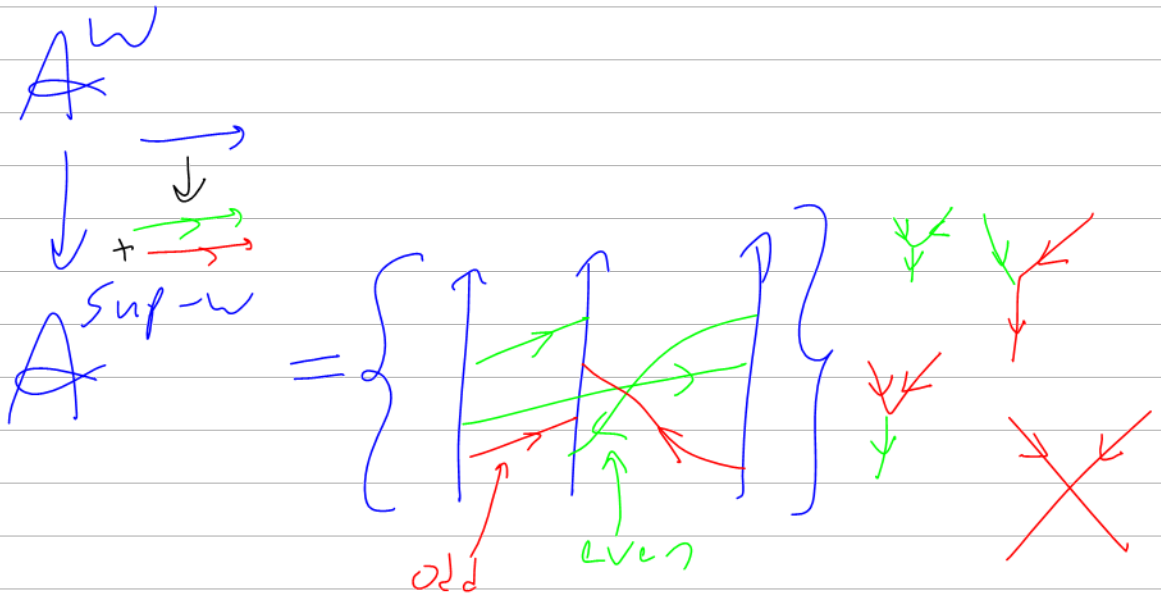
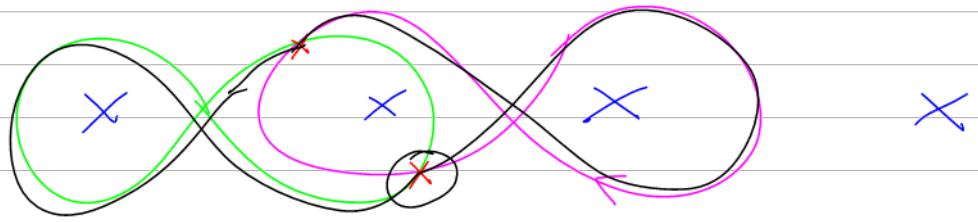
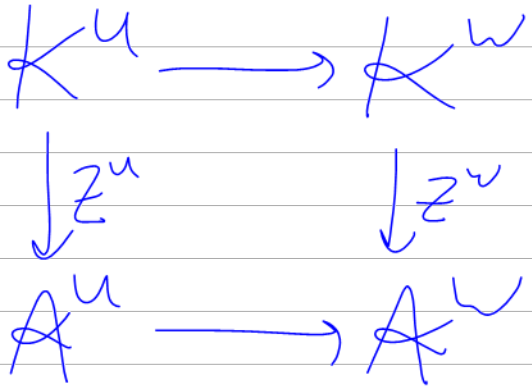
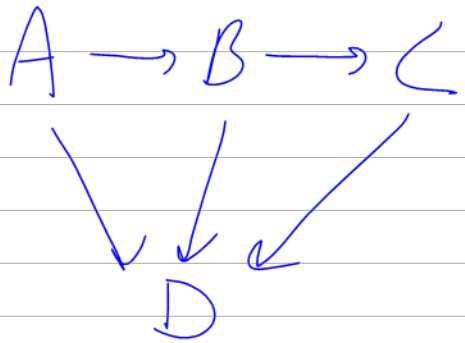


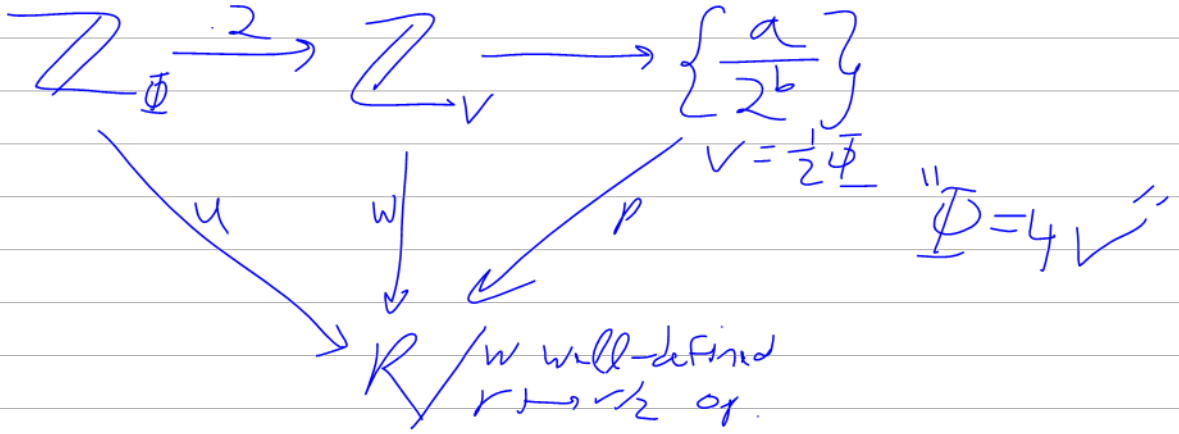
$A \in M_{n\times n}(R)$ $\begin{pmatrix} 1+\epsilon & \epsilon \\ \epsilon & 1 \end{pmatrix}$

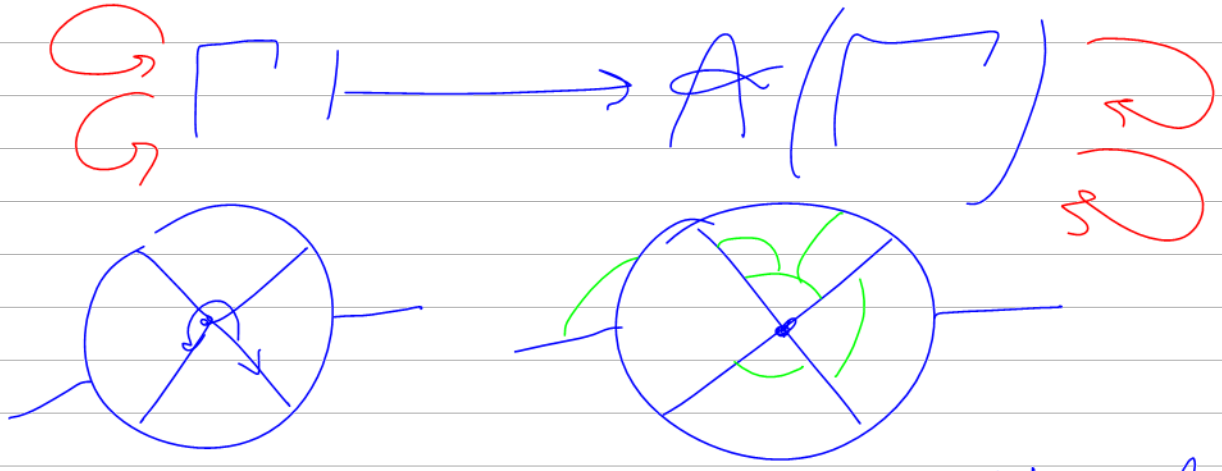
$\det(1-\epsilon(1-A))$ using Gaussian elimination in $R[[\epsilon]]/\epsilon^{n+1}$





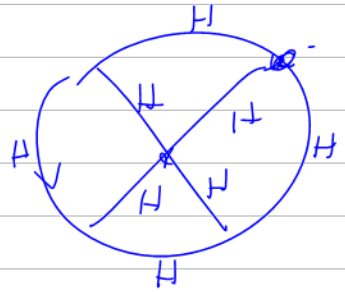



$\mathbb{Z}_\Phi$
2
$\mathbb{Z}_V$
$\left\{\frac{a}{2^b}\right\}$
$V=\frac{1}{2}\Phi$
$\Phi=4V$
u
w
p
R
w well-defined
$r \mapsto r/2$ of.

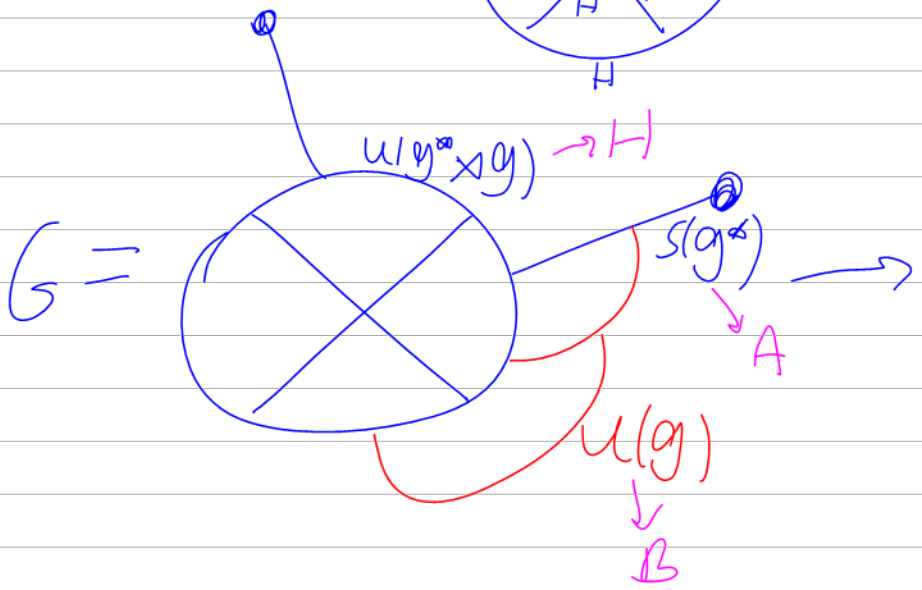


Given a Hopf algebra $\left(\begin{array}{l}\text{involutive}\\ \text{co-commutative}\\ S^2=I\end{array}\right)$

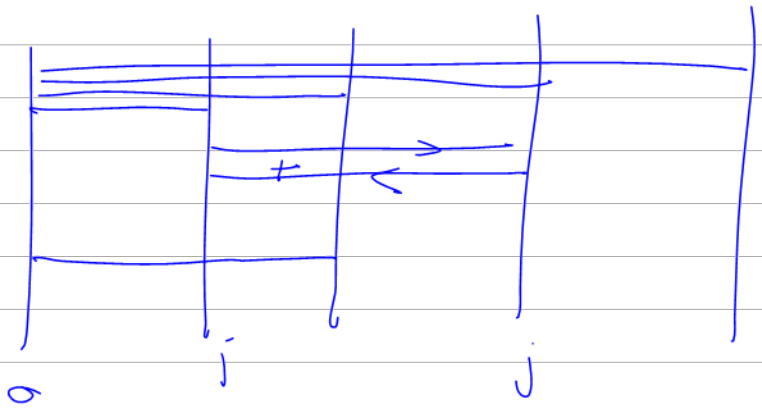
$G \longmapsto H(G)$



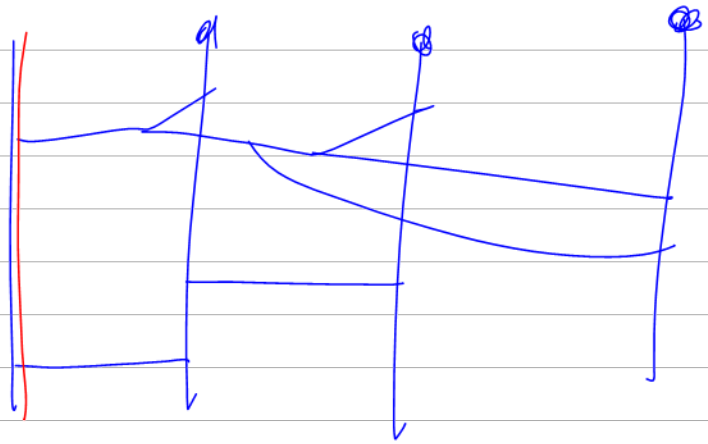
$= H^{\otimes |E|} / H^{\otimes V}$

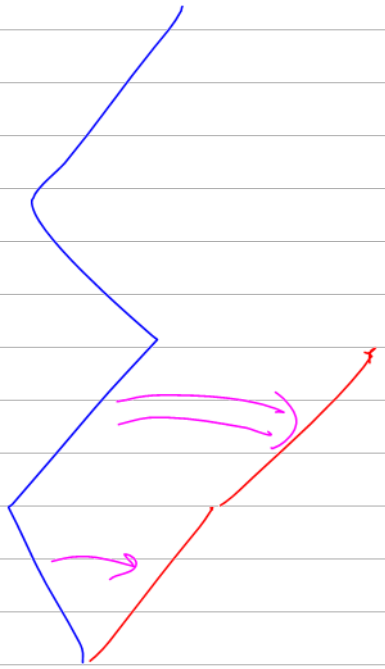


$\longrightarrow A^V(G)$



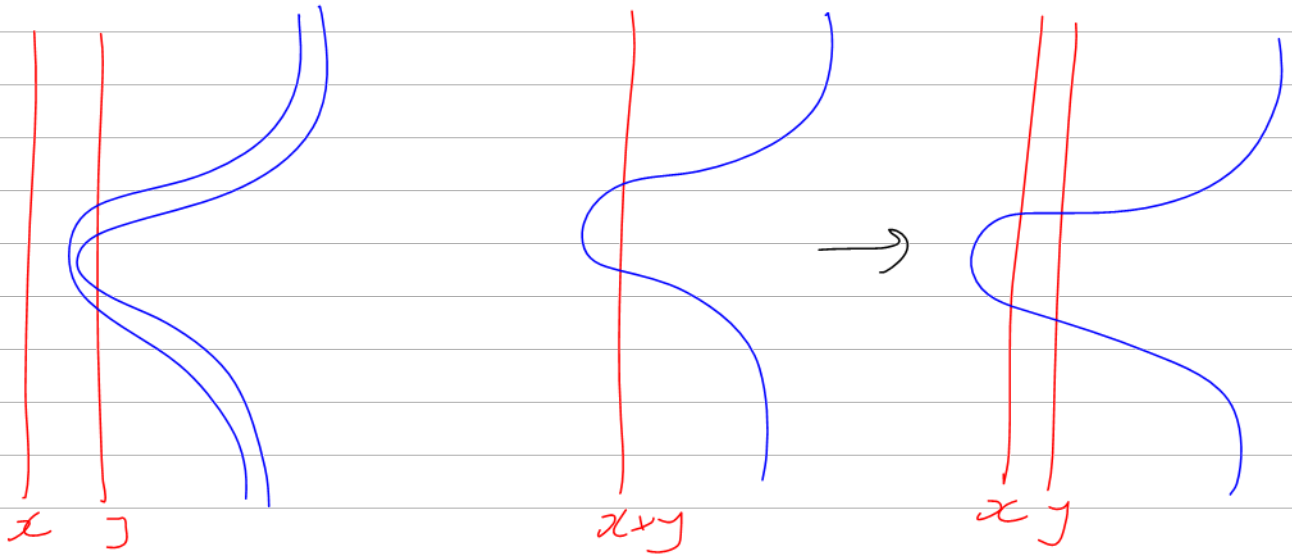
$f_{ij} = FL(t_{ij})$





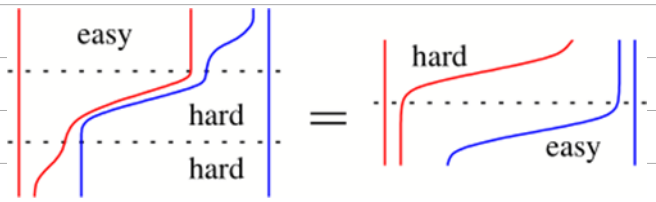
$GRT_7 \Rightarrow GRT_6$

$grt_7 \overset{?}{\Rightarrow} grt_6$



$$A \oplus A \otimes_A A$$

0 1


easy
hard
hard
=
hard
easy

$\varphi_a : x \to [x, a]$

$\varphi_b : x \mapsto [x, b]$

$$[\varphi_a, \varphi_b](x) = \varphi_a(\varphi_b(x)) - \varphi_b(\varphi_a(x))$$

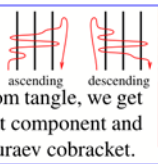
$$= [[x, b], a] - [[x, a], b]$$

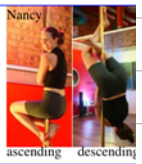
$$= -[[b, a], x] = -[x, [a, b]] = -\varphi_{[a,b]}(x)$$

$$[D, D'](x_i) = (DD' - D'D)(x_i) = \underline{D[x_i, a_i']} - D'[x_i, a_i] =$$
$$= [[x_i, a_i], a_i'] + [x_i, Da_i'] - [[x_i, a_i'], a_i] - [x_i, D'a_i] = [x_i, Da_i' - D'a_i + [a_i, a_i']].$$

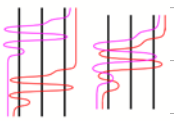
$[Dx_i, a_i'] + [x_i, Da_i']$

Example 3. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3 : \bar{\pi} \to \bar{\pi} \otimes |\bar{\pi}|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.



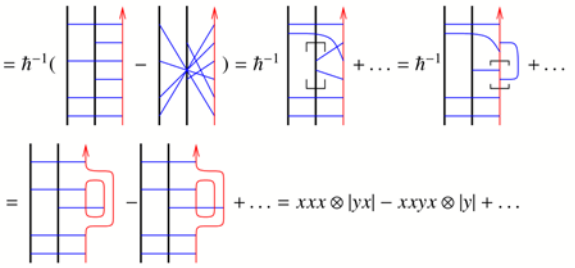


Example 4 [Ma]. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4 : \bar{\pi} \to \bar{\pi} \otimes \bar{\pi}$. After some massaging, it too becomes the Turaev cobracket.



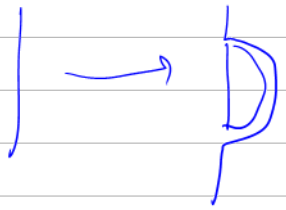
Applying ∂_r

Example 3^a. ~~Ignoring complications,~~ $\eta_3^a(xxyxyx) =$



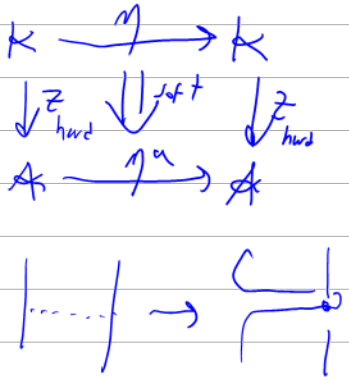
$\eta_3 : \overline{\pi}_{\geq t} \to (\overline{\pi} \otimes |\overline{\pi}|)_{\geq t-1}$ degree decreasing filtered.

$gr_{-1}\eta_3 : \frac{\overline{\pi}_{\geq t}}{\overline{\pi}_{\geq t+1}} \to \frac{(\overline{\pi} \otimes |\overline{\pi}|)_{\geq t-1}}{(\overline{\pi} \otimes |\overline{\pi}|)_{\geq t}}$



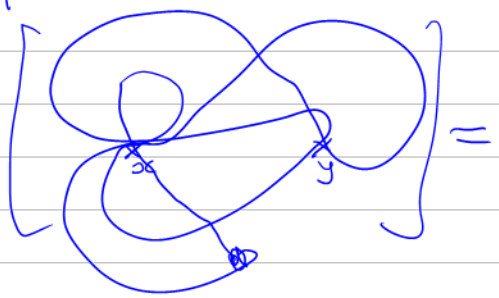
$\parallel$ is

$A \xrightarrow{\eta_a^3} A \otimes |A|$ of degree (-1)

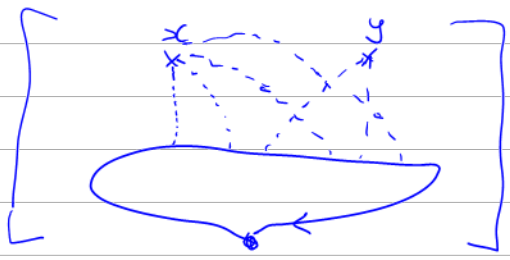


$\forall D \quad u[K_D] = [K_{u^a D}]$

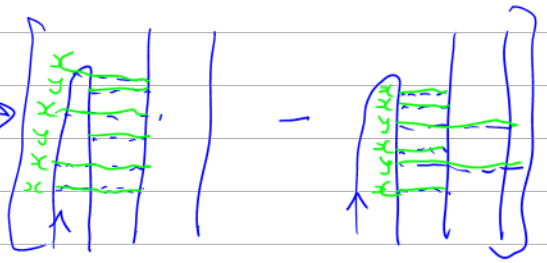
$D \rightsquigarrow xxyxyx \to K_{xxyxyx} =$



$=$



$\xrightarrow{\eta_3}$



$\cdots$

$\overline{\pi} \xrightarrow{\eta} \overline{\pi} \otimes |\overline{\pi}|$

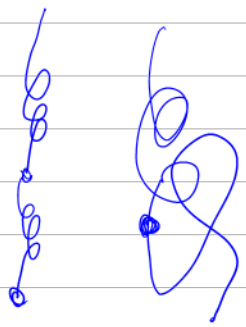
$\downarrow tr \qquad \downarrow tr /\!/ Alt$

$|\overline{\pi}| \xrightarrow{\delta} |\overline{\pi}| \otimes |\overline{\pi}|$

$\overline{\pi} \xrightarrow{\eta} \overline{\pi} \otimes |\overline{\pi}|$

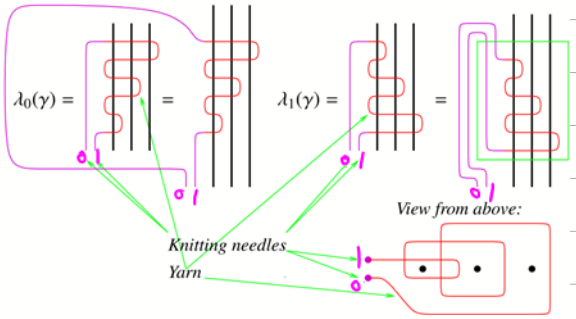
$\downarrow tr \qquad \downarrow tr$

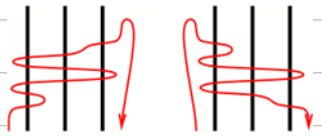
$|\overline{\pi}| \qquad |\overline{\pi}| \otimes |\overline{\pi}|$



Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. ~~Up to conjugation~~ temporarily $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.





$\eta = \hbar^{-1}(\lambda_0 - \lambda_1)$

$\exists \xi_i$ s.t. $Z(\lambda_i(\delta)) = \xi_i(W(\delta))$

$\overline{\pi} \xrightarrow{\tau \text{ (for Turaev)}} \overline{\pi} \otimes |\overline{\pi}|$

$\eta = \hbar^{-1}(\lambda_0 - \lambda_1) : \overline{\pi} \to \overline{\pi} \otimes |\overline{\pi}|$ the Turaev operation

$\delta = \eta /\!/ tr_1 /\!/ Alt \qquad tr_1 : \overline{\pi} \otimes |\overline{\pi}| \to |\overline{\pi}| \otimes |\overline{\pi}|$

$W /\!/ \delta^2 = W /\!/ \eta^2 /\!/ tr_1 /\!/ Alt = W /\!/ \frac{\lambda_0^2 - \lambda_1^2}{\hbar} /\!/ tr_1 /\!/ Alt$

$\delta /\!/ W = \delta /\!/ Z^{/1} = \frac{\lambda_0 - \lambda_1}{\hbar} /\!/ tr_1 /\!/ Alt /\!/ Z^{/1} = (\lambda_0 - \lambda_1) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1}$

$= (\lambda_0 - \lambda_0 /\!/ F /\!/ C) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1}$

$= (\lambda_0 - \lambda_0 /\!/ F) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1} = (\lambda_0 - \lambda_0 /\!/ F) /\!/ Z^{/2} /\!/ tr_1 /\!/ Alt /\!/ \hbar^{-1}$

$= \frac{1}{\hbar} \lambda_0 /\!/ Z^{/2} /\!/ (1 - F) /\!/ tr_1 /\!/ Alt = \frac{1}{\hbar} W /\!/ (1 - F) /\!/$

Key 1. $W\colon |\bar{\pi}| \to |A|$ is $Z^{/1}_H\colon \mathcal{K}^{/1}_H(\bigcirc) \to \mathcal{A}^{/1}_H(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar{\pi}| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar{\pi}|$,

Lemma 1. "Division by $\hbar$" is well-defined.

$$\eta(\gamma) \coloneqq (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}^{/1}_H(\bigcirc\bigcirc) = |\bar{\pi}| \otimes |\bar{\pi}|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ^a_i with $Z^{/2}(\lambda_i(\gamma)) = \lambda^a_i(W(\gamma))$), then η will have a compatible algebraic companion η^a:

$$\eta^a(\alpha) \coloneqq (\lambda^a_0(\alpha) - \lambda^a_1(\alpha))/\hbar \in \mathcal{A}^{/1}_H(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}^{/2}_H$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda^a_0(W(\gamma)) - \lambda^a_1(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

$K(S)$ $K(\infty)$ $K(/0)$

$\downarrow$

$F(S)$ $|\bar{\pi}| \times \bar{|\pi|}$

$K(S)$ knots

$G(S)$ curves $G(\infty) = |\bar{\pi}| \otimes |\bar{\pi}|$

$K(S)$

$\downarrow P$

$G(S)$

Example 1

$K(\infty) \xrightarrow{\lambda_0, \lambda_1} K(\infty)$

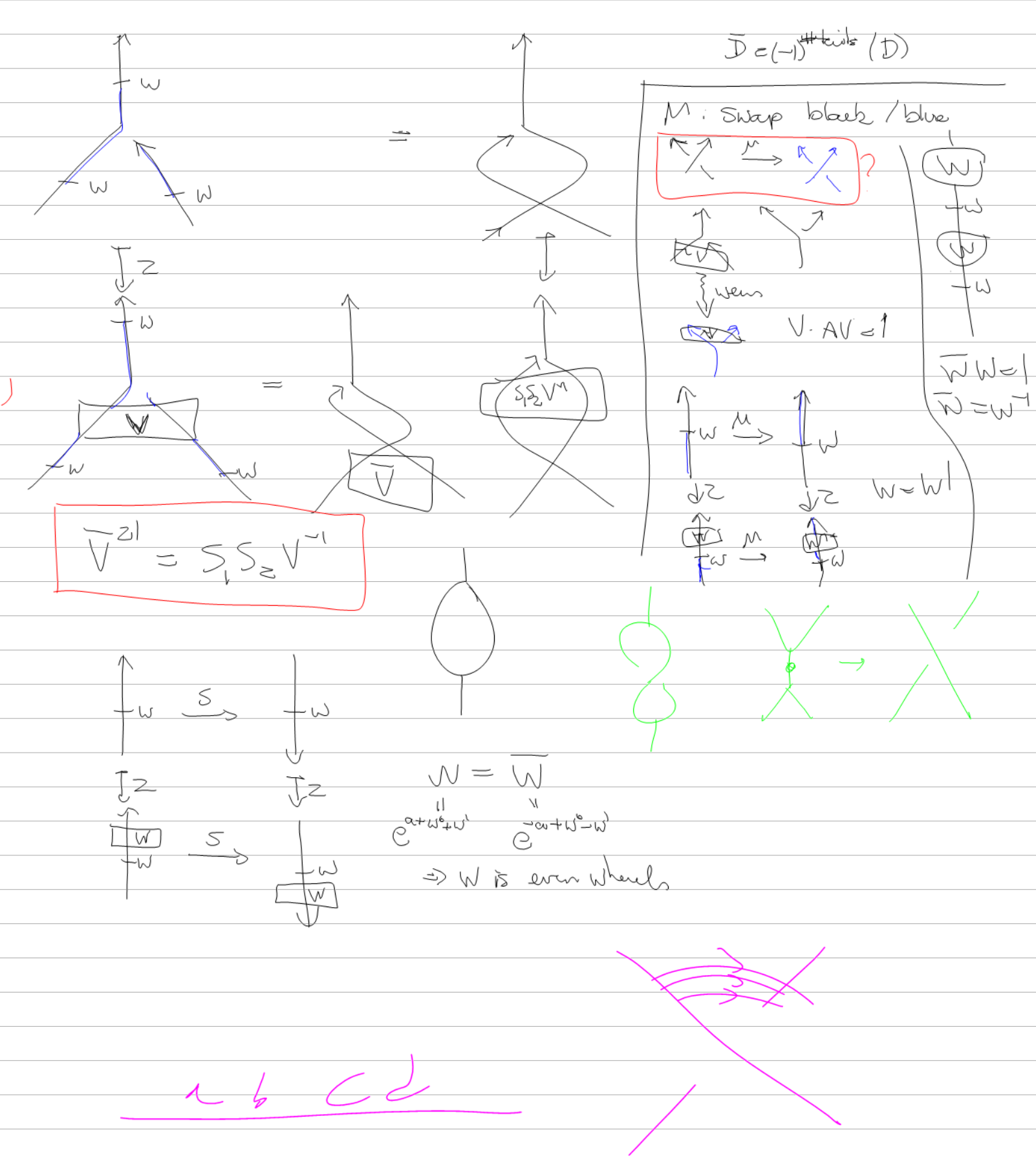
$\downarrow P$

$G(0,1) \xrightarrow{id} G(\infty)$

$K(\infty) \xrightarrow{\lambda_0, \lambda_1} K(\infty)$

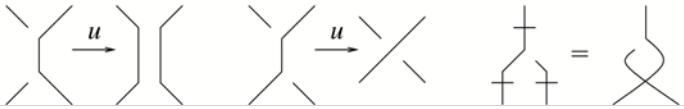
$\downarrow$ $\downarrow P$

$G(\infty) \xrightarrow{I} G(\infty)$


$\overline{D} = (-1)^{\#\text{twists}}(D)$
M: swap black / blue
$V \cdot AV = 1$
$\overline{V}W = 1$
$\overline{W} = W^T$
$W = \overline{W}$
$e^{a+w^0+w^1}$
$e^{-a+w^0-w^1}$
$\Rightarrow W$ is even wheels

231009 Best w-practices:

1. Trivalent tangles are end-labeled, to make a circuit algebra.
2. Tubes are bare (no colours and/or orientations).
3. Crossings have no signs; filtration is by comparison with virtual crossings.
4. Vertices are oriented and have marked legs: stem (s), upper (u), and lower (l). They are classical: they satisfy both R4s.
5. Wenjugating interchanges the two vertex types, and adds a virtual $l \leftrightarrow u$ crossing.
6. Only stems can be unzipped. Unzipping untwisted edges connects u to u above a connection of l to l. Unzipping through a wen is defined by wenjugating it out.



"R4 type"

$slu = usl = lus = l^{-1}s^{-1}u^{-1}$

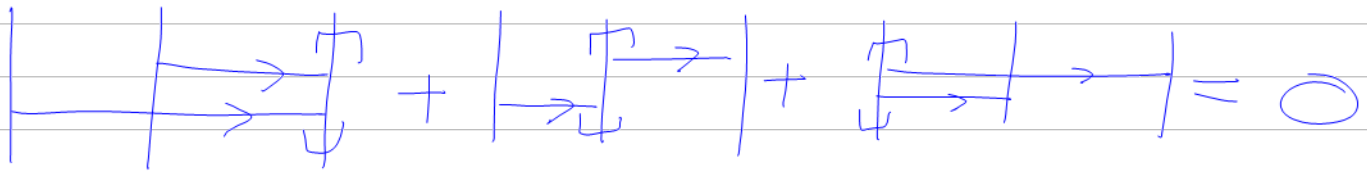
$slu \quad sul$

$s^{-1}lu \quad s^{-1}ul$

$sl^{-1}u \quad sl^{-1}u$

$slu^{-1} \quad slu^{-1}$

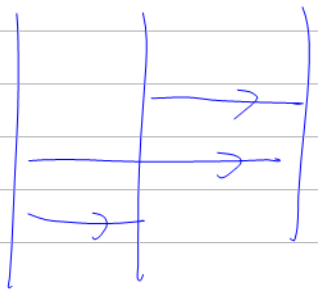
$slu \to s^{-1}l^{-1}u^{-1} = uls$



$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

$$(1+a^{12})(1+a^{13})(1+a^{23}) =$$

$$(1+a^{23})(1+a^{13})(1+a^{12})$$



What are all possible operations on 2D curves?