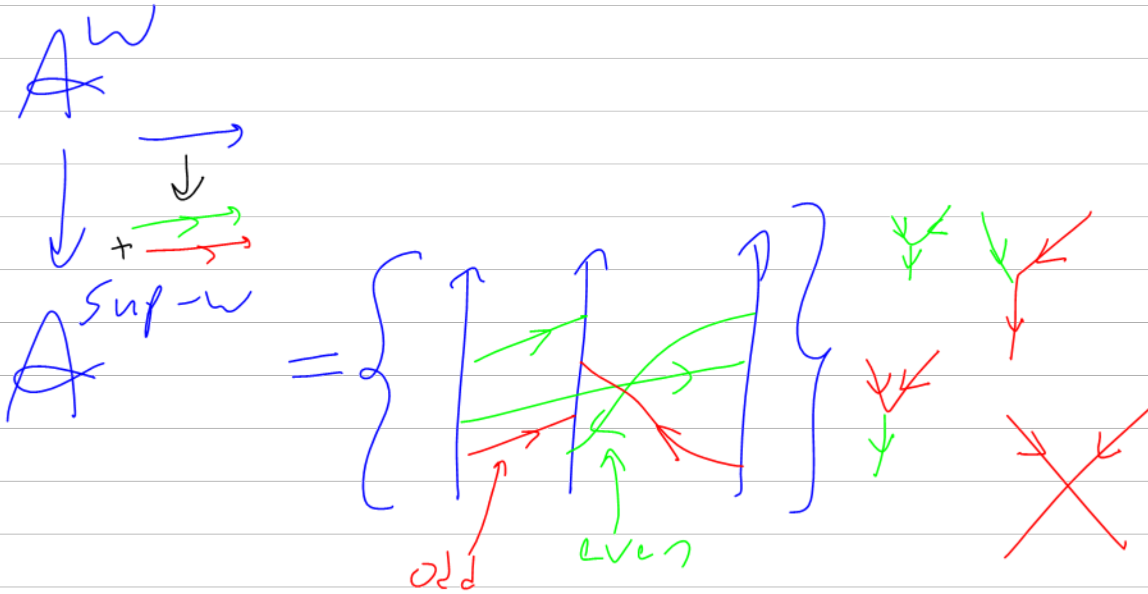
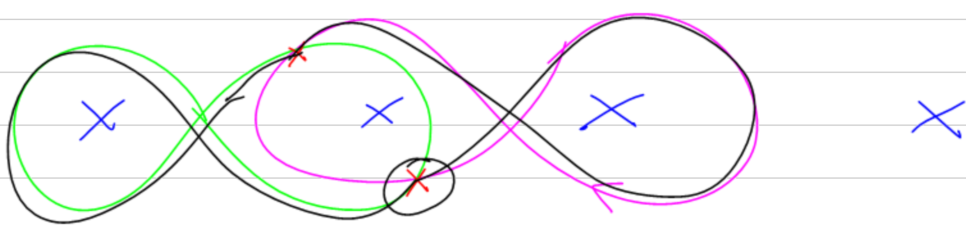


$A \in M_{n\times n}(R)$ $\quad \begin{pmatrix} 1+\epsilon & \epsilon & \\ \epsilon & 1 & \\ & & 1 \end{pmatrix}$

$\det(1-\epsilon(1-A))$ using Gaussian elimination in $R[\epsilon]/\epsilon^{n+1}$



$$
\begin{tikzcd}
K^u \arrow[r] \arrow[d, "Z^u"] & K^w \arrow[d, "Z^w"] \\
A^u \arrow[r] & A^w
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] & C \arrow[dl] \\
& D &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}_{\Phi} \arrow[r, "2"] \arrow[dr, "u"'] & \mathbb{Z}_{V} \arrow[r] \arrow[d, "w"] & \left\{ \frac{a}{2^b} \right\} \arrow[dl, "p"] \\
& \mathbb{R} &
\end{tikzcd}
$$

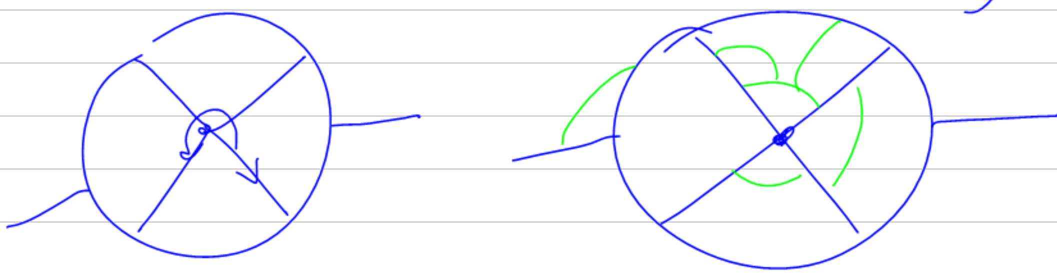
$V = \frac{1}{2}\Phi$

"$\Phi = 4V$"

$\mathbb{R}$ $\checkmark$ w well-defined

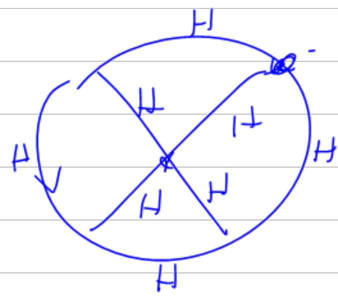
$r \mapsto r/2$ op.

$$\Gamma \longmapsto A(\Gamma)$$

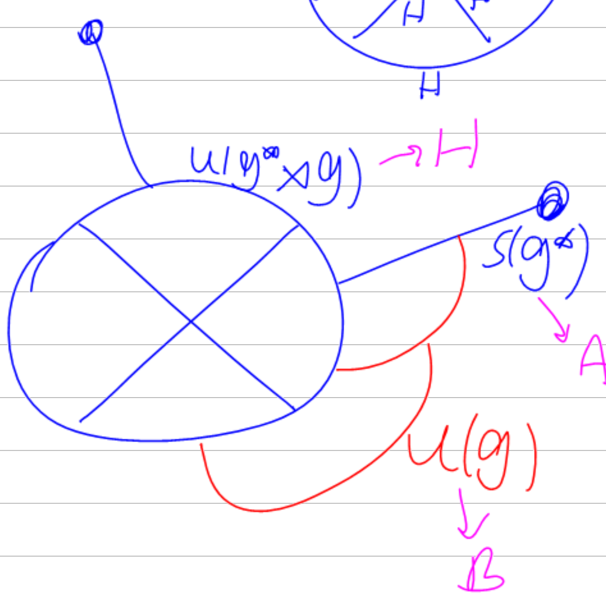


Given a Hopf algebra $\left(\begin{array}{c}\text{involutive} \\ \text{co-commutative} \\ S^2 = I\end{array}\right)$

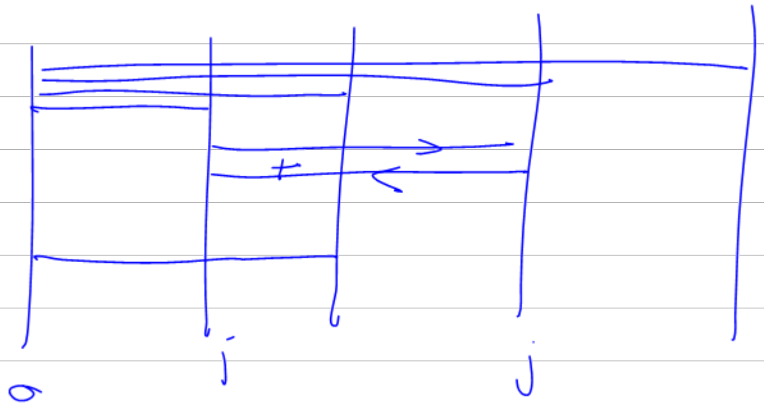
$$G \longmapsto H(G)$$



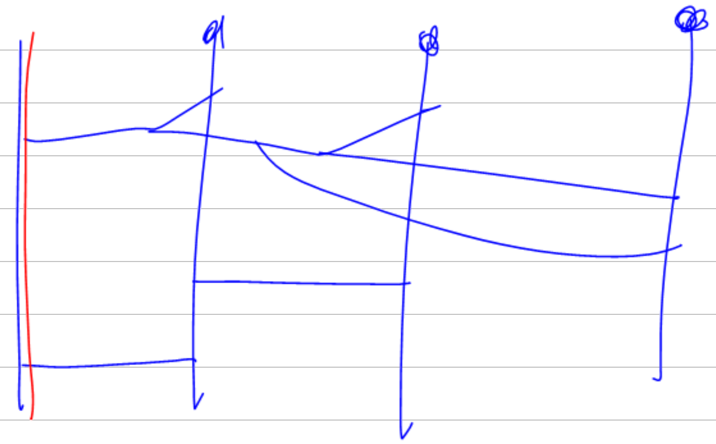
$= H^{\otimes E} / H^{\otimes V}$

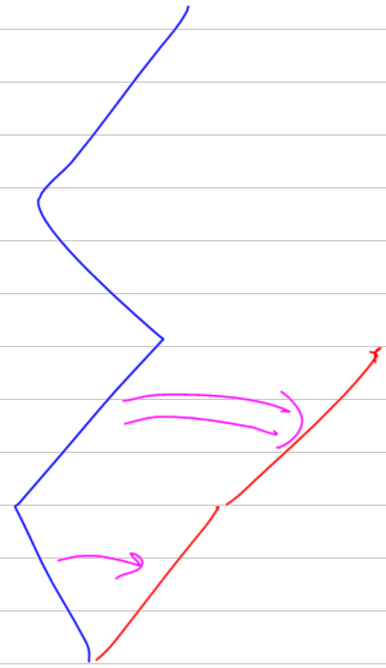
$G =$ 

$\longrightarrow A^\omega(G)$



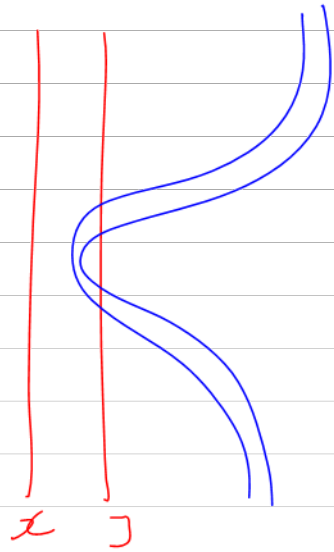
$f_n = FL\langle t_{0j} \rangle$

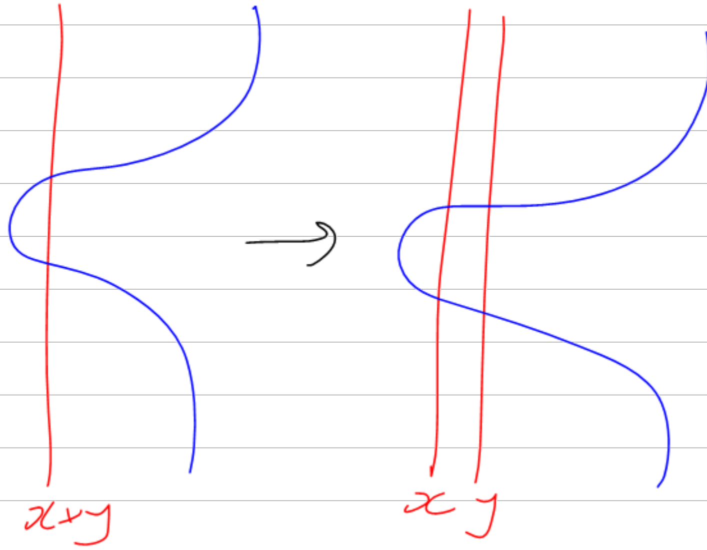




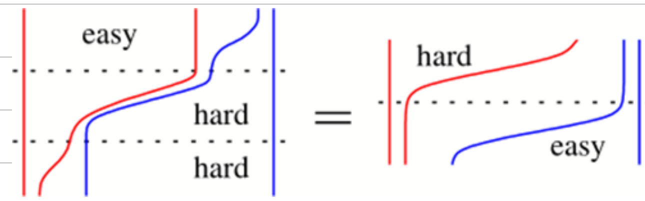
$$GRT_7 \twoheadrightarrow GRT_6$$

$$\mathfrak{grt}_7 \overset{?}{\twoheadrightarrow} \mathfrak{grt}_6$$





$$A_0 \oplus A_1 \otimes_A A$$


easy
hard
hard
hard
easy

$\varphi_a : x \to [x, a]$

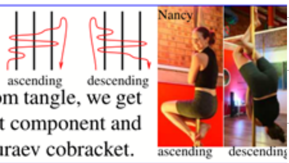
$\varphi_b : x \longmapsto [x, b]$

$$[\varphi_a, \varphi_b](x) = \varphi_a(\varphi_b(x)) - \varphi_b(\varphi_a(x))$$
$$= [[x,b],a] - [[x,a],b]$$
$$= -[[b,a],x] = -[x,[a,b]] = -\varphi_{[a,b]}(x)$$

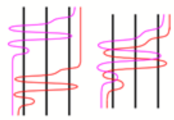
$$[D, D'](x_i) = (DD' - D'D)(x_i) = \underline{D[x_i, a_i']} - D'[x_i, a_i] =$$
$$= [[x_i, a_i], a_i'] + [x_i, Da_i'] - [[x_i, a_i'], a_i] - [x_i, D'a_i] = [x_i, Da_i' - D'a_i + [a_i, a_i']].$$

$[Dx_i, a_i'] + [x_i, Da_i']$

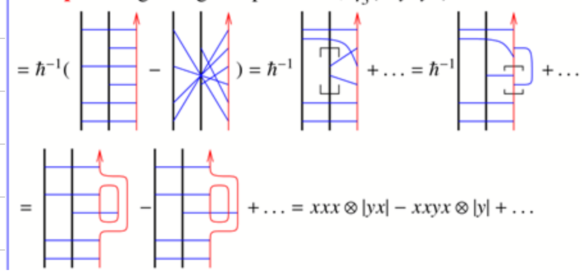
Example 3. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3 \colon \bar{\pi} \to \bar{\pi} \otimes |\bar{\pi}|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.



Example 4 [Ma]. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4 \colon \bar{\pi} \to \bar{\pi} \otimes \bar{\pi}$. After some massaging, it too becomes the Turaev cobracket.



Example 3^a. Applying gr ~~Ignoring complications~~, $\eta_3^a(xxyxyx) =$



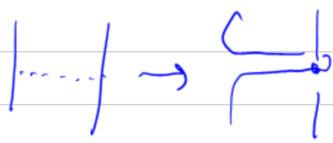
$+ \ldots = xxx \otimes |yx| - xxyx \otimes |y| + \ldots$

$\eta_3 : \bar{\pi}_{\geq t} \longrightarrow (\bar{\pi} \otimes |\bar{\pi}|)_{\geq t-1}$ degree decreasing filtered.

$$\mathrm{gr}_{-1}\eta_3 : \prod \frac{\bar{\pi}_{\geq t}}{\bar{\pi}_{\geq t+1}} \longrightarrow \prod \frac{(\bar{\pi} \otimes |\bar{\pi}|)_{\geq t-1}}{(\bar{\pi} \otimes |\bar{\pi}|)_{\geq t}}$$

$$\cong \qquad A \xrightarrow{\eta^3_a} A \otimes |A| \quad \text{of degree } (-1)$$

$$\begin{array}{ccc} K & \xrightarrow{\eta} & K \\ \downarrow Z_{hwd} & \Downarrow \text{soft} & \downarrow Z_{hwd} \\ A & \xrightarrow{\eta^a} & A \end{array}$$



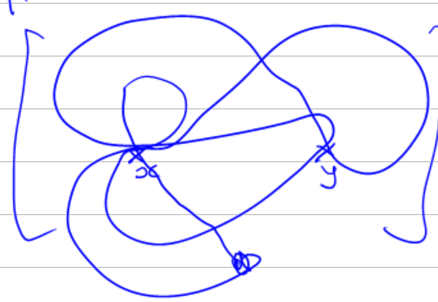
$$\begin{array}{ccc} K_1 & \xrightarrow{u} & K_2 \\ & \Downarrow & \\ \mathrm{gr} K_1 & \xrightarrow{\mathrm{gr}\, u} & \mathrm{gr} K_2 \\ s \uparrow & & \uparrow \\ A & \xrightarrow{u^a} & A \\ \downarrow & & \\ D & & \end{array}$$

$A \longrightarrow \mathrm{gr} K$

$D \longmapsto [K_D]$

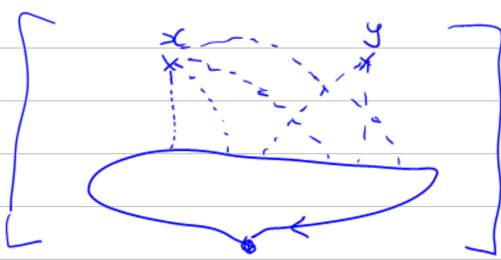
$\forall D \quad u[K_D] = [K_{u^a D}]$

$D \rightsquigarrow xxyxyx \longrightarrow K_{xxyxyx} =$

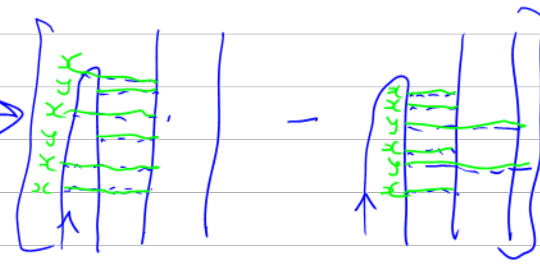


$=$

$=$

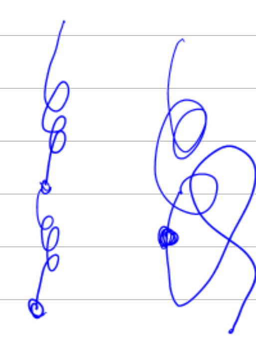


$\xrightarrow{\eta_3}$



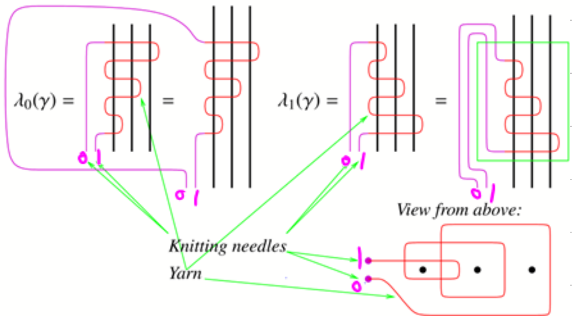
\cdots

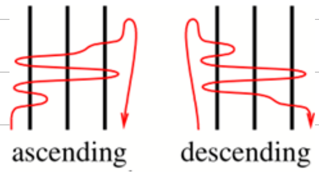
$$\begin{array}{ccc} \bar{\pi} & \xrightarrow{\eta} & \bar{\pi} \otimes |\bar{\pi}| \\ \downarrow tr & & \downarrow tr // Alt \\ |\bar{\pi}| & \xrightarrow{\delta} & |\bar{\pi}| \otimes |\bar{\pi}| \end{array} \qquad \begin{array}{ccc} \bar{\pi} & \xrightarrow{\eta} & \bar{\pi} \otimes |\bar{\pi}| \\ \downarrow tr & & \downarrow tr \\ |\bar{\pi}| & & |\bar{\pi}| \otimes |\bar{\pi}| \end{array}$$



Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. (~~Up to conjugation~~) temporarily $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.





$\eta = \hbar^{-1}(\lambda_0 - \lambda_1)$

$\exists \zeta_i$ s.t. $Z(\lambda_i(\gamma)) = \zeta_i(W(\gamma))$

$\pi \xrightarrow{\tau \text{ (for Turaev)}} \pi \otimes |\pi|$

$\eta = \hbar^{-1}(\lambda_0 - \lambda_1) : \pi \longrightarrow \pi \otimes |\pi|$ the Turaev operation

$\delta = \eta /\!/ tr_1 /\!/ Alt$ $\qquad tr_1 : \pi \otimes |\pi| \longrightarrow |\pi| \otimes |\pi|$

$W /\!/ \delta^a = W /\!/ \eta^a /\!/ tr_1 /\!/ Alt = W /\!/ \frac{\lambda_0^a - \lambda_1^a}{\hbar} /\!/ tr_1 /\!/ Alt$

$\delta /\!/ W = \delta /\!/ Z^{/1} = \frac{\lambda_0 - \lambda_1}{\hbar} /\!/ tr_1 /\!/ Alt /\!/ Z^{/1} = (\lambda_0 - \lambda_1) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1}$

$= (\lambda_0 - \lambda_0 /\!/ F /\!/ C) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1}$

$= (\lambda_0 - \lambda_0 /\!/ F) /\!/ tr_1 /\!/ Alt /\!/ Z^{/2} /\!/ \hbar^{-1} = (\lambda_0 - \lambda_0 /\!/ F) /\!/ Z^{/2} /\!/ tr_1 /\!/ Alt /\!/ \hbar^{-1}$

$= \frac{1}{\hbar} \lambda_0 /\!/ Z^{/2} /\!/ (1-F) /\!/ tr_1 /\!/ Alt = \frac{1}{\hbar} W /\!/ (1-F) /\!/$

Key 1. $W\colon |\bar\pi| \to |A|$ is $Z^{/1}_H\colon \mathcal{K}^{/1}_H(\bigcirc) \to \mathcal{A}^{/1}_H(\bigcirc)$.
Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$,

Lemma 1. "Division by $\hbar$" is well-defined.

$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}^{/1}_H(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ^a_i with $Z^{/2}(\lambda_i(\gamma)) = \lambda^a_i(W(\gamma))$), then η will have a compatible algebraic companion η^a:

$$\eta^a(\alpha) := (\lambda^a_0(\alpha) - \lambda^a_1(\alpha))/\hbar \in \mathcal{A}^{/1}_H(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}^{/2}_H$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda^a_0(W(\gamma)) - \lambda^a_1(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

$K(S)$ $\quad K(OO)$ $\quad K(/O)$

$$\begin{array}{c} K(S) \\ \downarrow \\ F(S) \end{array} \qquad |\bar\pi| \times |\bar\pi|$$

$K(S)$ knots

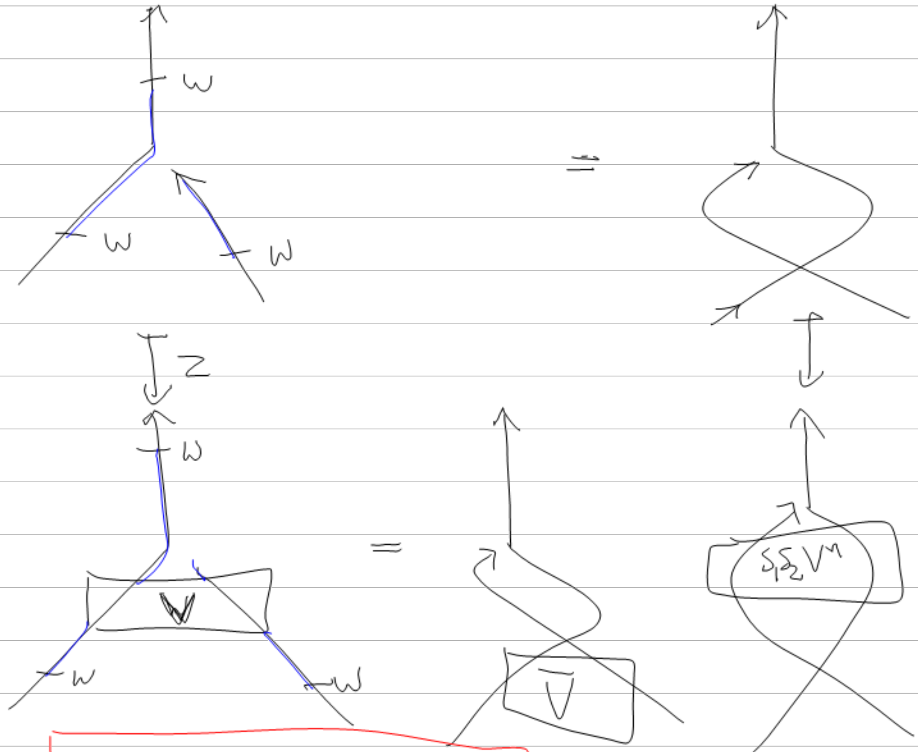
$C(S)$ curves $\quad C(OO) = |\bar\pi| \otimes |\bar\pi|$

$$\begin{array}{c} K(S) \\ \downarrow P \\ C(S) \end{array}$$

Example 1

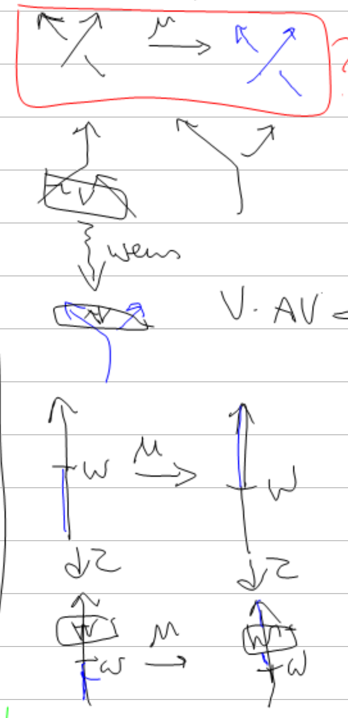
$$\begin{tikzcd} & K(OO) \arrow[d, "P"] \\ C(OO) \arrow[ur, "\lambda_0, \lambda_1"] \arrow[r, "id"] & C(OO) \end{tikzcd} \qquad \begin{tikzcd} K(OO) \arrow[r, "\lambda_0, \lambda_1"] \arrow[d] & K(OO) \arrow[d, "P"] \\ C(OO) \arrow[r, "I"] & C(OO) \end{tikzcd}$$

$\bar{D} = (-1)^{\#\text{tails}}(D)$



$$\boxed{\bar{V}^{21} = S_1 S_2 V^{-1}}$$

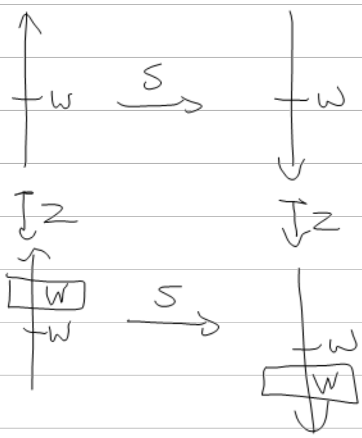
M: Swap black / blue

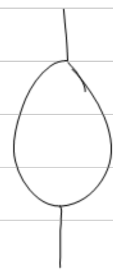


$\bar{W} W = 1$

$\bar{W} = W^{-1}$

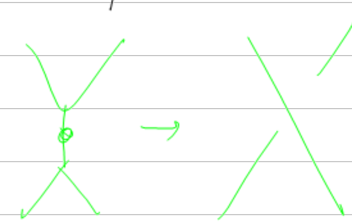
$W = W'$

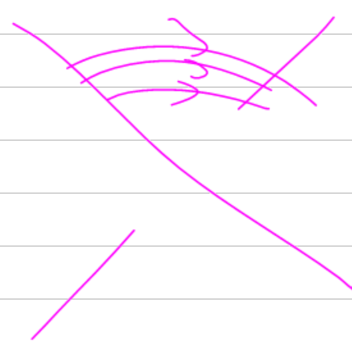




$$W = \bar{W}$$
$$e^{a+w^0+w'} = e^{-a+w^0-w'}$$

$\Rightarrow$ W is even wheels

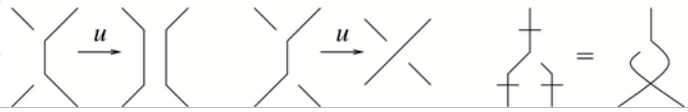


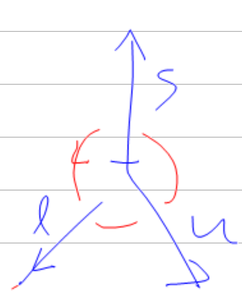


a b c d

231009 Best w-practices:

1. Trivalent tangles are end-labeled, to make a circuit algebra.
2. Tubes are bare (no colours and/or orientations).
3. Crossings have no signs; filtration is by comparison with virtual crossings.
4. Vertices are oriented and have marked legs: stem (s), upper (u), and lower (l). They are classical: they satisfy both R4s.
5. Wenjugating interchanges the two vertex types, and adds a virtual $l \leftrightarrow u$ crossing.
6. Only stems can be unzipped. Unzipping untwisted edges connects u to u above a connection of l to l. Unzipping through a wen is defined by wenjugating it out.



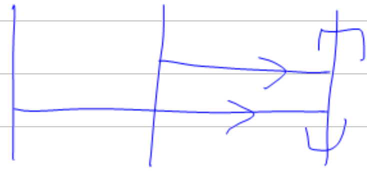
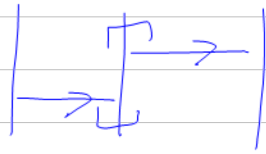


"R4 type"

$slu = usl = lus = l^{-1}s^{-1}u^{-1}$

slu	sul
$s^{-1}lu$	$s^{-1}ul$
$sl^{-1}u$	$sl^{-1}u$
slu^{-1}	slu^{-1}

$slu \to s^{-1}l^{-1}u^{-1} = uls$

 $+$  $+$  $= 0$

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$$

$$(1+a^{12})(1+a^{13})(1+a^{23}) =$$
$$(1+a^{23})(1+a^{13})(1+a^{12})$$