)^2} - \frac{6 h^2 t^4 d_1 z_1}{(t^2-t^4+t^6)^2} + \frac{12 h t^6 d_1 z_1}{(t^2-t^4+t^6)^2} + \cdots 503 \cdots +$$

$$\frac{6 h^2 t^{12} d_1^4 z_1^4}{t^2-t^4+t^6} - \frac{18 h^2 t^{14} d_1^4 z_1^4}{t^2-t^4+t^6} + \frac{32 h^2 t^{16} d_1^4 z_1^4}{t^2-t^4+t^6} - \frac{34 h^2 t^{18} d_1^4 z_1^4}{t^2-t^4+t^6} + \frac{22 h^2 t^{20} d_1^4 z_1^4}{t^2-t^4+t^6} - \frac{7 h^2 t^{22} d_1^4 z_1^4}{t^2-t^4+t^6}\Big\}\Big\}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
AbsoluteTiming[tr = trace_n[poly2];]
```
Out[∘]= {0.024945, Null}

In[∘]:=
```
test1 = Together[Coefficient[tr, h, 1]]
```
Out[∘]=

$$\frac{2 t^2 \left(-1 + 2 t^2 - 3 t^4 + 2 t^6\right)}{\left(1 - t^2 + t^4\right)^2}$$

In[∘]:=
```
test2 = Together[Coefficient[tr, h, 2]]
```
Out[∘]=

$$\frac{2 t^2 \left(1 - 2 t^2 + 4 t^4 - 2 t^6 + 6 t^{10} - 11 t^{12} + 4 t^{14}\right)}{\left(1 - t^2 + t^4\right)^4}$$

In[∘]:=
```
p = If[amp == True, Cancel[Denominator[test1] / alex], Cancel[Denominator[test1] / alex^2]]
q = If[amp == True, Cancel[Denominator[test2] / alex^3],
  Cancel[Denominator[test2] / alex^4]]
```

Out[∘]= t^4

Out[∘]= t^8

In[ ]:=
```
series = If[amp == True, Normal[Series[
    ((((1 + h test1 + h^2 test2 + h^3 test3 + h^4 test4) (1/alex)) /. {t → t Exp[h]}) (alex) /.
      {h → h/2 - h^2/4}) /. {h → -Sqrt[1 + h] + 1/Sqrt[1 + h]}, {h, 0, 2}]], Normal[Series[
    ((((1 + h test1 + h^2 test2 + h^3 test3 + h^4 test4) (1/alex)) /. {t → t Exp[h]}) alex) /.
    {h → h/2 - h^2/4}, {h, 0, 2}]]]
```

Out[ ]= $1+\frac{h\left(1-2t^2+2t^4-2t^6+t^8\right)}{\left(1-t^2+t^4\right)^2}-\frac{h^2t^4\left(1-t^2-t^4-t^6+t^8\right)}{\left(1-t^2+t^4\right)^4}$

In[ ]:=
```
p1 = Expand[Numerator[Coefficient[series, h, 1]] / p]
```

Out[ ]= $2+\frac{1}{t^4}-\frac{2}{t^2}-2t^2+t^4$

In[ ]:=
```
p2 = Expand[Numerator[Coefficient[series, h, 2]] / q]
```

Out[ ]= $1-\frac{1}{t^4}+\frac{1}{t^2}+t^2-t^4$

In[ ]:=
```
SessionTime[]
```

Out[ ]= 60.7435421