Pensieve header: A fast program for the Kauffman bracket.

In[•]:=
```
BarcodeImage["https://drorbn.net/AcademicPensieve/People/Burton"]
```

Out[•]=

From https://en.wikipedia.org/wiki/Bracket_polynomial:

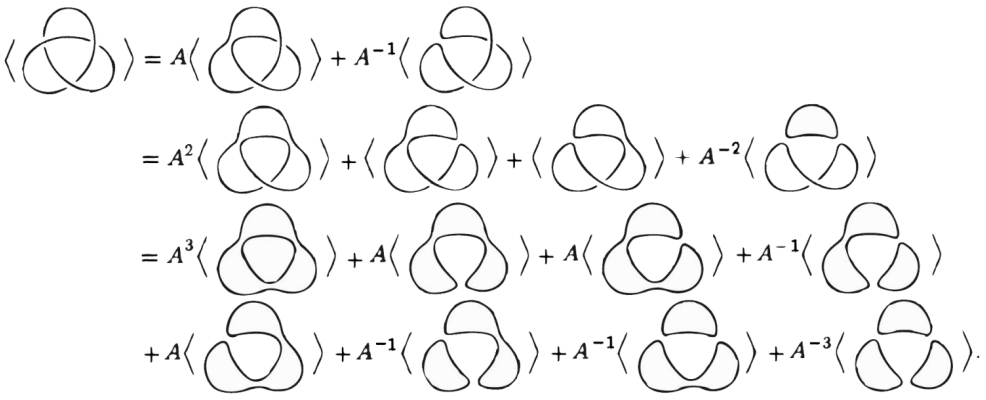
Definition [edit]

The bracket polynomial of any (unoriented) link diagram L, denoted $\langle L \rangle$, is a polynomial in the variable A, characterized by the three rules:

- $\langle \bigcirc \rangle = 1$, where $\bigcirc$ is the standard diagram of the unknot
- 
- $\langle \bigcirc \sqcup L \rangle = (-A^2 - A^{-2})\langle L \rangle$

From https://www.math.uni-hamburg.de/home/runkel/Material/SS20/L1hand.pdf:

Example 3.2. Let D be the canonical diagram of the Trefoil knot. We calculat the Kaufmann bracket of the Trefoil knot $\langle D \rangle$. At first resolve all crossings with the first formula.



Since a diagram without crossings is the disjoint union of loops we obtail the value of the bracket recursively by (II) and (III).

$$\langle D \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7}).$$

In[ ]:=
```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

DirectoryName: String expected at position 1 in DirectoryName[File].

ParentDirectory: Argument DirectoryName[File] should be a positive machine–size integer, a nonempty string, or a File specification.

DeclarePackage: Symbol CreateWikiConnection in DeclarePackage[WikiLink`, {CreateWikiConnection, WikiGetPageText, WikiGetPageTexts, WikiSetPageText, WikiSetPageTexts, WikiUploadFile, WikiUserName, WikiPageMatchQ, WikiPageFreeQ, WikiStringReplace, WikiStringCases, WikiAllPages}] has already been declared.

DeclarePackage: Symbol WikiGetPageText in DeclarePackage[WikiLink`, {CreateWikiConnection, WikiGetPageText, WikiGetPageTexts, WikiSetPageText, WikiSetPageTexts, WikiUploadFile, WikiUserName, WikiPageMatchQ, WikiPageFreeQ, WikiStringReplace, WikiStringCases, WikiAllPages}] has already been declared.

DeclarePackage: Symbol WikiGetPageTexts in DeclarePackage[WikiLink`, {CreateWikiConnection, WikiGetPageText, WikiGetPageTexts, WikiSetPageText, WikiSetPageTexts, WikiUploadFile, WikiUserName, WikiPageMatchQ, WikiPageFreeQ, WikiStringReplace, WikiStringCases, WikiAllPages}] has already been declared.

General: Further output of DeclarePackage::aldec will be suppressed during this calculation.

```
Loading QuantumGroups` version 2.0
Read more at http://katlas.math.toronto.edu/wiki/QuantumGroups
```

A: Symbol A appears in multiple contexts {QuantumGroups`, Global`}; definitions in context QuantumGroups` may shadow or be shadowed by other definitions.

StringDrop: String or list of strings expected at position 1 in StringDrop[File, −14].

StringTake: String or list of strings expected at position 1 in StringTake[StringDrop[File, −14], −7].

Get: StringDrop[File, −14] in $Path is not a string.

Get: StringDrop[File, −14] in $Path is not a string.

Get: StringDrop[File, −14] in $Path is not a string.

General: Further output of Get::path will be suppressed during this calculation.

WikiGetPageText: You must call CreateWikiConnection before using WikiGetPageText or WikiSetPageText

StringSplit: String or list of strings expected at position 1 in StringSplit[Null, <tr>].

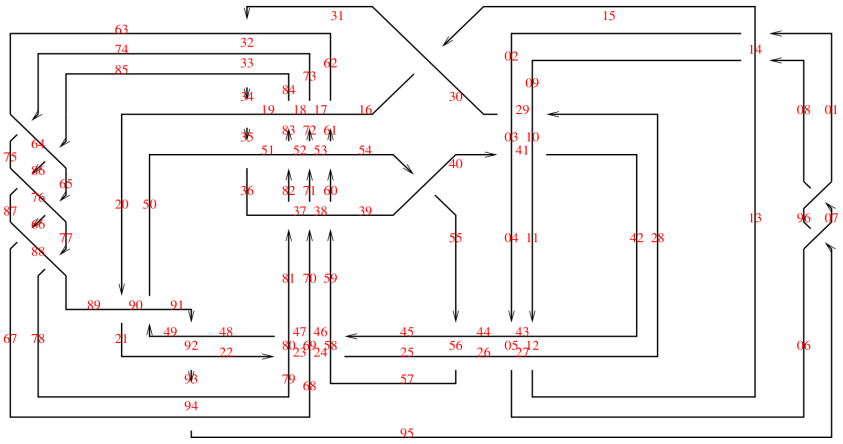
StringSplit: StringSplit called with 0 arguments; between 1 and 3 arguments are expected.

Join: Heads StringSplit and List at positions 1 and 2 are expected to be the same.

Syntax: Warning: comma encountered with no adjacent expression. The expression will be treated as Null. (line 68 of "KnotTheory`UniversalKh`").

The 48-crossing Gompf-Scharlemann-Thompson knot [GST] is significant because it may be a counterexample to the slice-ribbon conjecture:

Gompf Scharlemann Thompson



[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

In[•]:=
```
PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

GordianUnknot from mail/Burton/250415:

In[•]:=
```
PD[GordianUnknot] = PD[X[1, 156, 2, 157], X[3, 225, 4, 224], X[8, 149, 9, 150],
   X[9, 179, 10, 178], X[13, 207, 14, 206], X[15, 184, 16, 185], X[18, 211, 19, 212],
   X[20, 137, 21, 138], X[21, 7, 22, 6], X[25, 248, 26, 249], X[26, 277, 27, 278],
   X[27, 130, 28, 131], X[29, 78, 30, 79], X[32, 167, 33, 168], X[33, 124, 34, 125],
   X[34, 272, 35, 271], X[37, 106, 38, 107], X[38, 52, 39, 51], X[39, 119, 40, 118],
   X[40, 195, 41, 196], X[41, 245, 42, 244], X[44, 113, 45, 114], X[48, 200, 49, 199],
   X[49, 108, 50, 109], X[50, 241, 51, 242], X[52, 120, 53, 119], X[54, 248, 55, 247],
   X[55, 192, 56, 193], X[56, 101, 57, 102], X[57, 43, 58, 42], X[58, 115, 59, 116],
   X[61, 46, 62, 47], X[62, 178, 63, 177], X[63, 94, 64, 95], X[64, 263, 65, 264],
   X[65, 203, 66, 202], X[67, 90, 68, 91], X[68, 268, 69, 267], X[69, 172, 70, 173],
   X[70, 239, 71, 240], X[72, 35, 73, 36], X[73, 84, 74, 85], X[75, 276, 76, 277],
   X[76, 164, 77, 163], X[79, 127, 80, 126], X[80, 31, 81, 32], X[81, 166, 82, 167],
   X[82, 274, 83, 273], X[85, 37, 86, 36], X[86, 71, 87, 72], X[87, 171, 88, 170],
   X[88, 238, 89, 237], X[89, 268, 90, 269], X[91, 66, 92, 67], X[92, 203, 93, 204],
   X[93, 263, 94, 262], X[95, 176, 96, 177], X[96, 48, 97, 47], X[98, 59, 99, 60],
   X[99, 115, 100, 114], X[100, 43, 101, 44], X[102, 194, 103, 193], X[103, 246, 104, 247],
   X[104, 53, 105, 54], X[105, 120, 106, 121], X[107, 241, 108, 240], X[109, 198, 110, 199],
   X[110, 98, 111, 97], X[111, 60, 112, 61], X[112, 45, 113, 46], X[116, 243, 117, 244],
   X[117, 197, 118, 196], X[121, 74, 122, 75], X[122, 84, 123, 83], X[123, 272, 124, 273],
   X[125, 169, 126, 168], X[127, 30, 128, 31], X[128, 78, 129, 77], X[131, 279, 132, 278],
   X[132, 250, 133, 249], X[134, 23, 135, 24], X[136, 7, 137, 8], X[139, 213, 140, 212],
   X[140, 17, 141, 18], X[142, 184, 143, 183], X[143, 15, 144, 14], X[144, 205, 145, 206],
   X[146, 11, 147, 12], X[148, 179, 149, 180], X[150, 135, 151, 136], X[151, 23, 152, 22],
   X[153, 4, 154, 5], X[154, 225, 155, 226], X[157, 252, 158, 253], X[158, 281, 159, 282],
   X[160, 219, 161, 220], X[161, 28, 162, 29], X[162, 130, 163, 129], X[165, 274, 166, 275],
   X[171, 239, 172, 238], X[173, 266, 174, 267], X[180, 259, 181, 260],
   X[181, 230, 182, 231], X[186, 155, 187, 156], X[187, 3, 188, 2], X[190, 134, 191, 133],
   X[191, 24, 192, 25], X[194, 245, 195, 246], X[200, 176, 201, 175],
   X[201, 264, 202, 265], X[204, 235, 205, 236], X[208, 231, 209, 232],
   X[209, 182, 210, 183], X[210, 258, 211, 257], X[214, 6, 215, 5], X[215, 152, 216, 153],
   X[216, 190, 217, 189], X[220, 159, 221, 160], X[221, 281, 222, 280],
   X[223, 188, 224, 189], X[226, 213, 227, 214], X[227, 139, 228, 138],
   X[228, 19, 229, 20], X[232, 207, 233, 208], X[233, 13, 234, 12], X[234, 145, 235, 146],
   X[242, 197, 243, 198], X[250, 218, 251, 217], X[251, 222, 252, 223],
   X[254, 185, 255, 186], X[255, 16, 256, 17], X[256, 142, 257, 141], X[258, 230, 259, 229],
   X[260, 147, 261, 148], X[261, 11, 262, 10], X[265, 174, 266, 175], X[269, 236, 270, 237],
   X[270, 169, 271, 170], X[275, 164, 276, 165], X[279, 219, 280, 218], X[282, 254, 1, 253]];
```

In[•]:=
```
1 + 1
```

Out[•]=
```
2
```

In[•]:=
```
SetAttributes[p, Orderless]
```

```
In[•]:= K = Knot[3, 1]
```

```
Out[•]=
Knot[3, 1]
```

```
In[•]:= PD[K]
```

```
KnotTheory: Loading precomputed data in PD4Knots`.
```

```
Out[•]=
PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]
```

```
In[•]:= PD[K] /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]
```

```
Out[•]=
PD[p[1, 5] p[2, 4] / A + A p[1, 4] p[2, 5],
 A p[1, 4] p[3, 6] + p[1, 3] p[4, 6] / A, p[2, 6] p[3, 5] / A + A p[2, 5] p[3, 6]]
```

```
In[•]:= Expand[Times @@ PD[K] /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]]
```

```
Out[•]=
p[1, 4] p[1, 5] p[2, 4] p[2, 6] p[3, 5] p[3, 6] / A +
 A p[1, 4]^2 p[2, 5] p[2, 6] p[3, 5] p[3, 6] + A p[1, 4] p[1, 5] p[2, 4] p[2, 5] p[3, 6]^2 +
 A^3 p[1, 4]^2 p[2, 5]^2 p[3, 6]^2 + p[1, 3] p[1, 5] p[2, 4] p[2, 6] p[3, 5] p[4, 6] / A^3 +
 p[1, 3] p[1, 4] p[2, 5] p[2, 6] p[3, 5] p[4, 6] / A +
 p[1, 3] p[1, 5] p[2, 4] p[2, 5] p[3, 6] p[4, 6] / A + A p[1, 3] p[1, 4] p[2, 5]^2 p[3, 6] p[4, 6]
```

```
In[•]:= Expand[Times @@ PD[K] /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]] //.
  p[i_, j_] p[j_, k_] :> p[i, k]
```

```
Out[•]=
A^3 p[1, 4]^2 p[2, 5]^2 p[3, 6]^2 + A p[3, 6]^2 p[4, 5]^2 +
 A p[2, 5]^2 p[4, 6]^2 + p[3, 5]^2 p[4, 6]^2 / A^3 + 3 p[5, 6]^2 / A + A p[1, 4]^2 p[5, 6]^2
```

```
In[•]:= Expand[Expand[Times @@ PD[K] /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]] //.
    p[i_, j_] p[j_, k_] :> p[i, k] /. p[_, _]^2 → (-A^2 - A^-2)]
```

```
Out[•]=
1/A^7 + 1/A^3 + A - A^9
```

```
In[•]:= K = Knot[10, 165]
```

```
Out[•]=
Knot[10, 165]
```

```
In[•]:= Expand[Expand[Times @@ PD[K] /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]] //.
          p[i_, j_] p[j_, k_] :> p[i, k] /. p[_, _]^2 → (-A^2 - A^-2)]
```

Out[•]=

$$-1-\frac{1}{A^{20}}+\frac{2}{A^{16}}-\frac{1}{A^{12}}+\frac{2}{A^8}-\frac{1}{A^4}-2A^8+2A^{12}-2A^{16}$$

```
In[•]:= FKB[K_] := Module[{front = {}, v, todo, x, kb = 1},
          v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}];
          todo = List @@ PD[K];
          While[todo =!= {},
           x = RandomChoice[MaximalBy[todo, v]];
           kb = Expand[Expand[(x /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^-1 p[i, l] p[j, k]) kb] //.
                p[i_, j_] p[j_, k_] :> p[i, k] /. p[_, _]^2 → (-A^2 - A^-2)];
           todo = DeleteCases[todo, x];
           front = front ⋃ (List @@ x);
          ];
          Expand@Together[kb / (-A^2 - A^-2)]
         ]
```

```
In[•]:= FKB[Knot[3, 1]]
```

Out[•]=

$$-\frac{1}{A^5}-A^3+A^7$$

```
In[•]:= FKB[Knot[10, 165]]
```

Out[•]=

$$\frac{1}{A^{18}}-\frac{3}{A^{14}}+\frac{4}{A^{10}}-\frac{6}{A^6}+\frac{7}{A^2}-6A^2+6A^6-4A^{10}+2A^{14}$$

```
In[•]:= FKB[GST48]
```

Out[•]=

$$-\frac{1}{A^{58}}+\frac{3}{A^{54}}-\frac{4}{A^{50}}+\frac{3}{A^{46}}-\frac{1}{A^{42}}+\frac{1}{A^{38}}-\frac{1}{A^{34}}+\frac{1}{A^{30}}-\frac{1}{A^{26}}+\frac{2}{A^{22}}-$$
$$\frac{4}{A^{18}}+\frac{3}{A^{14}}-\frac{1}{A^{10}}-\frac{3}{A^6}+\frac{5}{A^2}-5A^2+5A^6-3A^{10}+3A^{14}-A^{18}+A^{30}-A^{34}$$

```
In[•]:= NaiveWidth[K_] := Module[{front = {}, v, todo, x, w = 0},
          v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}];
          todo = List @@ PD[K];
          While[todo =!= {},
           x = RandomChoice[MaximalBy[todo, v]];
           todo = DeleteCases[todo, x];
           front = Complement[front ⋃ (List @@ x), front ⋂ (List @@ x)];
           w = Max[w, Length@front]
          ];
          w
         ]
```

```
In[•]:= Table[NaiveWidth[GST48], {32}]
```

```
Out[•]=
{12, 14, 14, 14, 16, 12, 12, 16, 12, 12, 18, 12, 16, 12, 16,
 12, 12, 16, 16, 12, 14, 14, 14, 14, 12, 14, 16, 16, 16, 12, 12, 12}
```

```
In[•]:= Min@Table[NaiveWidth[GST48], {1000}]
```

```
Out[•]=
12
```

```
In[•]:= Table[NaiveWidth[GordianUnknot], {32}]
```

```
Out[•]=
{12, 22, 14, 18, 14, 24, 18, 14, 16, 18, 18, 22, 18, 20, 16,
 20, 26, 20, 20, 20, 18, 20, 22, 20, 20, 22, 18, 16, 20, 16, 20, 22}
```

```
In[•]:= Min@Table[NaiveWidth[GordianUnknot], {1000}]
```

```
Out[•]=
12
```

```
In[•]:= AbsoluteTiming@FKB[GordianUnknot]
```

```
Out[•]=
{45.2552, -1/A^99}
```