Summary:

Let $\varphi \in \mathrm{SolEMP}_{\mathrm{ent}}$ of degree ≥ 3 & assume that $\overline{\partial_x \varphi} = (\partial_x \varphi)(y,x)$. Then, $\nu(\varphi) := (\varphi(y,x), \varphi(x,y))$ $\in \mathrm{tder}_2$ satisfies (KV1) & (KV2).

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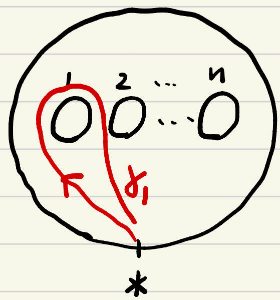
1. the homotopy intersection form & (KV1)

$\pi = \pi_1(\Sigma_{0,n+1}) \cong \langle \gamma_1, \cdots, \gamma_n \rangle$



$\eta : \mathbb{Q}\pi \otimes \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$ (Massuyeau–Turaev)

$$\eta(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \alpha_{*p} \beta_{p*}$$

η is a Fox pairing: $\eta(a, bc) = \eta(a,b)c + \varepsilon(b)\eta(a,c)$

$\eta(ab, c) = \varepsilon(b)\eta(a,c) + a\,\eta(b,c)$

$\mathrm{gr}\,\mathbb{Q}\pi = \mathrm{Ass}(z_1, \cdots, z_n)$ $\quad z_i = [\gamma_i - 1]$ $\quad H := \bigoplus_i \mathbb{Q} z_i \cong H_1(\Sigma_{0,n+1})$

$\zeta : H \times H \longrightarrow H, \quad \zeta(z_i, z_j) := \delta_{ij} \cdot z_i$

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$\eta \rightsquigarrow \eta_{gr} : Ass^{\otimes 2} \longrightarrow Ass$

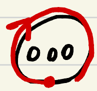
$$\eta_{gr}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_{m-1}\, \zeta(a_m, b_1)\, b_2 \cdots b_n$$

<u>Thm</u> (Massuyeau–Turaev, Naef)

$\varphi \in tAut(Lie(z_1, \dots, z_n))$

$$\varphi \circ \eta_{gr} = \eta_{gr} \circ \varphi \iff \varphi(z_1 + \cdots + z_n) = z_1 + \cdots + z_n$$

Also true: $\varphi \in tAut(Prim(\widehat{\mathbb{Q}\pi}))$

$\varphi \circ \eta = \eta \circ \varphi \iff \varphi($  $) =$ 

logarithm

$u = (u_1, \dots, u_n) \in tder(Lie(z_1, \dots, z_n))$

$$u \circ \eta_{gr} = \eta_{gr} \circ u \iff u(z_1 + \cdots + z_n) = 0$$

$\Updownarrow$

$$u(\eta_{gr}(z_i, z_j)) = \eta_{gr}(u(z_i), z_j) + \eta_{gr}(z_i, u(z_j))$$

$$u(z_1 + \cdots + z_n) = 0$$

$$\Updownarrow$$

$$u\big(\eta_{gr}(z_i, z_j)\big) = \eta_{gr}(u(z_i), z_j) + \eta_{gr}(z_i, u(z_j))$$

η

$$\text{LHS} = u(\zeta(z_i, z_j)) = \begin{cases} 0 & i \neq j \\ u(z_i) = [z_i, u_i] & i = j \end{cases}$$

$$\text{RHS} = \eta([z_i, u_i], z_j) + \eta(z_i, [z_j, u_j])$$

$z_i u_i - u_i z_i$ $\qquad$ $z_j u_j - u_j z_j$

$$\left(\eta_{gr}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_{m-1} \zeta(a_m, b_1) b_2 \cdots b_n \right)$$

$$= z_i(\partial_j u_i) z_j - u_i \zeta(z_i, z_j) + \zeta(z_i, z_j) u_j - z_i(\partial^i u_j) z_j$$

$$= \begin{cases} z_i(\partial_j u_i - \partial^i u_j) z_j & i \neq j \\ z_i(\partial_j u_i - \partial^i u_j) z_j + [z_i, u_i] & i = j \end{cases}$$

[**Lem** $u = (u_1, \cdots, u_n) \in \text{tder}$

$u(z_1 + \cdots + z_n) = 0 \iff \partial_j u_i = \partial^i u_j \quad \forall i, \forall j$]

$$\overline{\partial_i u_j}$$

$$\left[\begin{array}{l}\underline{\text{Lem}}\quad u=(u_1,\cdots,u_n)\in \text{tder} \\ \quad u(z_1+\cdots+z_n)=0 \iff \partial_j u_i = \partial^i u_j \quad \forall_i, \forall_j \end{array}\right]$$

(with $\partial^i u_j = \overline{\partial_i u_j}$)

The case $n=2$ & $u = v(\varphi) = (\underset{u_1}{\varphi(y,x)}, \underset{u_2}{\varphi(x,y)})$ $\quad z_1 = x,\ z_2 = y$

$$u(x+y)=0 \iff \begin{cases} \partial_x(\varphi(y,x)) = \overline{\partial_x\varphi(x,y)} & (1,1) \\ \partial_y(\varphi(x,y)) = \overline{\partial_y(\varphi(x,y))} & \\ \partial_y(\varphi(y,x)) = \overline{\partial_x(\varphi(x,y))} & (2,1) \\ \cdots & \end{cases}$$

$$\iff \partial_y\varphi = \overline{\partial_y\varphi} \ \& \ (\partial_x\varphi)(y,x) = \overline{\partial_x\varphi}$$

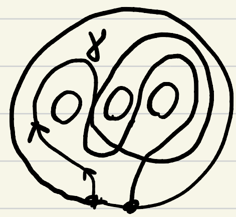
$\varphi \in$ SolEMPent ⇝ first condition (✓); ⇝ second condition (?)

2. The map R & self-intersection of loop

<u>Recall</u> $R : \mathrm{Lie} \longrightarrow \mathrm{Ass}$, $R(z_i) = 0$ &

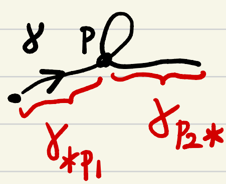
$$R([u,v]) = [u, R(v)] + [R(u), v] + \frac{1}{2}\sum_i \left((\partial_i v) z_i \overline{\partial_i u} - (\partial_i u) z_i \overline{\partial_i v} \right)$$

$\mu_0 : \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$



$$\mu_0(\gamma) = \sum_{p \in \mathrm{Self}(\gamma)} \varepsilon_p \, \gamma_{*p_1} \gamma_{p_2 *}$$

(γ: rot $= -1/2$; γ_{*p_1}: first; γ_{p_2*}: second)



<u>product formula</u> : $\mu_0(ab) = \mu_0(a) b + a \mu_0(b) + \eta(a,b)$

(η: homotopy int. form)

ass. gr

$$(\mu_0)_{gr}(a_1 \cdots a_m) = \sum_{j=1}^{m-1} a_1 \cdots a_{j-1} \xi(a_j, a_{j+1}) a_{j+2} \cdots a_m$$

[**Prop** $R = -\frac{1}{2}(\mu_0)_{gr}$] $\mu_0 = (\mu_0)_{gr}$

proof $\mu_0(z_i) = 0$

$$\left(\begin{array}{l} \eta_{gr}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_{m-1} \zeta(a_m, b_1) b_2 \cdots b_n \\ \eta_{gr}(a, b) = \sum_i (\partial_i a) z_i (\partial^i b) \underset{a, b \in Lie}{=} \sum_i (\partial_i a) z_i \overline{(\partial_i b)} \end{array} \right)$$

$a, b \in Lie$. Then

$$\begin{aligned} \mu_0([a,b]) &= \mu_0(a) b + a \mu_0(b) + \eta(a, b) \\ &\quad - \mu_0(b) a - b \mu_0(a) - \eta(b, a) \\ &= [\mu_0(a), b] + [a, \mu_0(b)] \\ &\quad + \sum_i \left((\partial_i a) z_i \overline{\partial_i b} - (\partial_i b) z_i \overline{\partial_i a} \right) \end{aligned}$$

//

μ_0 recovers the Turaev cobracket:

$$a \overset{\Delta}{\longmapsto} a' \otimes a'' \overset{id \otimes \mu_0}{\longmapsto} a' \otimes \mu_0(a'')$$

$a \in \mathbb{Q}\pi$

$$\overset{id \otimes ((2 \otimes id)\Delta)}{\longmapsto} a' \otimes \tilde{\Delta}'(\mu_0(a'')) \otimes \tilde{\Delta}''(\mu_0(a''))$$

$$\overset{(|\,|\circ mult)\otimes id}{\longmapsto} |a' \tilde{\Delta}'(\mu_0(a''))| \otimes \tilde{\Delta}''(\mu_0(a''))$$

$$\| \quad \mu_r(a) \in |\mathbb{Q}\pi| \otimes \mathbb{Q}\pi$$

Then,

$$\delta(|a|) = Alt \circ (id \otimes |\,|)(\mu_r(a)) + |a| \wedge \mathbb{1}$$

δ: (framed) Turaev cobracket

$|a| \wedge \mathbb{1}$: does not contribute to ass. gr.

Thm (AKKN)

$\varphi \in tAut(Lie(z_1, \cdots, z_n))$

$\varphi \in KRV_n \iff \begin{cases} \varphi \circ \eta_{gr} = \eta_{gr} \circ \varphi & \rightsquigarrow (KV1) \\ \varphi \circ \delta_{gr} = \delta_{gr} \circ \varphi & \rightsquigarrow (KV2) \ (KV1) \end{cases}$

logarithm $\longrightarrow$ $u \in krv_n \iff \cdots\cdots$

3. EM 5-gon & KV equations.

$$\left\{ (\partial_y\varphi)(x,y) + (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0) - 2R(x,y) = 0 \right.$$

$\varphi \in \mathrm{SolEMPent} \qquad \nu(\varphi) := (\varphi(y,x), \varphi(x,y))$

$$R(\nu(\varphi)(y)) = R([y, \varphi(x,y)])$$

$$= [y, \underset{R(\varphi)}{R(x,y)}] + \frac{1}{2}\left((\partial_y\varphi)y - y\,\overline{\partial_y\varphi}\right)$$

$$\overline{\partial_y\varphi} = \partial_y\varphi$$

$$= [y, R(x,y) - \frac{1}{2}\partial_y\varphi]$$

$$\underset{\text{EM 5-gon}}{=} \left[y, \frac{1}{2}\left((\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0)\right)\right]$$

$$= [y, f(x+y)]$$

the same !

$$R(\nu(\varphi)(x)) = R([x, \varphi(y,x)]) = \cdots = [x, f(x+y)]$$

Prop $u \in \mathrm{sder}(\mathrm{Lie}(z_1, \cdots, z_n))$, $\forall a = a_1 \cdots a_m \in \mathrm{Ass}$

$$\mu_r(u(a)) - u.(\mu_r(a)) = \sum_{\lambda=1}^{m} (1 \otimes a_1 \cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1 \otimes a_{\lambda+1} \cdots a_m)$$

To be continued....