Emergent version of Drinfeld's associator equations

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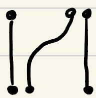
work in progress, joint with Dror Bar-Natan (Toronto)

§ Introduction

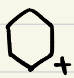
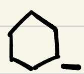
Φ : Drinfeld associator $\rightsquigarrow$ $\quad \mathbf{PaB} \xrightarrow{\cong} \mathbf{PaCD}$,

 $\mapsto \left(\exp\left(\frac{1}{2} H\right), X \right)$

 $\mapsto \Phi \in \exp(DK_3)$

$$\begin{cases} \text{5-gon eqn} & \\ \text{6-gon eqns} & \end{cases}$$

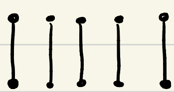
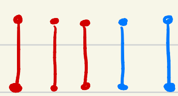
5-gon eqn  $\quad \Phi\Phi\Phi = \Phi\Phi \in \exp(DK_4)$

6-gon eqns  , 

$\Leftarrow$ Furusho

"Tangles in a pole dance studio"

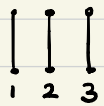
Bar-Natan - Dancso - Hogan - Liu - Scheirich

 $\rightsquigarrow$ 

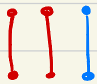
poles strands

Emergent version

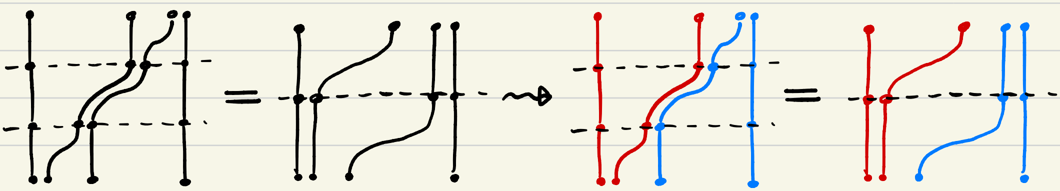
- no chords between poles
- at most one chord between strands



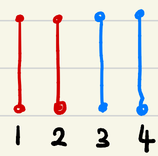
$$DK_3 \cong \mathbb{Q}(t_{12}+t_{13}+t_{23}) \oplus FL(t_{13},t_{23})$$

$\rightsquigarrow$ 

$$EDK_{2,1} \cong FL(t_{13},t_{23})$$



The emergent 5-gon eqn takes place in



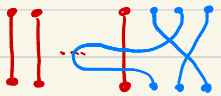
$$EDK_{2,2} \cong \frac{FL(\cancel{t_{12}}, t_{13}, t_{14}, t_{23}, t_{24}, t_{34})}{4T,\ [t_{ij},t_{kl}]=0,\ \deg t_{34} \geq 2}$$

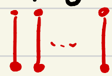
$$\cong FL(x,y)_1 \oplus FL(x,y)_2 \oplus FA(x,y)_{1,2}$$

Aim:
- more tractable equ's
- revisit the map $\nu : \mathfrak{grt}_1 \hookrightarrow \mathfrak{krv}_2$ (Alekseev-Torossian)

§ mixed braids & emergent braids

mixed braids

$B_{p,s} = \Big\{$ 

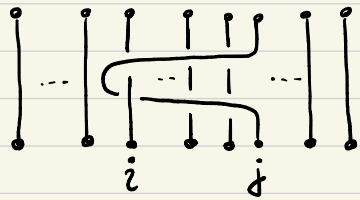
$\Big|$ $(p+s)$-braid which projects to the trivial p-braid 

$\Big\}$

$\cup$

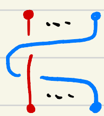
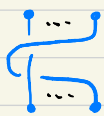
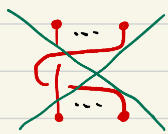
$PB_{p,s} = \mathrm{Ker}\left(B_{p,s} \longrightarrow \mathfrak{S}_s\right)$ the pure part

Recall: generators for PB_n



α_{ij}

Presentation of $PB_{p,s}$ (Lambropoulou)

generators  ,  ()

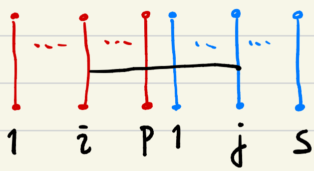
relations The pure group relations among the above α_{ij}'s

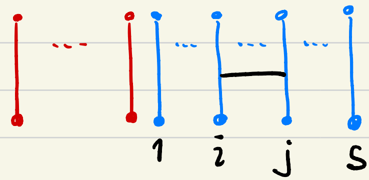
$\mathcal{A}_{p,s} := \mathrm{gr}\, \mathbb{Q} PB_{p,s}$ (w.r.t. the aug. ideal)

$\mathcal{A}_{p,s} \supset DK_{p,s}$ the primitives

$DK_{p,s}$

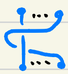
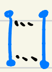
<u>generators</u> a_{ij} $\begin{cases} 1 \le i \le p \\ 1 \le j \le s \end{cases}$, c_{ij} $1 \le i < j \le s$





<u>relations</u> 4T, $\left[\begin{matrix} a_{ij} \\ c_{ij} \end{matrix}, c_{kl} \right] = 0$

<u>emergent braids</u>

$J := \langle$  $-$  $\rangle \subset \mathbb{Q}B_{p,s}$

$\mathbb{Q}B^{1/2}_{p,s} := \mathbb{Q}B_{p,s} / J^2$, $\quad \mathbb{Q}PB^{1/2}_{p,s} := \mathbb{Q}B_{p,s} / (J^2 \cap \mathbb{Q}PB_{p,s})$

$\mathcal{EA}_{p,s} := \mathrm{gr}\, \mathbb{Q}PB^{1/2}_{p,s}$ $\qquad \mathcal{EA}_{p,s} \supset EDK_{p,s}$ the primitives

$EDK_{p,s}$

<u>generators</u> the same as $DK_{p,s}$

<u>relations</u> the same as $DK_{p,s}$ + $\deg_c \le 1$

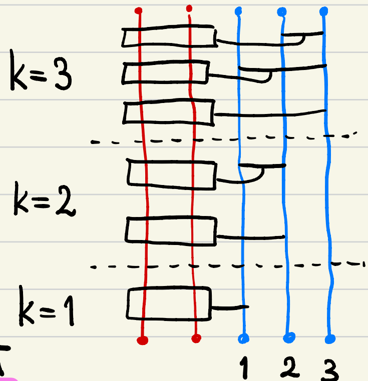
Combing

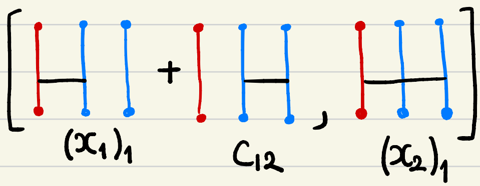
$$FL = FL(x_1, \dots, x_p), \qquad FA = FA(x_1, \dots, x_p)$$

$$EDK_{p,s} \underset{\text{u.s.}}{\cong} \bigoplus_{1 \le k \le s} \Big(FL \oplus \bigoplus_{j<k} FA \Big)$$

$$\mathcal{U}(x_1, \dots, x_p)_k \longleftrightarrow \mathcal{U}(a_{1k}, \dots, a_{pk})$$

$$(x_{i_1} \cdots x_{i_m})_{jk} \longleftrightarrow [a_{i_1 k}, \cdots [a_{i_m k}, c_{jk}] \cdots]$$





$= 0$

$$\rightsquigarrow [(x_1)_1, (x_2)_1] = [-(x_2)_1, c_{12}]$$

Lie bracket

- FL vs. FL : $\begin{cases} [u_j, v_k] = \Big(\sum_i (\partial_i v) x_i \overline{(\partial_i u)} \Big)_{jk} & (j<k), \\ [u_k, v_k] = [u, v]_k \end{cases}$

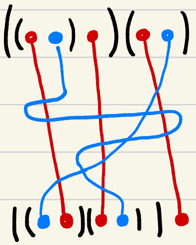
- FL vs. FA :

$$[u_i, w_{jk}] = \begin{cases} 0 & i \notin \{j,k\} \\ -(wu)_{ij} & i = j \\ (uw)_{ij} & i = k \end{cases}$$

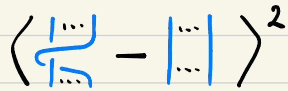
- FA vs. FA : $[FA, FA] = 0$

§ Category of emergent braids

PaEB object: $((\bullet\ \bullet)\ \bullet)(\bullet\ \bullet)$ with labels $1\ 2\ 2\ 3\ 1$ — par. of p poles and s strands

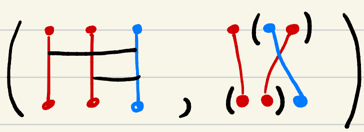
morphism: 

$$\frac{\mathbb{Q}\left\{\begin{array}{l}\text{par. } (p+s)\text{-braids which project} \\ \text{to the trivial par. } p\text{-braids}\end{array}\right\}}{\text{(see figure below)}}$$



PaECD object: the same as PaEB

morphism: $\mathcal{E}A_{p,s} \times \left\{\begin{array}{l}\text{par. permutations of } p \text{ poles and} \\ s \text{ strands which project to the} \\ \text{trivial par. perm. of } p \text{ poles}\end{array}\right\}$

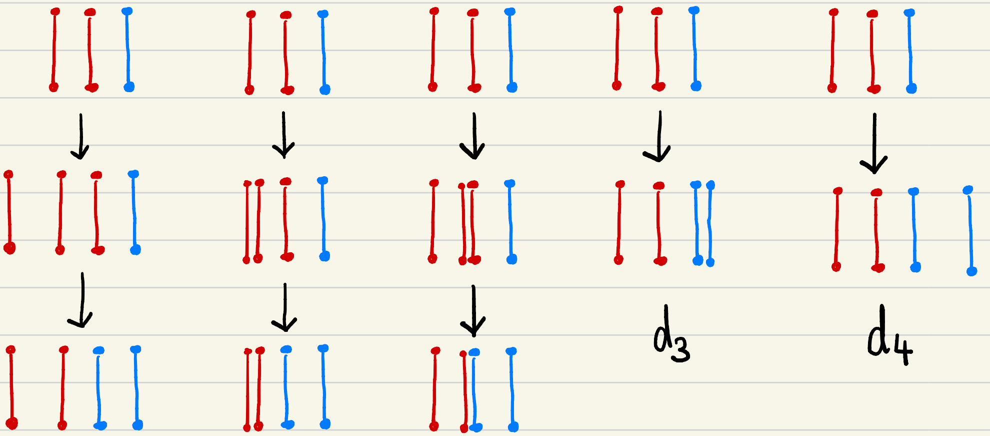


operations

- adding a pole or a strand to the left / right
- doubing a pole or a strand
- changing a pole to a strand
- deleting a pole or a strand

coface maps $\quad d_i : EDK_{p,s} \longrightarrow EDK_{p,s+1} \quad i = 0, 1, \cdots, p+s+1$

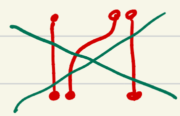
<u>Example</u>: $p=2,\ s=1 \qquad d_i : EDK_{2,1} \longrightarrow EDK_{2,2}$

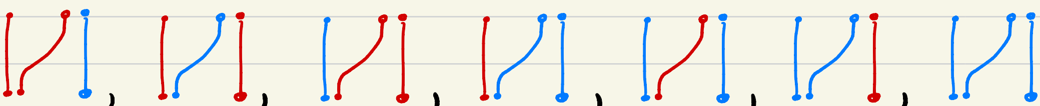


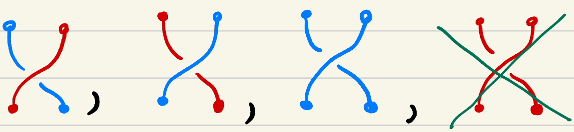
$d_0 \qquad d_1 \qquad d_2 \qquad d_3 \qquad d_4$

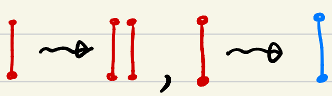
<u>Problem</u> Construct an expansion $Z : PaEB \xrightarrow{\cong} PaECD$

building blocks of PaEB



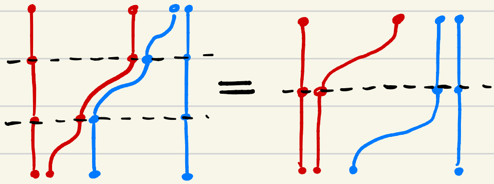




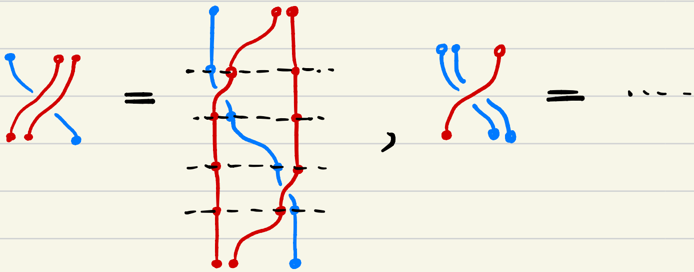
and applying  etc., we get more.

<u>relations</u> 5-gons and 6-gons with various colors

pps 5-gon :



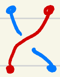
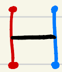
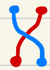
6-gon :

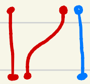


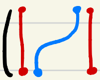
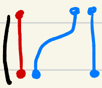
<u>Problem</u> Give a presentation of PaEB. (We will need a pole-strand version of the coherence theorem.)

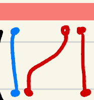
§ 5-gon and 6-gon equations

An expansion $Z : PaEB \longrightarrow PaECD$ will be specified by

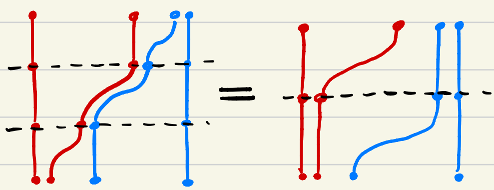
$Z\Big($  $\Big) = \Big(\exp\big(\tfrac{1}{2}$  $\big),$  $\Big),$ $\qquad x = x_1 \quad y = x_2$

$\Phi_{pps} := Z\Big($  $\Big) \in \exp(EDK_{2,1}) \equiv \exp(FL(x,y)),$

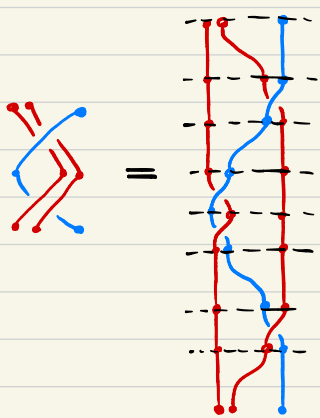
$\Phi_{psp} := Z\Big($  $\Big),\quad \Phi_{pss} := Z\Big($  $\Big),$ etc.

Assume $\Phi_{spp} = Z\Big($  $\Big) = \Phi_{pps}(y,x)^{-1}$ and focus on

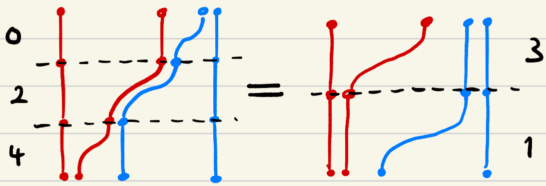
pps 5-gon:



pps doubled 6-gon:



pps 5-gon



$\Phi = \Phi_{pps}$

$$\rightsquigarrow \quad d_4\Phi \cdot d_2\Phi \cdot d_0\Phi = d_1\Phi \cdot d_3\Phi$$

$$\overset{\text{linearization}}{\rightsquigarrow} \quad d_4\varphi + d_2\varphi + d_0\varphi = d_1\varphi + d_3\varphi$$

<u>Key term</u> $\quad d_3\varphi = \varphi_1 + \varphi_2 + R(\varphi)_{12}$,

where $R : FL \longrightarrow FA$ is a unique map such that

$$R(x_i) = 0 \quad \& \quad R([u,v]) = \sum_i \left((\partial_i v) x_i \overline{(\partial_i u)} - (\partial_i u) x_i \overline{(\partial_i v)} \right)$$

$x, y, \ldots = x_1, x_2, \ldots$

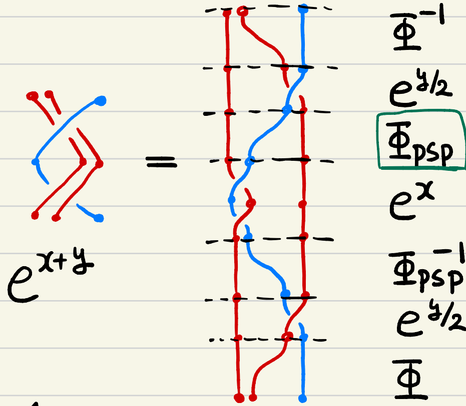
linearized pps 5-gon $\Longleftrightarrow$

$$\underbrace{\varphi(x,y)_1}_{d_4\varphi} + \underbrace{\left(\varphi(x,y)_2 + (\partial_y\varphi)(x,y)_{1,2}\right)}_{d_2\varphi} + \underbrace{\left(\varphi(y,0)_2 + (\partial_y\varphi)(y,0)_{1,2}\right)}_{d_0\varphi}$$

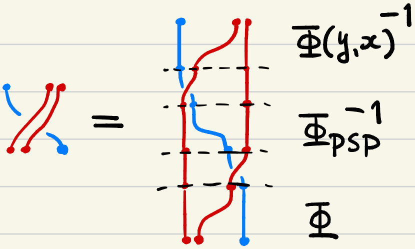
$$= \underbrace{\left(\varphi(x+y,0)_2 + (\partial_y\varphi)(x+y,0)_{1,2}\right)}_{d_1\varphi} + \underbrace{\left(\varphi(x,y)_1 + \varphi(x,y)_2 + R(\varphi)_{1,2}\right)}_{d_3\varphi}$$

$$\Longleftrightarrow \begin{cases} \varphi(y,0) - \varphi(x+y,0) = 0 \\ (\partial_y\varphi)(x,y) + (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0) - R(\varphi) = 0 \end{cases}$$

pps doubled 6-gon



... ✩



$$\rightsquigarrow \Phi_{psp} = e^{x/2}\, \Phi(y,x)^{-1} e^{\frac{-(x+y)}{2}}\, \Phi\, e^{y/2}$$

✩ $\Longleftrightarrow$

$$e^{x+y} = \exp\Big(\exp\big(\mathrm{ad}_{\frac{x+y}{2}}\big)\exp\big(\mathrm{ad}_{\varphi(y,x)}\big)(x)\Big) \times \exp\big(\mathrm{ad}_{\varphi(x,y)}\big)(y)$$

linearization $\rightsquigarrow$ $0 = [\varphi(y,x), x] + [\varphi(x,y), y]$

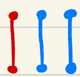
Summary linearized pps 5-gon and doubled 6-gon

$$\Longleftrightarrow \left\{ \begin{array}{l} \left\{ \begin{array}{ll} \varphi(y,0) - \varphi(x+y,0) = 0 & \cdots (P1) \\ (\partial_y\varphi)(x,y) + (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0) - R(\varphi) = 0 & \cdots (P2) \end{array} \right. \\ 0 = [\varphi(y,x), x] + [\varphi(x,y), y] \qquad \cdots (H) \end{array} \right.$$

Computer experiment : up to degree 17,

$$\{\text{solutions to } (P2)\} = \{\text{solutions to } (P1), (P2), (H)\}$$
$$= FL(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \cdots)$$

Tentatively, $\boxed{\mathfrak{grt}_1^{EM} := \{\text{solutions to } (P1), (P2), (H)\}}$

<u>Rem</u> For the pss case , $\Phi = \Phi_{pss} \in \exp(EDK_{1,2}) \cong \exp(\mathbb{Q}[[x]]_{\geq 1})$

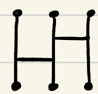
$$\{\text{sol. to 5-gon}\} = \mathbb{Q}[[x]]_{\geq 1}$$

$$\left\{\text{sol. to } \begin{matrix}\text{5-gon} \\ \text{doubled 6-gon}\end{matrix}\right\} = \left\{ f(x) \;\middle|\; f_{odd}(x) = \frac{1}{2}\sum_{k\geq 2} \frac{B_k}{k!} x^{k-1} \right\}$$

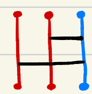
§ Relation to KV

$$\begin{tikzcd}[column sep=small]
 & \left\{\begin{matrix}\text{Sol. to various}\\ \text{pole-strand 5-gons}\\ \text{\& 6-gons}\end{matrix}\right\} \arrow[d, dashed] \arrow[r, phantom, "\underset{\text{hope}}{\cong}"] & \left\{\begin{matrix}\text{expansions}\\ \text{for PaEB}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{Drinfeld}\\ \text{associators}\end{matrix}\right\} \arrow[ur, dashed] \arrow[r] \arrow[d, squiggly, "\text{linearization}"] & \left\{\begin{matrix}\text{Sol. to PPS 5-gon}\\ \text{\& doubled 6-gon}\end{matrix}\right\} \arrow[d, squiggly, "\text{linearization}"] & \\
\mathfrak{grt}_1 \arrow[r, hook] \arrow[d, "\nu"', "\text{Alekseev-Torossian}"] & \mathfrak{grt}_1^{EM} \arrow[dl, dashed, bend left, "?"] & \\
\mathfrak{krv}_2^{sym} & &
\end{tikzcd}$$

$\psi(x,y) \longmapsto \psi(-x-y,y)$



$x = t_{12}$, $y = t_{23}$



$x = t_{13}$, $y = t_{23}$

Thm (Bar-Natan - K.)

(1) There is an embedding $\nu^{EM}: \mathfrak{grt}_1^{EM} \hookrightarrow \mathfrak{krv}_2$,
$\varphi \longmapsto (\varphi(y,x), \varphi(x,y))$ compatible with ν.

(2) In deg ≥ 3, $\operatorname{Im}\nu^{EM} = \mathfrak{krv}_2^{sym}$

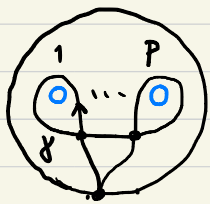
(y,x) deg 1 elm. in $\mathfrak{krv}_2^{sym}$

§ Details about the proof

The map R and self-intersection of loops

(P2) $(\partial_y\varphi)(x,y) + (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0) - R(\varphi) = 0$

$\pi = \pi_1(D^2 - p \text{ pts})$



Def (based loop ver. of the Goldman bracket, the Turaev cobracket)

$\eta : \mathbb{Q}\pi^{\otimes 2} \to \mathbb{Q}\pi, \qquad \eta(\alpha,\beta) = \sum_{p\in\alpha\cap\beta} \varepsilon_p\, \alpha_{*p}\beta_{p*}$

$\mu_0 : \mathbb{Q}\pi \to \mathbb{Q}\pi, \qquad \mu_0(\gamma) = \sum_{p\in \mathrm{Self}(\gamma)} \varepsilon_p\, \gamma_{*t_1^p}\, \gamma_{t_2^p *}$

Rem • η & μ_0 are equivalent to κ & μ in AKKN

• μ_0 recovers the (framed) Turaev cobracket

$\rightsquigarrow$ induced operations on $\mathrm{gr}\,\mathbb{Q}\pi = FA = FA(z_1,\cdots,z_p)$

Prop $R = -(\mu_0)_{gr} : FL \to FA$ (true for any p)

KV equations and loop operations

$$\mathrm{gr}\,\mathbb{Q}\pi = FA = FA(z_1, \cdots, \underset{P}{z_n}), \qquad z_0 = z_1 + \cdots + z_n$$

$$\mathrm{tder}_n \supset \mathrm{sder}_n \supset \mathrm{krv}_n \supset \mathrm{krv}_n^0$$

sder_n: $\boxed{\tilde{u}(z_0) = 0}$

krv_n: $\boxed{\mathrm{div}(\tilde{u}) \in \mathrm{Span}(|z_i^m|)}$

krv_n^0: $\boxed{\mathrm{div}(\tilde{u}) \in \mathrm{Span}(|z_i|)}$

$\Sigma = \Sigma_{g,1}$ general case

Thm (Massuyeau-Turaev, Naef)

$\tilde{u} \in \mathrm{sder}_n \iff \tilde{u}$ commutes with η_{gr}

Thm (AKKN)

framed Turaev cobracket

(1) $\tilde{u} \in \mathrm{krv}_n \iff \tilde{u}$ commutes with η_{gr} & δ^f_{gr}

(2) $\tilde{u} \in \mathrm{krv}_n^0 \iff$ ——— " ——— η_{gr} & $(\mu_0)_{gr}$

Outline of proof

Thm (1) : $\varphi \in \mathfrak{grt}_1^{EM} \Longrightarrow \nu^{EM}(\varphi) = \big(\varphi(y,x), \varphi(x,y)\big) \in \mathfrak{krv}_2$

$$(P1)(P2)(H) \Longleftrightarrow \nu^{EM}(\varphi) \in \mathfrak{sder}_2$$

Lem 1 If $\tilde{u} \in \mathfrak{sder}_n$, then $\mu_0 \circ \tilde{u} - \tilde{u} \circ \mu_0$ is a derivation on FA.

Lem 2 If $\tilde{u} \in \mathfrak{sder}_n$ and $\mu_0(\tilde{u}(z_i)) = [z_i, c]$ $1 \le i \le n$ for some $c \in$ FA, then $\tilde{u}$ commutes with δ_{gr}^f.

$$\begin{aligned}
\underbrace{R}_{= -(\mu_0)_{gr}}(\nu^{EM}(\varphi)(y)) &= R([y, \varphi(x,y)]) \\
&= [y, R(\varphi)] + (\partial_y\varphi)y - y\overline{(\partial_y\varphi)} \quad \overset{(H)}{=} \partial_y\varphi \\
&= [y, R(\varphi) - \partial_y\varphi] \\
&\overset{(P2)}{=} [y, (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0)] \\
&= [y, -(\partial_y\varphi)(x+y,0)]
\end{aligned}$$

Similarly, $R(\nu^{EM}(\varphi)(x)) = R([x, \varphi(y,x)]) = \cdots = [x, -(\partial_y\varphi)(x+y,0)]$

Applying Lem 2 to $\tilde{u} = \nu^{EM}(\varphi)$, one obtains $\nu^{EM}(\varphi) \in \mathfrak{krv}_2$ //

Thm (2) : $\mathrm{Im}\, \nu^{EM} \supset \mathfrak{krv}_2^{sym}$

Let $\tilde{u} = (\varphi(y,x), \varphi(x,y)) \in \mathfrak{krv}_2^{sym}$ of deg ≥ 3.

<u>Case 1</u> $\tilde{u} \in \mathfrak{krv}_2^{0}$. Then, $\tilde{u}$ commutes with $(\mu_0)_{gr} = -R$.

$$0 = \tilde{u}(R(y)) = R(\tilde{u}(y)) = R([y,\varphi])$$

$$= [y, R(\varphi)] + (\partial_y \varphi) y - y\overline{(\partial_y \varphi)} = [y, R(\varphi) - \partial_y \varphi]$$

$$\rightsquigarrow R(\varphi) - \partial_y \varphi \in \mathbb{Q}[[y]]_{\geq 2}$$

$$\xrightarrow[\text{more analysis}]{} R(\varphi) - \partial_y \varphi = 0 \text{ and } (\partial_y \varphi)(x,0) = 0 \rightsquigarrow (P2) \text{ for } \varphi$$

<u>Case 2</u> general case. Let $\tilde{u}$ be homogeneous of deg $l \geq 3$.

<u>l: even</u> $\rightsquigarrow \mathrm{div}(\tilde{u}) = 0$ (cf. AT Prop 4.5)

$\rightsquigarrow \tilde{u} \in \mathfrak{krv}_2^{0}$ Case 1 applies

<u>l: odd</u> $\exists c \in \mathbb{Q}$ $\mathrm{div}(\tilde{u}) = c\,|x^l + y^l - (x+y)^l|$

Let $\sigma_l \in \mathfrak{grt}_1$ be the Drinfeld-Ihara generator.

$$\mathrm{div}(\nu(\sigma_l)) = |x^l + y^l - (x+y)^l| \quad \text{(cf. AT Prop 4.10)}$$

$$\begin{array}{ccc} \sigma_l \in \mathfrak{grt}_1 & \hookrightarrow & \mathfrak{grt}_1^{EM} \ni \psi_l \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\nu^{EM}} & \\ & \mathfrak{krv}_2 & \end{array} \qquad \sigma_l \longmapsto \psi_l$$

$$\varphi = (\varphi - c\psi_l) + c\psi_l \in \mathfrak{grt}_1^{EM}$$

$$\begin{array}{ccc} (\varphi - c\psi_l) & & c\psi_l \\ \updownarrow & & \cap \\ \tilde{u} - c\nu(\sigma_l) & & \mathfrak{grt}_1^{EM} \\ \cap & & \\ \mathfrak{krv}_2^{sym} \cap \mathfrak{krv}_2^{0} & & \end{array}$$

//