$$\eta_{gr} : \mathrm{Ass} \otimes \mathrm{Ass} \longrightarrow \mathrm{Ass}$$

Recall: $\tilde{u} \in \mathrm{tder}(\mathrm{Lie}(z_1, \cdots, z_n))$

$$\tilde{u} \in \mathfrak{krv}_n \iff \tilde{u} \circ \eta_{gr} = \eta_{gr} \circ \tilde{u} \quad \& \quad \tilde{u} \circ \delta_{gr} = \delta_{gr} \circ \tilde{u}$$

$$\mathfrak{krv}_n^0 := \left\{ \tilde{u} \in \mathfrak{krv}_n \;\middle|\; \mathrm{div}(\tilde{u}) \in \bigoplus_{i=1}^{n} \mathbb{K}|z_i| \right\}$$

$$\boxed{\text{Thm (AKKN)} \quad \tilde{u} \in \mathfrak{krv}_n^0 \iff \tilde{u} \circ \eta_{gr} = \eta_{gr} \circ \tilde{u} \quad \& \quad \tilde{u} \circ (\mu_r)_{gr} = (\mu_r)_{gr} \circ \tilde{u}}$$

can be replaced with $(\mu_0)_{gr}$

$\mu_0 \doteq R$

Symmetric KV (Alekseev-Torossian §8)

$$\mathfrak{krv}_2^{sym} := \{ (a(x,y), b(x,y)) \in \mathfrak{krv}_2 \mid a(x,y) = b(y,x) \}$$

- The AT map $\nu : \mathfrak{grt}_1 \longrightarrow \mathfrak{krv}_2$ takes values in $\mathfrak{krv}_2^{sym}$.
- No non-symmetric elements in $\mathfrak{krv}_2$ are known

$$\boxed{\text{Thm1 Let } \varphi \in \mathrm{Lie}(x,y) \text{ such that } \deg \varphi \geq 3 \ \& \ (\varphi(y,x), \varphi(x,y)) \in \mathfrak{krv}_2^{sym}. \text{ Then, } \varphi \in \mathrm{SolEMPent}.}$$

($\rightsquigarrow$ surjectivity of $\mathfrak{gnt}_1^{EM} \longrightarrow \mathfrak{krv}_2^{sym}$)

if properly defined

$\mathfrak{gnt}_1^{EM} \longrightarrow \mathfrak{krv}_2$

R

Φ

Some preliminary considerations about $\mu_0 = (\mu_0)_{gr}$

Prop2 Let $\tilde{u} \in \mathrm{sder}(\mathrm{Lie}(z_1, \cdots, z_n))$. Then, $\mu_0 \circ \tilde{u} - \tilde{u} \circ \mu_0$ is a derivation on $\mathrm{Ass}(z_1, \cdots, z_n)$. Namely, for $\forall a, \forall b \in \mathrm{Ass}$

$$(\mu_0 \circ \tilde{u} - \tilde{u} \circ \mu_0)(ab) = \big(\mu_0(\tilde{u}(a)) - \tilde{u}(\mu_0(a))\big)\, b + a\,\big(\mu_0(\tilde{u}(b)) - \tilde{u}(\mu_0(b))\big)$$

proof

$$\mu_0(\tilde{u}(ab)) = \mu_0\big(\tilde{u}(a)b + a\,\tilde{u}(b)\big) = \cdots \quad \mu_0(\hat{u}(a))b + \tilde{u}(a)\mu_0(b) + \eta(\hat{u}(a), b) + \cdots$$

$$\tilde{u}(\mu_0(ab)) = \tilde{u}\big(\mu_0(a)b + a\mu_0(b) + \eta(a,b)\big) = \cdots$$

Use the relation $\tilde{u}(\eta(a,b)) = \eta(\tilde{u}(a), b) + \eta(a, \tilde{u}(b))$, which follows from the assumption $\tilde{u} \in \mathrm{sder}$. //

Since μ_0 vanishes on degree 1 elements, we obtain

Cor3 $a_1, \cdots, a_m \in \mathrm{Ass}$ degree 1

$$(\mu_0 \circ \tilde{u} - \tilde{u} \circ \mu_0)(a_1 \cdots a_m) = \sum_{i=1}^{m} (a_1 \cdots a_{i-1})\, \mu_0(u(a_i))\, (a_{i+1} \cdots a_m)$$

$\cancel{-\tilde{u}(\mu_0(a_i))}$

cf. Prop 1 in the notes "20240520 ⋯", which is about μ_r.
Proof is a bit simplified.

Consider the case where $\tilde{u} = (\varphi(y,x), \varphi(x,y)) \in \mathrm{sder}_2$ for some $\varphi \in \mathrm{Lie}(x,y)$. Then, (recall $\mu_0 = -2R$)

$$\mu_0(\tilde{u}(y)) = \mu_0([y,\varphi]) = [y, \mu_0(\varphi)] + y\,\underbrace{\overline{\partial_y \varphi}}_{= \partial_y\varphi} - (\partial_y \varphi)\,y$$

$$= [y, \underbrace{\mu_0(\varphi) + \partial_y \varphi}_{=: q(x,y)}]$$

and

$$\mu_0(\tilde{u}(x)) = [x, \underbrace{\mu_0(\varphi)(y,x) + (\partial_y\varphi)(y,x)}_{=: p(x,y) = q(y,x)}]$$

$\Big[$ <u>Thm1</u> Let $\varphi \in \mathrm{Lie}(x,y)$ such that $\deg \varphi \geq 3$ & $(\varphi(y,x), \varphi(x,y)) \in \mathrm{krv}_2^{\mathrm{sym}}$. Then, $\varphi \in \mathrm{SolEMPent}$. $\Big]$

<u>proof</u> ① Assume that $\tilde{u} = (\varphi(y,x), \varphi(x,y)) \in \mathrm{krv}_2^0 \cap \mathrm{krv}_2^{\mathrm{sym}}$.

Then, $\tilde{u} \in \mathrm{sder}_2$ and $\mu_0 \circ \tilde{u} - \tilde{u} \circ \mu_0 = 0$.

suffices to consider y !

For $\forall m, \forall n$, applying Cor 3 to $x^m y^n \in \mathrm{Ass}$, we obtain

$$0 = \sum_{i=1}^{m} x^{i-1}[x,p]\,x^{m-i} y^n + \sum_{j=1}^{n} x^m y^{j-1}[y,q]\,y^{n-j}$$

$$= [x^m, p]\,y^n + x^m [y^n, q]$$

$y \rightsquigarrow [y, q] = 0$

$x^m y^n = a_1 \cdots a_m$

$$\sum_{i=1}^{m} (a_1 \cdots a_{i-1})\,\mu_0(u(a_i))\,(a_{i+1} \cdots a_m) = 0$$

not needed!

Lem4 Let $g = g(x,y)$ & $p = g(y,x) \in$ Ass satisfy

$$[x^m, p]\, y^n + x^m [y^n, g] = 0 \quad \cdots \quad ✩$$

for any $m, n > 0$. Then, $g \in \mathbb{Q}[[y]]$.

proof May assume that g is homogeneous. Let $m, n \gg \deg g = \ell$.

Write $g = g' + c\, y^{\ell}$ with g' a $\mathbb{Q}$-span of monomials $\neq y^{\ell}$.

✩ expands to

$$x^m p' y^n - \underset{\text{contains } y}{p'} x^m y^n + x^m y^n \underset{\text{contains } x}{g'} - x^m g' y^n = 0$$

Taking terms spanned by monomials of the form $x^m \boxed{\cdots} y^n$,

we obtain $x^m p' y^n - x^m g' y^n = 0$ (and so $p' = g'$).

Thus, $-\underset{\text{contains } y}{p'} x^m y^n + x^m y^n g' = 0$, and so $g' = 0$ //

$\rightsquigarrow \quad g \in \mathbb{Q}[[y]]$

Applying Lem 4, we obtain $f = \mu_0(\varphi) + \partial_y\varphi \in \mathbb{Q}[[y]]_{\geq 2}$

However, any monomial appearing in $\mu_0(\varphi)$ contains at least one x, and so $\mu_0(\varphi)\Big|_{\substack{x=0\\y=y}} = 0$. Also, $\partial_y\varphi\Big|_{\substack{x=0\\y=y}} = 0$ since $\varphi \in \mathrm{Lie}(x,y)$ is of degree ≥ 2. We conclude that

$$\mu_0(\varphi) + \partial_y\varphi = 0.$$

Since $\mu_0 = -2R$, this is almost equivalent to (P3). To complete the proof, we show that $(\partial_y\varphi)(x,0) = 0$. But this follows from $(\partial_y\varphi)(x,0) = -\mu_0(\varphi)\Big|_{\substack{x=x\\y=0}} = 0$. So,

(P3) $\partial_y\varphi + \partial_y\varphi(y,0) - \partial_y\varphi(x+y,0) - 2R(\varphi) = 0$.

(P1) $\left(\varphi(y,0) - \varphi(x+y,0) = 0\right)$ is true since $\deg\varphi \geq 2$, and

(P2) $\left(R(y,0) - R(x+y,0) = 0\right)$ is true since any monomial in $R(\varphi)$ contains at least one y. Hence $\varphi \in \mathrm{SolEMPent}$ ① //

② Let $\tilde{u} = (\varphi(y,x), \varphi(x,y)) \in \mathrm{krv}_2^{\mathrm{sym}}$ be homogeneous of $\deg = l \geq 3$.

<u>l: even</u> It is known that (see e.g., Alekseev-Torossian Prop 4.5) $\mathrm{div}(\tilde{u}) = 0$. Then $\tilde{u} \in \mathrm{krv}_2^0$ and $\varphi \in$ SolEMPent by ①.

<u>l: odd</u> There is some $c \in \mathbb{Q}$ such that

$$\mathrm{div}(\tilde{u}) = c \, | x^l + y^l - (x+y)^l | .$$

Let $\sigma_l \in \mathfrak{grt}_1$ be the Drinfeld generator. Then, $\nu(\sigma_l) \in \mathrm{krv}_2^{\mathrm{sym}}$ and $\mathrm{div}(\nu(\sigma_l)) = | x^l + y^l - (x+y)^l |$. Also, writing $\nu(\sigma_l) = (\psi_l(y,x), \psi_l(x,y))$, we have $\psi_l \in$ SolEMPent.

Therefore, $\varphi = \underbrace{(\varphi - c\,\psi_l)}_{\in \text{ SolEMPent by ①}} + c\,\psi_l \in$ SolEMPent

// ②

// Thm 1.

Rem More direct proof for ② ? One has to consider the condition $\delta \circ \tilde{u} = \tilde{u} \circ \delta$. Need a tr-version of Lem 4, and maybe more... For instance,

Q. Let $c \in \mathrm{Ass}(x, y)$ satisfy

$$|x^m [c, y^n]| = 0$$

for $^\forall m, {}^\forall n > 0$. Then, $c \in \mathbb{Q}[[x]] + \mathbb{Q}[[y]]$?

(according to a computer experiment, true up to deg 10)