On the Gassner Invariant of Braids and String links

by

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Abstract

In this thesis we delve into the computation of the Gassner invariant for string links, which are a more generalized form than braids, utilizing a (co)homological approach. We restrict this (co)homology invariant, denoted as $\mathcal{G}_h$, to pure braids, leading to the derivation of the Gassner representation.

We introduce the concept of "flying cars," which assigns an invariant $\mathcal{C}(L)$ to an $(n+1)$-component string link L. This invariant, an $n \times n$ matrix, has entries in the field $\mathbb{Q}(t_0, t_1, \ldots, t_n)$. We establishes a connection between the invariant $\mathcal{C}(L)$ and the homology Gassner invariant $\mathcal{G}_h(L)$ of L through the formula $\mathcal{G}_h(L) = \left(D_n \cdot \mathcal{C}(L) \cdot D_n^{-1}\right)//\rho_{col}//m^t$. Here, D_n is a diagonal matrix, m^t denotes matrix transpose, and ρ_{col} represents column permutation. We prove that $\mathcal{C}(L)$ is indeed an invariant of string links under the Reidemeister moves, thereby directly verifying the invariance of the homology Gassner invariant.

Moreover, we provide formulas for the intersection product $\mu := \langle -, - \rangle : H_1(P; \mathcal{F}) \times H_1(P; \mathcal{F}) \to \mathcal{F}$, which is defined on the cycles of the homology group $H_1(P; \mathcal{F})$. In this context, P is an $(n+1)$-punctured disk viewed as a subspace of the complement X of an $n+1$ string link, and $\mathcal{F}$ is a local coefficient system on X determined by the abelianization map $\epsilon : \pi_1(X, x_0) \to \langle t_0, t_1, \cdots, t_n \rangle$. This map takes values in the free abelian group $\langle t_0, t_1, \cdots, t_n \rangle$. We conclude by verifying that the homology Gassner invariant is unitary with respect to this intersection product.

I dedicate this thesis to the memory of my mother Juliana Abena Asantewaa

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Contents

Chapter 1

Introduction

1.1 Background

The theory of braid groups is an interesting topic in topology that deals with braids, the groups formed by their equivalence classes, and other related concepts. It is one of the many areas in mathematics that weaves together the beauty of topology and the robustness of algebra. An n-braid is formed by intertwining n ***strands*** (see Definition 2.7) whose endpoints are attached to two fixed planes, P_0 and P_1. Each strand never backtracks. Figure 1.1 shows some examples of braids. More formally, Artin defines the braid

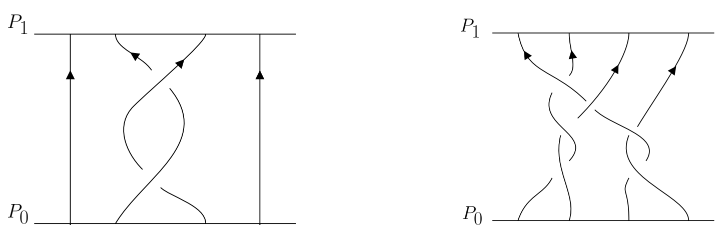
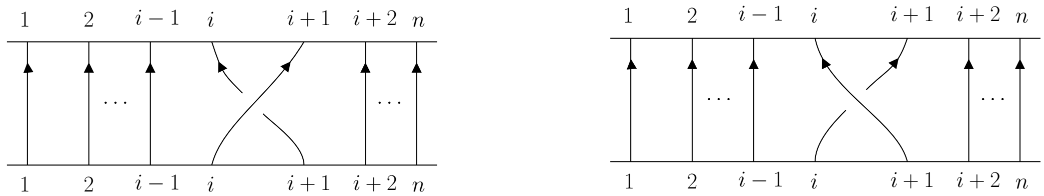


Figure 1.1: Examples of braids

group on n strands as the group generated by $n-1$ generators $\sigma_1, \cdots, \sigma_n$ satisfying the braid relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j = 1, 2, \ldots, n-1$ with $|i-j| \geq 2$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$. Figure 1.2 shows a geometric representation for the generator σ_i. The interest in braids stems from their role in knot theory, such as the close relation of braids to knots and links, as detailed by Alexander's theorem and Markov's theorem[1], and in physics, such as their connection to the Yang–Baxter equation. In particular, studying braids leads to various algebras and linear representations, which include the Burau and the Gassner representations. For further reading on the braid group, refer to [BC74] and [KT08]. The ***Gassner representation*** is a homomorphism $\Gamma : PB_n \to GL_n(\mathbb{Z}[\mathbb{Z}^n])$ defined on the ***pure braid*** (see Definition 2.8, Item 3) group PB_n, which is a subgroup of the braid group on n strands. Pure braids on n strands induce the identity permutation and they are generated by the set $\{A_{i,j}\}_{1\leq i<j\leq n}$. Here $A_{i,j}$ is the generator shown in Figure 1.3 and it can be expressed via the braid group generators as $A_{i,j} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}$. The Gassner representation has been the subject of study by many

[1] Alexander's Theorem: Any link in $\mathbb{R}^3$ can be represented as the closure of a braid.
Markov's Theorem: The closures of two braids are isotopic if and only if one can be obtained from the other by a sequence of moves called the Markov moves



(a) Over-crossing σ_i **(b)** Under-crossing σ_i^{-1}

Figure 1.2: Generators of a braid

authors, including [Knu05], [BC74], [Abd97], [KLW01] [BN14], [Mar20] and [Gas59], each employing different approaches. The classical Gassner representation, as presented in [BC74] is constructed using the ***Magnus representation*** and ***Fox free differential calculus*** as follows:

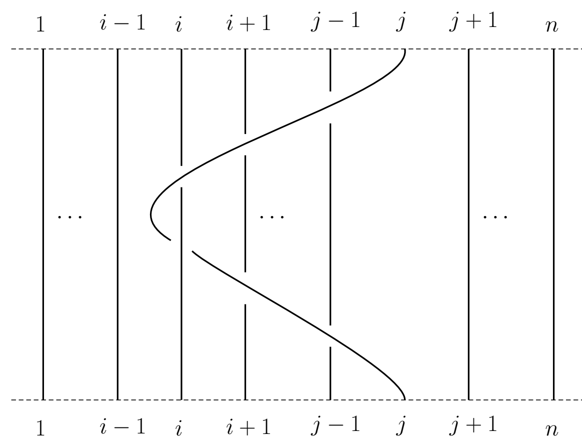


Figure 1.3: The n-braid $A_{i,j}$ with $1 \leq i < j \leq n$.

Definition 1.1 (Fox free differential calculus)**.** For each $j = 1, \cdots, n$, there is a map $\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$ given by

$$\frac{\partial}{\partial x_j}\left(x_{r_1}^{\epsilon_1} \cdots x_{r_k}^{\epsilon_k}\right) = \sum_{i=1}^{k} \epsilon_i \delta_{r_i,j} x_{r_1}^{\epsilon_1} \cdots x_{r_i}^{\frac{\epsilon_i - 1}{2}}, \quad \text{and} \quad \frac{\partial}{\partial x_j}\left(\sum_g a_g g\right) = \sum_g a_g \frac{\partial}{\partial x_j} g, \quad g \in F_n, a_g \in \mathbb{Z},$$

where $\epsilon = \pm 1$, δ is the Kronecker symbol, and $\mathbb{Z}[F_n]$ is the ***group ring*** (see Definition 2.3) of the free group F_n generated by the set $\{x_1, \cdots, x_n\}$.

Let Φ be an arbitrary homomorphism with domain the free group F_n, taking values in some free abelian group of $\mathcal{A}$ of rank n and A_Φ be the group of automorphisms of F_n satisfying $\Phi(x) = \Phi(\alpha(x))$ for each $x \in F_n$ and $\alpha \in A_\Phi$.

Definition 1.2 ([BC74], Theorem 3.9)**.** Let $\alpha \in A_\Phi$ and $[\alpha]^\Phi$ be the $n \times n$ matrix

$$[\alpha]^\Phi = \left[\Phi\left(\frac{\partial(\alpha(x_i))}{\partial x_j}\right)\right]_{i,j},$$

with entries in the group ring $\mathbb{Z}[\mathcal{A}]$. Then the morphism $A_\Phi \longrightarrow M_n(\mathbb{Z}[\mathcal{A}])$ defined by $\alpha \mapsto [\alpha]^\Phi$ is a well defined group homomorphism, and it a Magnus representation.

It is a well-known fact that the braid group B_n can be represented as a group of automorphisms of the free group F_n (see [KT08], Theorem 1.31). Since PB_n is a subgroup of B_n, it can also be represented as a subgroup of the automorphisms of F_n. Let Z_n be the free abelian group generated by the free basis $t_1, \cdots, t_n$, and $\phi : F_n \to Z_n$ be the homomorphism defined by $x_i \mapsto t_i$.

Definition 1.3 (The classical Gassner representation)**.** The morphism $PB_n \to M_n(\mathbb{Z}[Z_n])$ assigning to a pure braid β the matrix

$$[\beta]^{\phi} = \left[\phi\left(\frac{\partial(\widetilde{\beta}(x_i))}{\partial x_j}\right)\right]_{i,j},$$

where $\widetilde{\beta} \in A_{\phi}$ is the automorphism corresponding to β, is the Gassner representation of the pure braid group.

A formula for the Gassner representation for the generator $A_{i,j}$ is presented in [Knu05]:

$$[A_{i,j}]^{\phi} = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & 1-t_i+t_it_j & 0 & t_i(1-t_i) & 0 \\ 0 & \vec{u} & I_{j-i-1} & \vec{v} & 0 \\ 0 & 1-t_j & 0 & t_i & 0 \\ 0 & 0 & 0 & 0 & I_{n-j} \end{pmatrix}. \tag{1.1}$$

Here, I_{i-1}, I_{j-i-1} and I_{n-j} are identity matrices and $\vec{u}$ and $\vec{v}$ are column vectors

$$\vec{u} = \begin{pmatrix} (1-t_{i+1})(1-t_j) \\ \vdots \\ (1-t_{j-1})(1-t_j) \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} (1-t_{i+1})(t_i-1) \\ \vdots \\ (1-t_{j-1})(t_i-1) \end{pmatrix}.$$

Another approach discussed in [BN14] is as follows: Let t be a formal variable and let $U_i(t) = U_{n:i}(t)$ denote the $n \times n$ identity matrix with its 2×2 block at rows i and $i+1$ and column i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$. Let $U_i^{-1}(t)$ be the inverse of $U_i(t)$; it is the $n \times n$ identity matrix with block at $\{i, i+1\} \times \{i, i+1\}$ replaced by $\begin{pmatrix} 0 & \bar{t} \\ 1 & 1-\bar{t} \end{pmatrix}$, where $\bar{t}$ denotes t^{-1}. Let β be a braid $\beta = \prod_{a=1}^{k} \sigma_{i_a}^{s_a}$, where s_a are signs and where products are taken from the left to right. Let j_a be the index of the over strand at a crossing numbered a in β. Define $\Gamma : B_n \to M_n\left(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1} \ldots, t_n^{\pm 1}]\right)$ as the product of matrices $\Gamma(\beta) = \prod_{a=1}^{k} U_{i_a}^{s_a}(t_{j_a})$. The map Γ is not multiplicative, that is $\Gamma(\beta_1 \cdot \beta_2) \neq \Gamma(\beta_1)\Gamma(\beta_2)$ for braids β_1 and β_2 in general. However, it becomes multiplicative when restricted to the pure braids. The restriction, $\Gamma : PB_n \to M_n\left(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1} \ldots, t_n^{\pm 1}]\right)$ of Γ to PB_n is the Gassner representation.

The two constructions of the Gassner representation seen above are equivalent. Transposing the matrix representation $\Gamma(A_{i,j})$ yields a matrix which is equivalent to $[A_{i,j}]$ for appropriate values of $t_k's$. Although braids are topological objects, it is clear that the constructions do not involve any topological property of braids. Moreover, the classical definition involves complicated computations that result in matrices that need a bit of rewriting to be useful. Homology and cohomology present a natural way to utilize the topology of braids. The braid group B_n naturally acts on the (co)homology of topological spaces obtained from the n punctured disk by functorial constructions. One such construction is presented in [KLW01] using more general objects called ***string links*** as follows in the next two paragraphs.

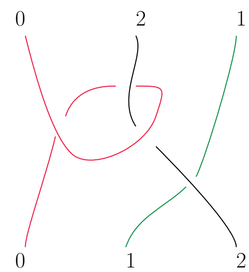
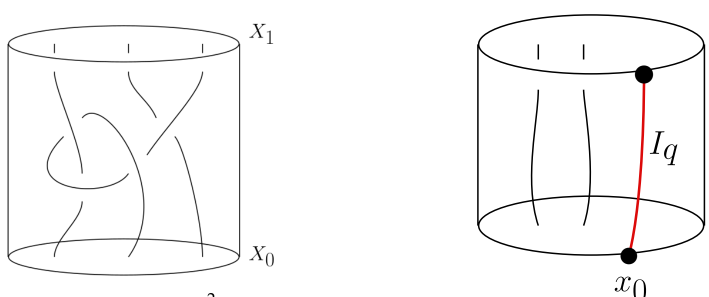


Figure 1.4: A labeled string link.

A ***string link*** [2] is a braid except that the strands are not required to be monotonic. Unlike braids, string links do not form a group since they need not have inverses. ***Labeling*** the strands of an $(n+1)$-string link L involves indexing the strands at the bottom from left to right, starting from 0 and going up to n. An indexed string link is called a ***labeled string link***. Let $T = \{0, 1, \cdots, n\}$ denote the indexing set; T is called the set of ***labels*** of L. An example of a labeled string link is shown in Figure 1.4. We denote the set of all string links on $n+1$ strands by SL_n. In this thesis, string links are not allowed to have circle components.

String links acts on the (co)homology of topological spaces obtained from the $n+1$ punctured disk. Let $D^2 \times [0,1]$ be the solid cylinder and let L be an $n+1$ string link embedded in the cylinder as shown in Figure 1.5a. The complement $X = D^2 \times [0,1] - L$ of L has two subspaces $X_0 = X \cap (D^2 \times \{0\})$ and $X_1 =$



(a) A string link in $D^2 \times [0,1]$.

(b) $I_q = \{x_0\} \times [0,1]$

Figure 1.5: String links in $D^2 \times [0,1]$.

$X \cap (D^2 \times \{1\})$; these are $n+1$ punctured disks, and they are canonically identified via the homeomorphism $(x,0) \mapsto (x,1)$. Let $\mathcal{F} = \mathbb{Q}(\{t_k\}_{k=0}^n)$ be the field of fractions of $\mathbb{Z}[Z_n]$, where $Z_n = \langle t_0, \cdots, t_n \rangle$ is the free abelian group generated by the free basis $\{t_k\}_{k=0}^n$. This field is a ***local coefficient system*** (see Section 2.3) on X determined by the abelianization map $\epsilon : \pi_1(X, x_0) \to \langle t_0, t_1, \cdots, t_n \rangle$. Some references for local coefficient system include [DK01] and [Hat02]. It turns out that the cohomology groups $H^1(X, I_q; \mathcal{F})$, $H^1(X_0, q; \mathcal{F})$ and $H^1(X_1, q; \mathcal{F})$ are pair-wise isomorphic (see Lemma 3.3) as vector spaces over the field $\mathcal{F}$, where $q = x_0$

[2] The term "***string links***" is typically used to refer to tangles that induce the identity permutation. However, in this thesis, we adopt the convention set forth in [KLW01]; string links do not necessarily induce the identity permutation. A string link that induces the identity permutation will be referred to as a "***pure string link***."

is a fixed basepoint on the boundary of X and $I_q = \{x_0\} \times [0,1]$. This construction assigns an automorphism

$$\mathcal{G}_c(L) : H^1(X_0, q; \mathcal{F}) \xleftarrow{\iota_0^*} H^1(X, I_q; \mathcal{F}) \xrightarrow{\iota_1^*} H^1(X_1, q; \mathcal{F}) \cong H^1(X_0, q; \mathcal{F})$$

to L of the homology group $H^1(X_0, q; \mathcal{F})$ of the $n+1$ punctured disk. The linear map $\mathcal{G}_c(L) = \iota_1^* \circ (\iota_0^*)^{-1}$ is called the cohomology Gassner invariant of L, see Definition 3.4. Restricting $\mathcal{G}_c$ to pure braids gives the Gassner representation. Also, one can define homology Gassner invariant

$$\mathcal{G}_h(L) : H_1(X_0, q; \mathcal{F}) \xrightarrow{\iota_{0*}} H_1(X, I_q; \mathcal{F}) \xrightarrow{\iota_{1*}^{-1}} H_1(X_1, q; \mathcal{F})$$

of L, see Definition 3.11. It is noteworthy that Le Dimet was the first to extend the Gassner representation to string links in [Dim92], broadening its scope to include $n \times n$ matrices whose entries are rational functions the variables $t_0, t_1, \cdots, t_n$.

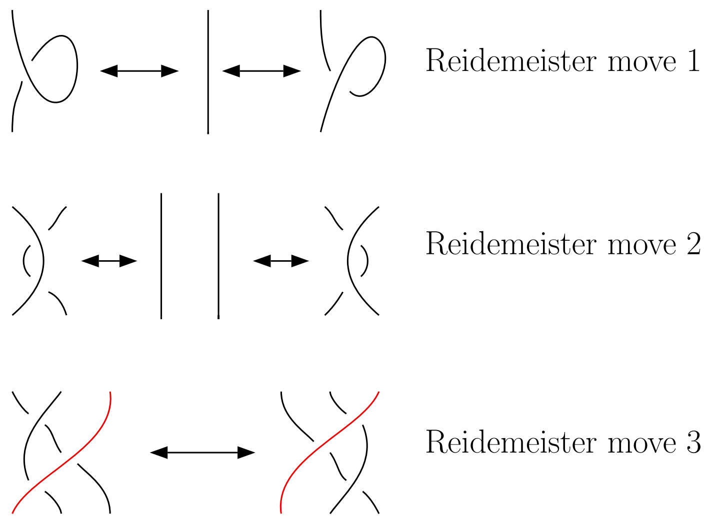


Figure 1.6: The 3 Reidemeister moves.

In addition, there is the notion of ***flying cars*** (see Definition 4.1) which is a modified version of the car concept discussed in [BNa]. It is based on a "probabilistic" interpretation of the Burau representation for string links studied in [LTW98], which is extended to give a similar interpretation of the Gassner representation in Section 8 of [KLW01]. A flying car associates a labeled $(n+1)$-string link L with an $n \times n$ matrix denoted as $\mathcal{C}(L)$ with entries in the field $\mathcal{F}' = \mathbb{Q}(\{t_k : k \in T'\})$ of rational functions in the variables $t_k \in T'$, where $T' = T - \{0\}$. This matrix serves as an ***invariant*** of L, meaning it remains unchanged the Reidemeister moves (see Figure 1.6). Specifically, if another string link L' can be derived from L through a sequence of Reidemeister moves, then $\mathcal{C}(L') = \mathcal{C}(L)$. This invariant is connected to the homology Gassner invariant, verifying its invariance. By employing flying cars in conjunction with the ***stitching operation*** defined in Section 4.4, one can compute the homology Gassner invariant of L from the homology Gassner invariant of a braid β, where L is the partial closure of β (see Lemma 4.4).

It has been established by several authors from different points of view that the Gassner invariant is unitary with respect to a skew hermitian matrix. In [Abd97] and [BN14], the authors explicitly define different Hermitian matrices to prove the unitary condition, but they do not provide details on how these matrices were derived. One advantage of cohomology is that it provides a natural way to define a hermitian matrix using cup product on the cocyles. However, it is quite difficult to find a suitable cell complex for the complement of a string link to define the cup product. Alternatively, we can use the dual version: the intersection product (see Section 5.2) defined on cycles which is easier to compute. Kirk et al. in [KLW01] tackled this problem from a geometric point of view. They showed that the Gassner invariant is unitary with respect to an intersection form with coordinate free arguments without providing explicit formulas.

Formulas are easy to remember and may not change over time; they are easy to implement using computer programs to reduce complexities in computations. In this thesis, motivated by the work in [BN14] and [KLW01], we provide formulas for the homology Gassner invariant, flying car invariant and the intersection form to complement the work in [KLW01].

Results of thesis

The results of this thesis are as follows:

1. **Formulas for the homology Gassner invariant (see Section 3.3)**

$$\mathcal{G}_h(\sigma_i) = \begin{pmatrix} 0 & t_{T[i]} \\ 1 & 1 - t_{T[i+1]} \end{pmatrix} \text{ and } \mathcal{G}_c(\sigma_i^{-1}) = \begin{pmatrix} \frac{t_{T[i+1]}-1}{t_{T[i]}} & 1 \\ \frac{1}{t_{T[i]}} & 0 \end{pmatrix},$$

   where i is the position of the over strand below the horizontal level of the crossing σ_i.

   These are matrix representations of the homology Gassner invariant for the generators σ_i and it inverse σ_i^{-1} of the braid group. They are related to the cohomology Gassner invariant given by inverse transpose in appropriate basis.

2. **A relation between homology Gassner invariant and flying cars (see Section 4.3):**

$$\begin{tikzcd} & SL_n \arrow[dl, "\mathcal{C}"'] \arrow[dr, "\mathcal{G}_h^r"] & \\ Mat_n(\mathcal{F}') \arrow[rr, "D_n(-)D_n^{-1}//\rho_{col}//m^t"'] & & Mat_n(\mathcal{F}') \end{tikzcd}$$

3. **The intersection product formulas (see Section 5.2):**

$$\langle \widetilde{\beta_i}, \widetilde{\beta_j} \rangle = \begin{cases} \frac{(t_0-1)(t_{T[i]}-1)(1-t_0 t_{T[i]})}{t_0 t_{T[i]}}, & i = j \quad (self-intersection) \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_{T[j]}}, & i < j \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_0 t_{T[j]}}, & i > j \end{cases},$$

4. **Unitary condition for the homology Gassner invariant (see Section 5.5):** The theorem

   Theorem 1.4. *Let L be an $(n+1)$-string link whose strands are labeled by $T = 0, 1, \cdots, n$. Suppose L is the partial closure of an $(n+2)$ braid β. If the homology Gassner invariant $\mathcal{G}_h(\beta)$ of β is unitary with* *respect to the intersection products Ω_0 and Ω_1, then $\mathcal{G}_h(L)$ is also unitary with respect to the intersection products* $\begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} \end{pmatrix} // t_{\rho(n)},\ t_n \mapsto t_{\rho(n)}$ *and* $\begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} \end{pmatrix}$.

is an alternative statement for Theorem 5.18, where Ω_0 and Ω_1 are the matrices for the intersection product on the spaces X_0 and X_1 corresponding L respectively, $\beta_{i,j} := \langle \widetilde{\beta_i}, \widetilde{\beta_j} \rangle$, $\rho(Z)$ represents the permutation of the element of Z, $\beta_{\rho(i,j)} := \langle \widetilde{\beta}_{\rho(i)}, \widetilde{\beta}_{\rho(j)} \rangle)$ and $\beta_{Z,Z} = \{\beta_{i,j} : i,j \in Z\}$. Here $T = Z \cup \{n-1, n\}$.

Refer to Chapter 6 for the concluding remarks on the thesis results.

1.2 Thesis Structure

In Chapter 2, we will explore fundamental concepts in algebraic topology. Specifically, we will introduce and define homology, cohomology, and local coefficient systems. References for these concepts can be found in Hatcher's "Algebraic Topology", [Hat02] and Brown's "Cohomology of Groups", [Bro12]. Additionally, we will define and discuss braids and string links, which serve as the focal points of this thesis. Furthermore, we will provide a cell structure for the complement of a given string link or braid, its ambient space being the solid cylinder, utilizing a group presentation known as the Wirtinger presentation. Lastly, we intend to expound upon a local coefficient system for the complement.

In Chapter 3, we will delve into the cohomology and homology Gassner invariants of string links and braids, and offer various examples to elucidate the computation of these invariants. It will be observed that these two invariants are inverse transpose of each other. Furthermore, we will verify that the homology Gassner invariant is a braid invariant. Subsequently, in Chapter 4, we will verify that it is also a string link invariant. Finally, a Mathematica implementation of the homology Gassner invariant will be presented. The main reference for this chapter is [KLW01].

In Chapter 4, we discuss the concept of flying cars, a slight modification of the one discussed in [BNa]. Flying cars involve assigning an $n \times n$ matrix to an $n+1$ string link or braid, where the leftmost strand is always free. It is demonstrated that this assignment serves as an invariant of string links and is connected to the homology Gassner invariant. Furthermore, the stitching operation is defined to establish a relationship between string links and braids. Finally, examples are provided to illustrate this concept.

In Chapter 5, we discuss the unitarity of the homology Gassner invariant with respect to a skew hermitian product given by an intersection product defined on the cycles of the first homology groups $H_1(X_0;\mathcal{F})$ and $H_1(X_1;\mathcal{F})$. We provide details on the computation of the intersection product defined on the elements of $H_1(X_0;\mathcal{F})$. Furthermore, we provide a detailed proof of the unitary statement in Theorem 3.2 of [KLW01] (see Theorem 5.18). We also provide an alternative proof of Theorem 5.18 (see Theorem 5.20). Finally, we present a Mathematica implementation of the unitarity of the homology Gassner invariant.

Finally, in Chapter 6, we give the concluding remarks.

1.3 Conventions and notations

- In most contexts, the term "***string links***" is commonly associated with tangles that result in the identity permutation. However, in the scope of this thesis, we adhere to the convention established [KLW01]. Accordingly, string links are not strictly limited to inducing the identity permutation. We will specifically use the term "***pure string link***" to denote a string link that does induce the identity permutation.

- The strands of a string link and a braid move from bottom to the top.
- In this thesis the Mathematica notation // is used to denote compositions of functions Specifically, $f \circ g := g \,//\, f$ or $f(g(x)) := x \,//\, g \,//\, f$.
- $\mathcal{G}_c$ represents cohomology Gassner invariant.
- $\mathcal{G}_h$ represents homology Gassner invariant.
- m^t stands for matrix transpose.
- A positive crossing is referred to as an over-crossing and a negative crossing is referred to as an under-crossing.
- The strands of an $n+1$ string link are indexed $0, 1, \cdots, n$, at the bottom from left to right. The set $T = \{0, 1, \cdots, n\}$ of indices will be the set of labels of the strands. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the horizontal level of that crossing. Here i is the position of the over strand below the horizontal level of the crossing σ_i. In the case of σ_i^{-1}, i is the position of the under strand instead. For example consider the braid b in the figure below:

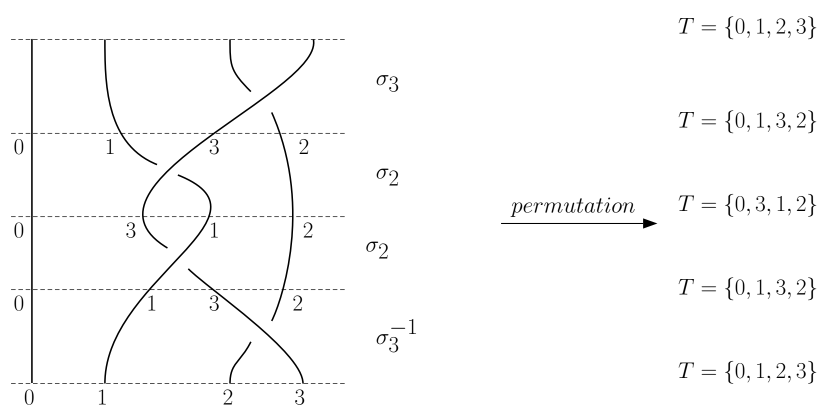


Figure 1.7: The braid b with permutations at each horizontal level.

The permutation in cycle notation at the horizontal level below the crossing σ_3 is $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}$. So, the labels of the two strands participating in the crossing are $3, 2$ in that order. The indices for the strands participating in the crossing σ_3^{-1} are $2, 3$. Permutation of the set T will also be denoted T.

- Braids are composed from bottom to the top. For example in the figure above. the compositionn is $b = \sigma_3^{-1}\sigma_2\sigma_2\sigma_3$. However, matrix multiplication is done in the opposite direction. For example

$$\mathcal{G}_h(b) = \mathcal{G}_h(\sigma_3)\mathcal{G}_h(\sigma_2)\mathcal{G}_h(\sigma_2)\mathcal{G}_h(\sigma_3)^{-1}.$$

Chapter 2

Preliminaries

2.1 Summary of Chapter

In this chapter, we will explore fundamental concepts in algebraic topology. Specifically, we will introduce and define homology, cohomology, and local coefficient systems. Some references for these concepts include [DK01], [Hat02], [Bro12] and [Rot09] . Additionally, we will define and discuss braids and string links, which serve as the focal points of this thesis. Furthermore, we will provide a cell structure for the complement of a given string link or braid, its ambient space being the solid cylinder, utilizing a group presentation known as the Wirtinger presentation. Lastly, we intend to expound upon a local coefficient system for the complement.

2.2 Homology and cohomology

In this section, we define and explore the basic concepts of homology and cohomology.

Definition 2.1.

1. A ***chain complex*** $(C_\bullet, \mathrm{d}_\bullet)$ is a sequence,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots ,$$

   of modules such that $\partial_n \circ \partial_{n+1} = 0$. The quotient $H_n(C_\bullet) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$ is called the nth ***homology*** group of the complex.

2. A ***cochain complex*** $(C^\bullet, \mathrm{d}^\bullet)$ is a sequence,

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots ,$$

   of modules such that $d^{n+1} \circ d^n = 0$. The quotient $H^n(C^\bullet) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}$ is called the nth ***cohomology*** group of the complex.

3. A ***chain map*** $f_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \mathrm{d}_\bullet)$ between two chain complexes $(C_\bullet, \partial_\bullet)$ and $(D_\bullet, \partial_\bullet)$ is a collection of homomprphisms $f_n : C_n \to D_n$ such that the $\partial_n \circ f_n = f_{n-1} \circ \partial_n$.

4. Given a space X and a subspace $A \subset X$, the kth ***relative chain group*** $C_k(X, A)$ is defined as the quotient group $C_k(X)/C_k(A)$ for each integer k. The kth homology group of the ***relative chain***

complex

$$\cdots \longrightarrow C_{k+1}(X,A) \xrightarrow{\partial_{k+1}} C_k(X,A) \xrightarrow{\partial_k} C_{k-1}(X,A) \longrightarrow \cdots$$

is called k***th relative homology*** and it is denoted by $H_k(X,A)$.

Let M be a group. Then applying $\mathrm{Hom}(-,M)$ to the relative chain complex, we get the relative cochain complex

$$\cdots \longrightarrow C^{k-1}(X,A;M) \xrightarrow{\mathrm{d}^{k-1}} C^k(X,A) \xrightarrow{\mathrm{d}^k} C^{k+1}(X,A;M) \longrightarrow \cdots,$$

where $C^k(X,A;M) = \mathrm{Hom}(C_k(X,A),M)$ is the set of functions $C_k(X,A) \to M$ with values in M. The kth homology group of the complex is called the k***th relative cohomology*** with coefficients in M denoted by $H^k(X,A;M)$.

5. Given two chain complexes $C_\bullet$ and $D_\bullet$, and two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, as shown in the diagram below, a ***chain homotopy*** from $f_\bullet$ to $g_\bullet$ is a sequence of homomorphisms $s_k : C_k \to D_{k+1}$ such that the maps satisfiy $\partial_{k+1} \circ s_k + s_{k-1} \circ \partial_k = f_k - g_k$. The chain maps $f_\bullet$ and $g_\bullet$ are then said to be ***homotopic*** and denoted by $f \simeq g$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, shift right, "f_{k+1}"'] \arrow[d, shift left, "g_{k+1}"] & C_k \arrow[r, "\partial_k"] \arrow[dl, "s_k"'] \arrow[d, shift right, "f_k"'] \arrow[d, shift left, "g_k"] & C_{k-1} \arrow[r] \arrow[dl, "s_{k-1}"'] \arrow[d, shift right, "f_{k-1}"'] \arrow[d, shift left, "g_{k-1}"] & \cdots \\
\cdots \arrow[r] & D_{k+1} \arrow[r, "\partial_{k+1}"'] & D_k \arrow[r, "\partial_k"'] & D_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

6. A chain map $f : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial_\bullet)$ is ***null-homotopic*** if $f \simeq 0$, where 0 is the zero chain map.

7. A chain complex $(C_\bullet, \partial_\bullet)$ has a ***contracting homotopy*** if the identity chain map $id_{C_\bullet} : C_\bullet \to C_\bullet$ is null-homotopic. Such a chain is called ***contractible***.

Proposition 2.2. *A chain complex having a contracting homotopy is an exact sequence.*

Proof. Suppose the chain complex $(C_\bullet, \partial_\bullet)$ has a contracting homotopy. Then, $id_{C_\bullet} \simeq 0$. That is the maps in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, shift right, "id"'] \arrow[d, shift left, "0"] & C_k \arrow[r, "\partial_k"] \arrow[dl, "s_k"'] \arrow[d, shift right, "id"'] \arrow[d, shift left, "0"] & C_{k-1} \arrow[r] \arrow[dl, "s_{k-1}"'] \arrow[d, shift right, "id"'] \arrow[d, shift left, "0"] & \cdots \\
\cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"'] & C_k \arrow[r, "\partial_k"'] & C_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

satisfy $\mathrm{d}_{k+1} \circ s_k + s_{k-1} \circ \partial_k = id$. Let z be a k-cycle. Then $id(z) = \mathrm{d}_{k+1} \circ s_k(z) + s_{k-1} \circ \partial_k(z) = \mathrm{d}_{k+1}(s_k(z)) \in im(\partial_{k+1})$. This implies that the induced map $id_* : H_k(C) \to H_k(C)$ on homology groups, which is an isomorphism, is equivalent to the 0 map. Thus, $H_k(C) = 0$, and the proposition follows. $\square$

2.3 Homology and Cohomology with Local Coefficients

Definition 2.3. 1. A (left) G-module is an abelian group A together with a group action $\rho : G \to \mathbf{Aut}(A)$ of G on A, where $\mathbf{Aut}(A)$ denote the automorphisms of A.

2. Let R be a commutative ring and let G be a multiplicative group. The **group ring** $R[G]$ associated to G is a ring with elements of the form $\sum_{g\in G} r_g g$ where $r_g \in R$ and $r_g = 0$ for all but finitely many $g \in G$. Addition in the group ring is given by

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G}(r_g + s_g)g,$$

where as multiplication is given by the distributive law and multiplication in G, that is

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right) = \sum_{g,h\in G}(r_g s_h)gh.$$

Next, we discuss local coefficients. There are two approaches to defining local coefficients on a space X. The first approach, which will be considered in this thesis, is via modules over group ring. The second approach is via a fibre bundle $p : E \longrightarrow X$ over X, whose fibres $p^{-1}(x)$ are identified with some fixed abelian group. Standard references for local coefficients include [DK01] and [Hat02].

In this thesis, we will be working with local coefficients via modules over the group ring $R[G]$, where R and G are ring and group respectively.

2.3.1 Homology and Cohomology with Local Coefficients via modules

In this section, we define homology and cohomology with local coefficients using modules. We also provide a proposition to help identify the (co)chain complex that will be used to compute these homologies.

Let G be a groups. Let X be a finite CW complex and let $\epsilon : \pi_1(X, x_0) \to G$ be a surjective group homomorpism. The correspondence between the conjugacy classes of subgroups of $\pi_1(X, x_0)$ and the covering spaces of X provides us with a covering space $p_G : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ which corresponds to the subgroup, $\ker\epsilon$, of $\pi_1(X, x_0)$, which is the kernel of ϵ, such that $p_{G*}(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \ker\epsilon$, where p_{G*} is the map $p_{G*} : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ induced by the covering space map. This makes the covering regular, since $\ker\epsilon$ is a normal subgroup. The automorphism group of $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is isomorphic to $\pi_1(X, x_0)/p_{G*}(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \pi_1(X, x_0)/\ker\epsilon \cong G$.

An action of G on $\widetilde{X}$ can be defined as follows. Take an element $g \in G$. Since ϵ is unto, there is an equivalence class $[\gamma] \in \pi_1(X, x_0)$ such that $\epsilon([\gamma]) = g$, where γ is a loop on X representing the class $[\gamma]$. By the lifting property, one can consider the unique lift $\widetilde{\gamma}$ of γ to $\widetilde{X}$ such that $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $p_G : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a regular cover, there exist a unique deck transformation $h : \widetilde{X} \to \widetilde{X}$ such that $h(\widetilde{x}_1) = \widetilde{x}_2$ for $\widetilde{x}_1, \widetilde{x}_2 \in p_G(x_o)$. Let $h_\gamma(\widetilde{x}_0) = \widetilde{\gamma}(1)$. So, for any $z \in \widetilde{X}$, define $z \cdot \epsilon([\gamma]) = z \cdot g = h_\gamma(z)$. This defines an action of G on the covering space $\widetilde{X}$ by acting on the points of $\widetilde{X}$ and by extension, an action on the cells of $\widetilde{X}$ via action on the points that make up the cells. Let $C_k(\widetilde{X}, \mathbb{Z}) = \left\langle e_i^k \right\rangle_{i=1}^{r}$ be the chain group generated by k-cells. Let $g \in G$, and let $\sigma^k \in C_k(\widetilde{X}, \mathbb{Z})$; $\sigma^k = \sum_{i=1}^{r} m_i e_i^k$. Then

$$\left(\sum_{i=1}^{r} m_i e_i^k\right) \cdot g = \sum_{i=1}^{r} m_i (e_i^k) \cdot g = \sum_{i=1}^{r} m_i h_\gamma (e_i^k),$$

where $[\gamma] \in \pi_i(X, x_0)$ such that $\epsilon([\gamma]) = g$. It follows that the action described above further extends to an action of G on the chain groups, $C_k(\widetilde{X}, \mathbb{Z})$, for $k \geq 0$.

Let $\mathbb{Z}[G]$ be the group ring associated with the group G. Then the action of G on $C_k(\widetilde{X}, \mathbb{Z})$ extends to an action of $\mathbb{Z}[G]$ on $C_k(\widetilde{X}, \mathbb{Z})$. It follows that $C_k(\widetilde{X}, \mathbb{Z})$ is a $\mathbb{Z}[G]$ module.

Definition 2.4. (**Homology and Cohomology with Local Coefficients**) Let $\mathcal{M}$ be a $\mathbb{Z}[\pi]$-module, where $\pi = \pi_1(X, x_0)$

1. The homology groups $H_*(X; \mathcal{M})$ of the chain complex $C_*(X; \mathcal{M}) = C_*(\widetilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathcal{M}$ are called **homology groups of X with local coefficients in $\mathcal{M}$** .

2. The homology groups of the cochain complex $C^*(X; M) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \mathbb{Z}), \mathcal{M})$ is called **cohomology of X with local coefficients in $\mathcal{M}$** denoted $H^*(X; \mathcal{M})$.

The following two propositions provide a description of how to find generators for the chain and cochain groups, $C_n(X; \mathcal{M}) = C_n(\widetilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[G]} \mathcal{M}$ and $C^*(X; M) = \mathrm{Hom}_{\mathbb{Z}[G]}(C_*(\widetilde{X}, \mathbb{Z}), \mathcal{M})$, respectively This is done by first finding the generators of the chain groups $C_k(\widetilde{X}, \mathbb{Z})$.

Proposition 2.5 (Proposition 1.33 of [Hat02]). *Suppose given a covering space $p : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ and a map $f : (Y, y_0) \longrightarrow (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\widetilde{f} : (Y, y_0) \longrightarrow (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Proposition 2.6. *Let Y be a connected finite cell complex with one 0-cell y_0 and let $\epsilon : \pi_1(Y, y_0) \twoheadrightarrow \pi_1(Y, y_0)^{\mathbf{ab}}$[1] be the abelianisation map. Let $C_k(Y, \mathbb{Z}) = \langle e_i \rangle_{i=1}^{r}$ be the free abelian group with basis the set of r k-cells of Y. Let $\widetilde{Y} \to Y$ denote the universal abelian cover determined by the abelianization map. Then the free $\mathbb{Z}[\pi_1(Y, y_0)^{\mathbf{ab}}]$-module $C_k(\widetilde{Y}, \mathbb{Z})$ is generated by the set $\{\widetilde{e_i}\}_{i=1}^{r}$ of r k-cells of the covering space $\widetilde{Y}$ of Y, where $\widetilde{e_i}$ is a lift of e_i for each $i = 1, \dots r$.*

Proof. Let Y be as in the hypothesis and let $C_k(Y, \mathbb{Z}) = \langle e_i \rangle_{i=1}^{r}$. Let e be a k-cell of Y and let $f : S^{k-1} \longrightarrow Y$ be the ***attaching* map**[2]. For each element $\alpha \in \pi_1(Y, y)^{\mathbf{ab}}$, there is a lift $\widetilde{y}_0^{\alpha}$ of y_0 by Proposition 2.5. The k-cell e is simply connected. It follows that there is a unique lift $\widetilde{e}^{\alpha}$ of e for each $\alpha \in \pi_1(Y, y)^{\mathbf{ab}}$ such that $\widetilde{f}(y_0) = \widetilde{y}_0^{\alpha}$.

Now, consider the unique lift $\widetilde{e}^{\mathbf{0}}$ of e such that $\widetilde{f}(y_0) = \widetilde{y}_0^{\mathbf{0}}$. Let $g \in \pi_1(Y, y)^{\mathbf{ab}}$ be the element corresponding to the deck transformation $h : \widetilde{Y} \to \widetilde{Y}$ such that $h(\widetilde{y}_0^{\mathbf{0}}) = \widetilde{y}_0^{\alpha}$. Then, $g \cdot \widetilde{e}^{\mathbf{0}} = h(\widetilde{e}^{\mathbf{0}})$ is a lift of e such that $\widetilde{f}(y_0) = \widetilde{y}_0^{\alpha}$. The uniqueness property of lifts implies that $g \cdot \widetilde{e}^{\mathbf{0}} = \widetilde{e}^{\alpha}$.

This argument can be repeated for each $k-$cell e of Y and thus, it is enough to consider a single lift of each k-cell of Y to $\widetilde{Y}$ to generate $C_k(\widetilde{Y}, \mathbb{Z})$ as $\mathbb{Z}[\pi_1(Y, y)^{\mathbf{ab}}]$-module. It follows that if $C_k(Y, \mathbb{Z}) = \langle e_i \rangle_{i=1}^{r}$, then $C_k(\widetilde{Y}, \mathbb{Z}) = \langle \widetilde{e_i} \rangle_{i=1}^{r}$. □

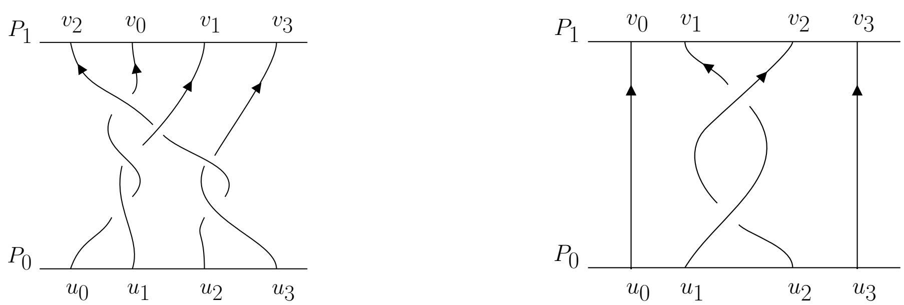
[1] Here, $\pi_1(Y, y_0)^{\mathbf{ab}}$ is the abelianization of the fundamental group. By definition, it is the quotient group $\pi_1(Y, y_0)/[\pi_1(Y, y_0), \pi_1(Y, y_0)]$, which is isomorphic to $H_1(Y; \mathbb{Z})$.

[2] Consider the pair (D^k, S^{k-1}) of an k-disk and its boundary. Let us think of D^k as an k-cell. Given a map $f : S^{k-1} \longrightarrow Y$, then we can attach an k-cell to X via f by the following identification: $Y \cup_f D^k := (Y \sqcup D^k)/(a \sim f(a))$ for all $s \in S^{k-1}$. We say $Y \cup_f D^k$ arises from attaching D^k to X along f and f is called an attaching map.

2.4 Braids, String links and their complements

In this section, we define and examine braids and string links. We present a cell structure for the complement of a string link or braid embedded in $D^2 \times [0,1]$. Furthermore, we analyze a local coefficient system on the complement of a string link or braid.

Definition 2.7. A ***strand*** is a continuous curve $f : [0,1] \to D^2 \times [0,1]$ that starts at a point on the disk D^2 at $x = 0$ and ends at a point on the disk at D^2 at $x = 1$. A ***free strand*** is a strand of a string link that is not involved in a crossing.



(a) A non pure braid **(b)** A pure braid

Figure 2.1: Pure and non-pure braids.

Definition 2.8.

1. A ***braid*** on $n+1$ strands is a geometric object consisting of two parallel planes P_0 and P_1 in three-dimensional space $\mathbb{R}^3$, containing two ordered sets of points $a_0, a_1, \ldots, a_n \in P_0$ and $b_0, b_1, \ldots, b_n \in P_1$, and $n+1$ simple non-intersecting strands $l_0, l_1, \ldots l_n$, intersecting each parallel plane P_t between P_0 and P_1 exactly once and joining the points $\{a_0, a_1, \ldots, a_n\}$ to $\{b_0, b_1, \ldots, b_n\}$. The points $a_0, a_1, \ldots, a_n$ are called the ***initial points*** and the points $b_0, b_1, \ldots, b_n$ are called ***end points***. Figure 2.1 shows some examples of braids.

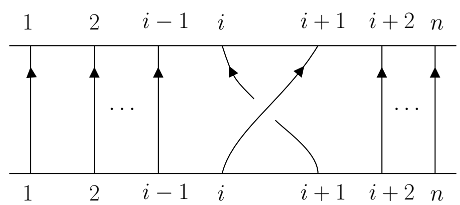
2. ***Labeling*** the strands of an $(n+1)$-braid β involves indexing the strands at the bottom from left to right, starting from 0 and going up to n. An indexed braid is called a ***labeled braid***. Let $T = \{0, 1, \cdots, n\}$ denote the indexing set; T is called the set of ***labels*** of β. The mapping of the initial positions to the final positions is a permutation of T.

   If we map T to the subscripts of the initial points, then each braid induces a permutation on the subscripts of the initial points $a_0, a_1, \ldots, a_n$. This is because the subscripts of the initial points $a_0, a_1, \ldots, a_n$ of the strands will generally be in a different order at the endpoints $b_0, b_1, \ldots, b_n$. For example the braid in Figure 2.1a induces the permutation $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$, where as the braid in Figure 2.1b induces the identity permutation.

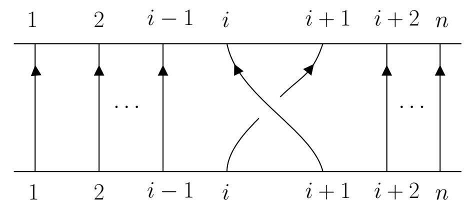
3. A ***pure braid***[3] is a braid that induces the identity permutation. That is, if the mapping of the initial positions to the final positions induces the identity permutation on T. Otherwise, the braid is called a ***non-pure braid***.

[3] In the literature, pure braids are sometimes referred to as ***coloured braids***, since each strand can be assigned a distinct colour (label) in a way compatible with composition

4. The ***Artin braid group*** B_n is the group generated by $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ such that the braid relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j = 1, 2, \ldots, n-1$ with $|i-j| \geq 2$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$ are satisfied.



(a) Over-crossing σ_i



(b) Under-crossing σ_i^{-1}

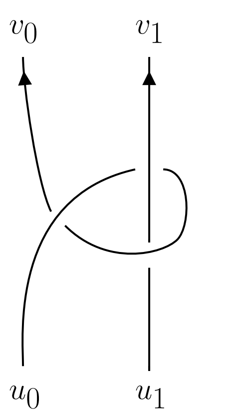
Figure 2.2: Geometric representation of σ_i and σ_i^{-1}

The generator σ_i denotes a positive crossing between the strand at position number i as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

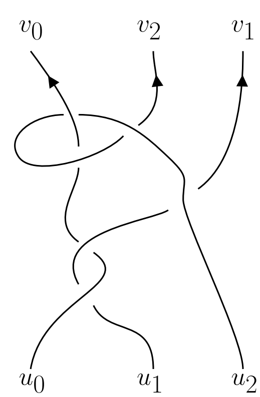
Definition 2.9. (**String Link**)

1. Let n be a positive integer and fix $n+1$ points, $p_0, p_1, \cdots p_n$, in the interior of the 2-disk, D^2. A ***string link*** L of n components is a smooth, proper, oriented 1-dimensional submanifold of $D^2 \times [0, 1]$ homeomorphic to the disjoint union of $n+1$ intervals (strands) such that the initial point of each interval (strand) coincides with some $p_i \times \{0\}$ and the endpoint coincides with $p_j \times \{1\}$.

   Just like braids, string links also induce permutations. See the diagram below.



(a) A string link with identity permutation.



(b) A string link with non-identity permutation.

Figure 2.3: String links inducing a permutation.

2. A ***pure string link*** is a string link that induces the identity permutation.

Remark 2.10. Braids form a group, where multiplication is achieved by stacking one braid on the other. String links on the other hand do not form a group since isotopy classes of string links need not have inverses. However, isotopy classes of pure n-string links form a semi-group.

There is a multiplication on labeled string links where one string link is stacked on the another. Two $n+1$- labeled string links can be multiplied if the labels of the bottom end of one equals the labels of the top end of the other. For example in Figure 2.4a the two string links are composable where as the two string

(a) The string links can be multiplied

(b) The string can not be multiplied.

Figure 2.4: Multiplication of string links

links in Figure 2.4b can not be multiplied.

By convention, all braids and string links will be oriented such that the strands move from the bottom to the top and we do not allow closed components. Also, the ambient space of braids and string links is taken to be the solid cylinder $D^2 \times [0,1]$; see Figure 1.5a. Unless otherwise stated, all string links will be denoted by L and the complement of the strands will be denoted $X = \left(D^2 \times [0,1]\right) - L$.

2.4.1 A Cell structure for the complement of a string link

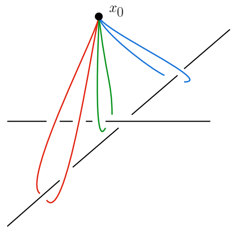
Let L be a labeled $(n+1)$-string link. L has positive crossings ( ) and negative crossings ( ). Suppose L has m crossings, then the fundamental group of the complement $X = \left(D^2 \times [0,1]\right) - L$ has a presentation $\langle\, g_1, \ldots, g_s \mid r_1, \ldots, r_m \,\rangle$ where the g_i's are closed loop around the strands of L and the r_j's are relations of the form $g_i g_j g_i^{-1} g_k^{-1}$ at each crossing. The closed loops are called ***meridians***. This presentation is called the ***Wirtinger presentation***.

Description 2.11 (Labelling scheme for Wirtinger generators)**.** The following is a labelling scheme for the generators of the Wirtinger presentation. Let $u_0, u_1, \ldots, u_n$ denote the generators corresponding to meridians which lie in $D^2 \times \{0\}$ and let $v_0, v_1, \ldots, v_n$ denote the generators corresponding to meridians which lie in $D^2 \times \{1\}$. Here, the subscripts of u_i and v_i are the labels of the corresponding strands. The remaining generators are denoted by $z_{1,i}, z_{2,i}, \ldots, z_{r_i,i}$, where r_i is the number of meridians $z_{k,i}$ on a strand with label i, $1 \le k \le r_i$. Let S be the set of the generators (meridians).

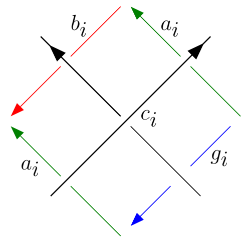
Ordering 2.12 (Ordering of generators(meridians))**.** The ordering S will always be

$$S : u_0, u_1, \ldots, u_n, z_{1,i}, z_{2,i}, \ldots, z_{r_i,i}, v_1, \ldots, v_n. \tag{2.1}$$

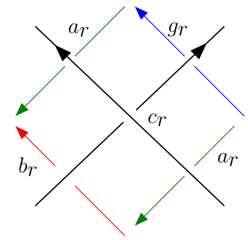
The complement X deformation retracts to a space Y which retains information at each crossing. The space Y can be constructed using the Wirtinger presentation as follows. At each positive crossing c_i (see



(a) A generic crossing



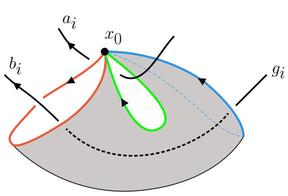
(b) An over-crossing (positive crossing)



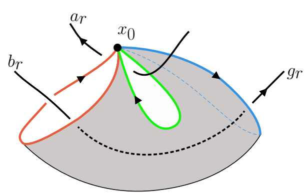
(c) An under-crossing (negative crossing)

Figure 2.5: Over and under-Crossing

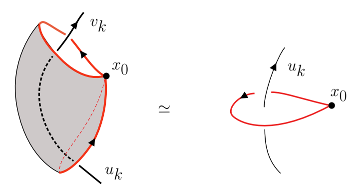
Figure 2.5b) there is a Wirtinger relation $a_i b_i a_i^{-1} g_i^{-1}$, and at each negative crossing c_r (see Figure 2.5c) there is a Wirtinger relation $a_r b_r a_r^{-1} g_r^{-1}$, where $a_i, a_r, b_i, b_r, g_i, g_r \in \left\{u_j, v_j, z_{k,s}\right\}_{j=0,1,\ldots,n;\ 1\leq k\leq r_s}$. Note that u_i, u_j, v_i, v_j are closed paths which form 1-cells; each path begins at the base point (0-cell) x_0, goes underneath a strand and back to x_0 (see Figure 2.5a). These relations form the boundaries of 2-cells that correspond to the m crossings. Figures 2.6a and 2.6b exhibits 2-cells at a positive and a negative crossing. If a strand of the string link is not involved in any of the crossings, then we have a 1-cell u_k (see Figure 2.6c).



(a) Cell structure at a positive crossing

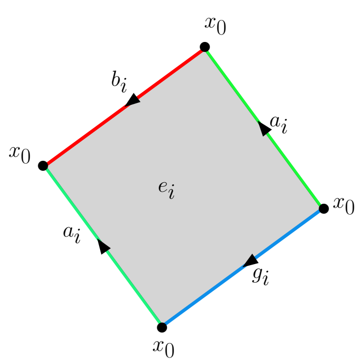


(b) Cell structure at a negative crossing

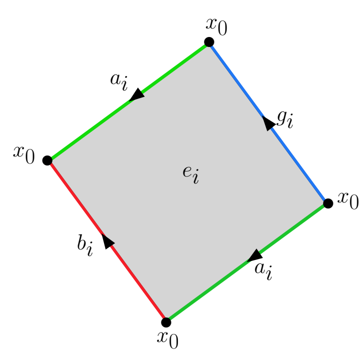


(c) Cell structure of a free strand

Figure 2.6: Cell structures at an over-crossing, under-crossing and a free strand.



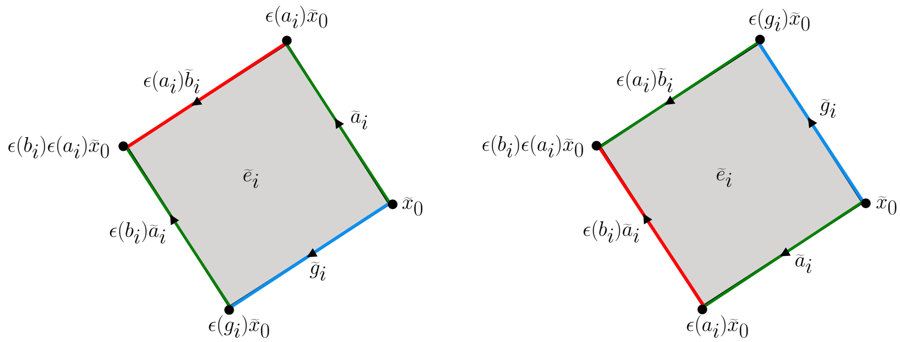
(a) The cell structure at a positive crossing viewed as a square



(b) The cell structure at a negative crossing viewed as a square

Figure 2.7: Cell structure at a positive and a negative crossing.

Gluing the cells together – using appropriate attaching maps – results in the desired cell-complex structure for the deformation retract Y, from which its chain complex is determined. Since the complement is connected, all 3-cells can be deformed to a point, and all these points can be identified. By Proposition 2.6, the chain complex of the covering space $\widetilde{Y}$ can be determined.



(a) The lift of e_i at a positive crossing

(b) The lift of e_i at a negative crossing

Figure 2.8: Cell structure of the lift of a 2-cell at a positive and a negative crossing.

2.4.2 A Local Coefficient System on the complement

Let L be an $n+1$ string link and let $X = \left(D^2 \times [0,1]\right) - L$. Let $G = \pi_1(X, x_0)$ be the fundamental group of X. The abelianisation, $G/[G,G]$, of G is isomorphic to the free group $Z_{n+1} \simeq \langle t_0, t_1, \ldots, t_n \rangle$ and the abelianisation map $\epsilon : G \to \langle t_0, t_1, \cdots, t_n \rangle$ is determined by assigning to a meridian of X its corresponding t_i, $i = 0, 1, \cdots, n$. Let $\Lambda = \mathbb{Z}[\langle t_0, t_1, \cdots, t_n \rangle]$ and let $\mathcal{F} = \mathbb{Q}(\langle t_0, t_1, \cdots, t_n \rangle)$ be the field of fractions for Λ. Let $g \in G$, then multiplication by $\epsilon(g)$ determines a local coefficient system on X with coefficients in Λ or $\mathcal{F}$, and hence the homology and cohomology groups $H_*(X; \mathcal{M})$ and $H^*(X; \mathcal{M})$ respectively (See Definition 2.4), where $\mathcal{M}$ is either Λ or $\mathcal{F}$.

Description 2.13 (Boundary of e_i and $\widetilde{e_i}$)**.** The cell structure of the 2-cells in Figure 2.6a and Figure 2.6b can be viewed as a square with the sides α_i identified as seen in Figure 2.7. The lifts of these cells are also given in Figure 2.8. Each $\widetilde{e_i}$ has boundary of the form

$$\partial(\widetilde{e_i}) = (1 - \epsilon(b_i))a_i + \epsilon(a_i)b_i - g_i \tag{2.2}$$

which is computed as follows:

The boundary $\partial_2(e_i)$ of each 2-cell e_i is a word (path) formed by the 1-cells of the deformation retract Y of the complement X. The boundary of each e_i at a crossing is of the form $a_i b_i a_i^{-1} g_i^{-1}$, regardless of the type of crossing. Label the 1-cells (meridians) with the labels of their corresponding strands. Note that b_i and g_i have the same labels since they are on the under strand of the crossing, so $\epsilon(b_i) = \epsilon(g_i)$. Following the path $a_i b_i a_i^{-1} g_i^{-1}$:

1. The lift of a_i is $\widetilde{a_i}$, a path from $\widetilde{x_0}$ to $\epsilon(a_i)\widetilde{x_0}$, where $\widetilde{x_0}$ is a lift of x_0.
2. The lift of b_i is the path $\epsilon(a_i)\widetilde{b_i}$, starting from $\epsilon(a_i)\widetilde{x_0}$ and ending at $\epsilon(a_i)\epsilon(b_i)\widetilde{x_0}$.
3. The lift of a_i^{-1} is the path $-\epsilon(b_i)\widetilde{a_i}$, starting from $\epsilon(a_i)\epsilon(b_i)\widetilde{x_0}$ and ending at $\epsilon(a_i)\epsilon(b_i)\epsilon(a_i)^{-1}\widetilde{x_0} = \epsilon(b_i)\widetilde{x_0} = \epsilon(g_i)\widetilde{x_0}$.
4. Finally, the lift of g_i^{-1} is the path $-\widetilde{g_i}$, from $\epsilon(g_i)\widetilde{x_0}$ to $\epsilon(c_i)\epsilon(g_i)^{-1}\widetilde{x_0} = \widetilde{x_0}$.

Since the lifts are elements in a free abelian group, the boundary of $\widetilde{e_i}$ is written additively as:

$$\widetilde{a_i} + \epsilon(a_i)\widetilde{b_i} - \epsilon(b_i)\widetilde{a_i} - \widetilde{g_i},$$

which simplifies to:

$$(1-\epsilon(b_i))\widetilde{a_i}+\epsilon(a_i)\widetilde{b_i}-\widetilde{g_i}.$$

Remark 2.14. Let $X_0 = X \cap D^2 \times \{0\}$ and $X_1 = X \cap D^2 \times \{1\}$. Both X_0 and X_1 are $n+1$-punctured discs (see Figure 3.1); they are canonically identified via the homeomorphism $\chi : X_0 \to X_1$ defined by $(x,0) \mapsto (x,1)$. They are subspaces of the complement X. Let ρ be a permutation induced by L. Then note that ρ also permutes the punctures of X_1 via the homeomorphism χ. Multiplication by $\epsilon(g)$ therefore determines local coefficient systems on X_0 and X_1, with coefficients in $\mathcal{F}$ and $\mathcal{F}^\rho$ respectively, where $\mathcal{F}^\rho = \mathbb{Q}(\langle\, t_{\rho(0)}, t_{\rho(1)}, \cdots, t_{\rho(n)}\,\rangle)$. So ρ permutes the set of labels (colors) $T = \{t_i\}_{i=0}^n$. Note that $\mathcal{F}$ and $\mathcal{F}^\rho$ are the same. For simplicity, we will also use $\mathcal{F}$ to denote $\mathcal{F}^\rho$.

Chapter 3

The Gassner and reduced Gassner invariant

3.1 Summary of Chapter

In this chapter, we will delve into the cohomology and homology Gassner invariants of string links and braids, and offer various examples to elucidate the computation of these invariants. It will be observed that these two invariants are inverse transpose of each other. Furthermore, we will verify that the homology Gassner invariant is a braid invariant. Subsequently, in Chapter 4, we will verify that it is also a string link invariant. Finally, a Mathematica implementation of the homology Gassner invariant will be presented. The main reference for this chapter is [KLW01].

3.2 The cohomology Gassner invariant

In this section, we present the cohomological approach to defining the Gassner invariant for a string link or braid as described in [KLW01]. We also provide several examples to illustrate this construction.

Given a string link L with $n+1$ strands, let $X = (D^2 \times I) - L$ be the complement of L, where D^2 is the 2dimensional disk, and let π_X denote the fundamental group $\pi_1(X, x_0)$ of the the complement X. The abelianisation of π_X, is isomorphic to the free abelian group, $\langle t_i \rangle_{i=1}^n$, generated by the set $\{t_0, t_1, \cdots, t_n\}$. The abelianisation map $\epsilon : \pi_X \to \langle t_i \rangle_{i=0}^n$ is determined by assigning to a meridian (a closed path around a strand of L) its corresponding t_i. Let $\Lambda = \mathbb{Z}[\langle t_i \rangle_{i=1}^n]$ and let $\mathcal{F} = \mathbb{Q}(\langle t_i \rangle_{i=1}^n)$ be the field of fractions of Λ. Let $X_0 = X \cap D \times \{0\}$ and $X_1 = X \cap D \times \{1\}$. Both X_0 and X_1 are $n-$punctured disks (see Figure 3.1); they are canonically identified via the homeomorphism $\chi : X_0 \to X_1$ defined by $(x,0) \mapsto (x,1)$. Fix a point $q \in \partial D$ and let $I_q \subset X$ be the arc $I_q = \{q\} \times [0,1]$.

The following ordering of the crossings will help in ordering the 2-cells.

Ordering 3.1. (Ordering the crossings) Order the crossings in the following manner: let the first crossing c_1 be the crossing at which the strand with Wirtinger generator u_1 ends. Next, the second crossing c_2 is the crossing at which the strand with Wirtinger generator u_2 ends. Repeat this process going through the Wirtinger generators $u_i, i = 2, 3, \cdots, n+1$ and then through $z_{1,i}, z_{2,i}, \ldots, z_{r_i,i}$, where r_i is the number of meridians $z_{k,i}$ on a strand with label i, $1 \le k \le r_i$.

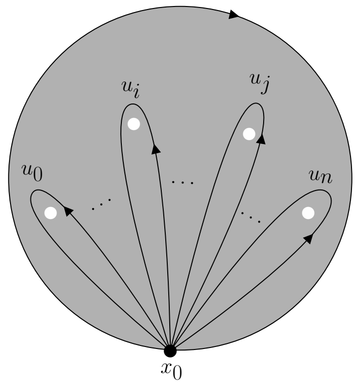


Figure 3.1: An n-punctured disk with loops $u_0, u_1, \cdots, u_n$. The white objects are the punctures.

Lemma 3.2 ([KLW01], Proposition 2.3). *Let (X, W) be a pair of path connected cell complexes and $\epsilon : \pi_1(X, x_0) \to \mathbb{Z}^n$ a homomorphism. Consider the corresponding local coefficients $\mathcal{F}$ on the pair (X, W). Suppose the inclusion of W in X induces an isomorphism on homology with (*untwisted*) $\mathbb{Q}$ coefficients. Then $H_*(X, W; \mathcal{F}) = 0$.*

Proof. Let $(C_*(X, W; \mathbb{Q}), \partial)$ denote the cellular chain complex with coefficients in $\mathbb{Q}$ for the pair and let (X, W) and let $(C_*(\widetilde{X}, \widetilde{W}; \mathbb{Q}), \widetilde{\partial})$ denote the cellular chain complex, also with $\mathbb{Q}$, of the covering space determined by the map $\epsilon : \pi_1(X, x_0) \to \mathbb{Z}^n$. Fix lifts of the cells of (X, W) to $(\widetilde{X}, \widetilde{W})$ to get a free $\mathcal{F}$-basis of $C_*(\widetilde{X}, \widetilde{W})$ by using Proposition 2.6.

Since the inclusion $W \hookrightarrow X$ induces an isomorphism on homology by the hypothesis, then $C_*(X, W)$ is acyclic, meaning that the homology group of the complex $C_*(X, W)$ are all 0. It follows that there exists a chain contraction $s : C_*(X, W) \to C_*(X, W)$, where s is a degree 1 map satisfying $\partial_{n+1} s_n + s_{n-1} \partial_n = Id$ (see Definition 2.1).

Using the $\mathcal{F}$-free basis for $C_*(\widetilde{X}, \widetilde{W})$ and the formula for $\partial_{n+1} s_n + s_{n-1} \partial_n = Id$, define a chain homotopy $\widetilde{s} : C_*(\widetilde{X}, \widetilde{W}) \to C_*(\widetilde{X}, \widetilde{W})$. That is, if $s(e) = \sum_i q_i y_i$ then, define $\widetilde{s}(\widetilde{e}) = \sum_i q_i \widetilde{y_i}$, where $\widetilde{e}, \widetilde{y_i}$ are the chosen lifts of e, y_i.

By construction $\Phi = \widetilde{\partial}_{n+1} \widetilde{s}_n + \widetilde{s}_{n-1} \widetilde{\partial}_n$ is a chain map whose matrix in the chosen basis augments to the identity map, meaning that if $a : \mathcal{F} \to \mathbb{Q}$ is the augmentation $t_i \mapsto 1$, then $a(\Phi) = Id$ is the identity map. Dualizing the complex $(C_*(\widetilde{X}, \widetilde{W}; \mathbb{Q}), \widetilde{\partial})$, the induced chain homotopy on the cochain complex $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Q}), \mathcal{F})$ is a chain homotopy from Φ^* to 0. Thus, Φ^* induces the zero map on the cohomology group $H^*(X, W; \mathcal{F})$ of the complex $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Q}), \mathcal{F})$. Note that the determinant of the matrix induced by Φ^* is a non-zero element of $\mathbb{Z}[\pi]$ since $a(\Phi) = Id$. Hence, Φ^* is an isomorphism. This implies the zero map is also an isomorphism. It follows that the cohomology group, $H^*(X, W; \mathcal{F})$ is 0. □

Lemma 3.3 ([KLW01], Lemma 2.1).

1. *$H^1(X_0; \mathcal{F}) \cong H^1(X_1; \mathcal{F}) \cong \mathcal{F}^n$ and $H^1(X_0, q; \mathcal{F}) \cong H^1(X_1, q; \mathcal{F}) \cong \mathcal{F}^{n+1}$.*

2. *Let $\iota_j : X_j \hookrightarrow X$ be the inclusion maps for $j = 0, 1$. The restriction maps $\iota_j^* : H^1(X; \mathcal{F}) \to H^1(X_j; \mathcal{F})$ and $\iota_j^* : H^1(X, I_q; \mathcal{F}) \to H^1(X_j, q; \mathcal{F})$, for $j = 0, 1$ are all isomorphisms.*

Proof. 1. The subspace $X_0 = X \cap D \times \{0\}$ is an $(n+1)$-punctured deformation retracts to the wedge product, $Y_0 = \vee_{i=0}^n S^1$, of $n+1$ copies of S^1. The cell structure of Y_0 consists of only one 0-cell, q, and $n+1$ 1-cells, u_i, for $i = 0, 1, \ldots, n$. Up to homotopy, both X_0 and Y_0 are the same, and it is much easier doing computations with Y_0. The fundamental group $\pi_1(Y_0, p)$ of Y_0 is the free group generated by the set $U = \{u_i\}_{i=0}^n$. The abelianisation map $\epsilon : \pi_1(Y_0, p) \longrightarrow \langle t_i \rangle_{i=0}^n$ sends each generator u_i to the corresponding t_i in $\langle t_i \rangle_{i=0}^n$. By Proposition 2.6, we have a cellular chain complex

$$\cdots 0 \xrightarrow{\partial_2} C_1(\widetilde{X_0};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{X_0};\mathbb{Z}) \xrightarrow{\partial_0} 0,$$

where $C_0(\widetilde{Y_0},\mathbb{Z}) = \langle\ \widetilde{q} \rangle$ and $C_1(\widetilde{Y},\mathbb{Z}) = \langle\ \widetilde{u_i}\ \rangle_{i=0}^n$. Dualizing the above complex with $\mathrm{Hom}_\Lambda(-,\mathcal{F})$, gives the corresponding cellular cochain complex

$$0 \to C^0(Y_0;\mathcal{F}) \xrightarrow{d^0} C^1(Y_0;\mathcal{F}) \xrightarrow{d^1} 0,$$

where $C^0(Y_0;\mathcal{F}) = \langle\ \widetilde{Q} \rangle$ and $C^1(Y_0;\mathcal{F}) = \left\langle\ \widetilde{U_i}\ \right\rangle_{i=0}^n$, and $\widetilde{Q}(\widetilde{q}) = 1$, $\widetilde{Q}(\widetilde{u_j}) = 0$, $\widetilde{U}_i(\widetilde{u_j}) = \delta_{ij}$ and $\widetilde{U}(\widetilde{q}) = 0$. The kernel of d^1 is $Z^1(Y_0;\mathcal{F}) = \ker d^1 = C^1(Y_0,\mathcal{F}) \cong \mathcal{F}^{n+1}$ and the image of d^0 is $\left\langle \sum_{i=0}^n (t_i - 1)\widetilde{U}_i \right\rangle \cong \mathcal{F}$, since $d^0(\widetilde{Q})(\widetilde{u_i}) = \widetilde{Q}(\partial_1(\widetilde{u_i})) = \widetilde{Q}((t_i - 1)\widetilde{q}) = t_i - 1$. The first cohomology $H^1(Y_0;\mathcal{F})$ is $\frac{\ker d^1}{\mathbf{im}\, d^0} = \left\langle\ \widetilde{U}_i\ \right\rangle_{i=1}^{n-1} \cong \mathcal{F}^n$. Hence, $H^1(X_0;\mathcal{F}) \cong H^1(Y_0;\mathcal{F}) \cong \mathcal{F}^n$. A similar argument shows that $H^1(X_1;\mathcal{F}) \cong \mathcal{F}^n$.

Next, the pair (X_0, q), induces a short exact sequence, which in turn induces a long exact sequences:

$$0 \to C^n(X_0,q;\mathcal{F}) \to C^n(X_0;\mathcal{F}) \to C^n(q;\mathcal{F}) \to 0,$$

$$\begin{array}{l} 0 \longrightarrow H^0(X_0,q;\mathcal{F}) \longrightarrow H^0(X_0;\mathcal{F}) \longrightarrow H^0(q;\mathcal{F}) \longrightarrow \\ \qquad \longrightarrow H^1(X_0,q;\mathcal{F}) \longrightarrow H^1(X_0;\mathcal{F}) \longrightarrow H^1(q;\mathcal{F}) \longrightarrow \dots . \end{array}$$

Note that in the cochain complex of Y_0 above, map d^0 is injective, so its kernel is 0. It follows that $H^0(X_0;\mathcal{F}) \cong 0$. Also, notice that the cochain groups of q are all 0 except $C^0(q;\mathcal{F}) \cong \mathcal{F}$. It follows that $H^1(q;\mathcal{F}) \cong 0$ and $H^0(q;\mathcal{F}) \cong \mathcal{F}$. Hence, the long exact sequence becomes

$$\begin{array}{l} 0 \longrightarrow H^0(X_0,q;\mathcal{F}) \longrightarrow 0 \longrightarrow \mathcal{F} \longrightarrow \\ \qquad \longrightarrow H^1(X_0,q;\mathcal{F}) \longrightarrow \mathcal{F}^n \longrightarrow 0 \longrightarrow \dots, \end{array}$$

which implies that $H^1(X_0,q;\mathcal{F}) \cong \mathcal{F} \oplus \mathcal{F}^n \cong \mathcal{F}^{n+1}$. A similar argument using the inclusion $q \hookrightarrow X_1$ shows that $H^1(X_1,q;\mathcal{F}) \cong \mathcal{F}^{n+1}$.

2. Since X_j deformation retracts to the wedge product of $n+1$ circles, we have $H^1(X_j;\mathbb{Q}) \cong \mathbb{Q}^{n+1}$. Also, the abelianization of the fundamental group of X is $H_1(X;\mathbb{Q})$, which is the free abelian group generated by $n+1$ generator. This is because each strand of the string link contributes a generator to $H_1(X,\mathbb{Q})$, corresponding to the meridian around each string. Thus, the inclusion maps $\iota_j : X_j \hookrightarrow X$ $\iota_j : (X_j, q) \hookrightarrow (X, I_q)$ satisfy the hypothesis of Lemma 3.2, so $H^*(X, X_j;\mathcal{F}) = 0$. It follows from the

long exact sequence of the pair (X, X_j):

$$\begin{tikzcd} \cdots \arrow[r] & \cancel{H^1(X,X_j;\mathcal{F})} \arrow[r] & H^1(X;\mathcal{F}) \arrow[r] & H^1(X_j;\mathcal{F}) \arrow[dll, rounded corners] \\ & \cancel{H^2(X,X_j;\mathcal{F})} \arrow[r] & H^2(X;\mathcal{F}) \arrow[r] & H^2(X_j;\mathcal{F}) \arrow[r] & \cdots. \end{tikzcd}$$

that the maps $H^1(X;\mathcal{F}) \xrightarrow{\iota_j^*} H^1(X_j;\mathcal{F})$ are isomorphisms for $j = 0, 1$. Similarly, $H^1(X, I_q;\mathcal{F}) \xrightarrow{\iota_j^*} H^1(X_j, q;\mathcal{F})$ are isomorphisms for $j = 0, 1$.

□

Definition 3.4 (Gassner Invariant-Cohomological definition [KLW01]). To a string link L assign the automorphism

$$\mathcal{G}_c((L): H^1(X_0, p;\mathcal{F}) \xleftarrow{\iota_0^*} H^1(X, I_p;\mathcal{F}) \xrightarrow{\iota_1^*} H^1(X_1, p;\mathcal{F}).$$

The composition $\mathcal{G}_c((L) = \iota_1^* \circ (\iota_0^*)^{-1}$ is called the ***cohomology Gassner invariant*** of the string link L

The following lemma provides a simple way to compute the relative (co)homology groups $H^1(X, I_q;\mathcal{F})$.

Lemma 3.5. *Let X be the complement of a string link L and $I_q = q \times [0,1]$, where $q = x_0$ is a fixed point on the boundary of D^2. Let Y be the deformation retract of X as described in Section 2.4.1 with one 0-cell q. Then the relative homology group $H^1(X, I_q;\mathcal{F})$ is isomorphic to $H^1(Y, q;\mathcal{F}) = \ker(d^1)$, where d^1 is the coboundary map $C^1(Y;\mathcal{F}) \xrightarrow{d^1} C^2(Y;\mathcal{F})$.*

Proof. Suppose X and Y satisfy the hypothesis. Then the inclusion map $(Y, q) \hookrightarrow (X, I_q)$ induces the isomorphism $H^1(X, I_q;\mathcal{F}) \cong H^1(Y, q;\mathcal{F})$. The relative complex,

$$\cdots\ C_2(\widetilde{Y}, \widetilde{q};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y}, \widetilde{q};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{Y}, \widetilde{q};\mathbb{Z}) \to 0,$$

of the pair (Y, q), is equivalent to

$$\cdots 0 \to C_2(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_1} 0,$$

since

$$\begin{aligned} C_0(\widetilde{Y}, \widetilde{q};\mathbb{Z}) &= C_0(\widetilde{Y};\mathbb{Z})/C_0(\widetilde{q};\mathbb{Z}) = \langle \widetilde{q} \rangle / \langle \widetilde{q} \rangle = 0, \\ C_1(\widetilde{Y}, \widetilde{q};\mathbb{Z}) &= C_1(\widetilde{Y};\mathbb{Z})/C_1(\widetilde{q};\mathbb{Z}) = C_1(\widetilde{Y};\mathbb{Z})/0 \cong C_1(\widetilde{Y};\mathbb{Z}), \\ C_2(\widetilde{Y}, \widetilde{q};\mathbb{Z}) &= C_2(\widetilde{Y};\mathbb{Z})/C_2(\widetilde{q};\mathbb{Z}) = C_2(\widetilde{Y};\mathbb{Z})/0 \cong C_2(\widetilde{Y};\mathbb{Z}) \\ C_3(\widetilde{Y}, \widetilde{q};\mathbb{Z}) &= 0/0 = 0 \\ &\vdots \end{aligned}$$

Dualizing with $\mathrm{Hom}_{\mathbb{Z}[\pi_Y]}(-,\mathcal{F})$, gives the cochain complex

$$0 \xrightarrow{d^0} C^1(Y;\mathcal{F}) \xrightarrow{d^1} C^2(Y;\mathcal{F}) \to 0 \cdots .$$

But $H^1(Y,q;\mathcal{F}) \cong \ker(d^1)$ and the lemma follows. □

Corollary 3.6. *Let $X_j, j = 0,1$, be the punctured disks associated with the complement of a string link. Then the relative cohomology groups $H^1(X_j,q;\mathcal{F})$ is isomorphic to $\ker(d^1) = C^1(X_j;\mathcal{F})$ for each $j = 0,1$.*

Proof. The corollary follows immediately from Lemma 3.5, since all but $C^1(X_j;\mathcal{F})$ is 0 for each $j = 0,1$. □

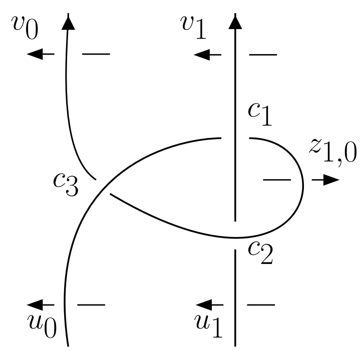
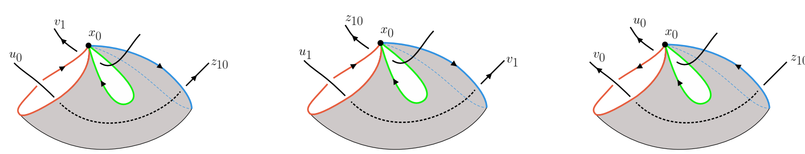
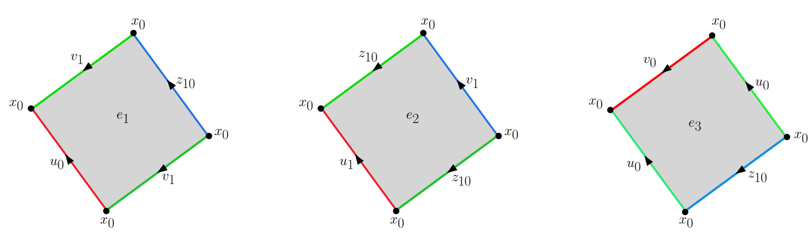


Figure 3.2: A 2-component string link L_1.



(a) 2-cell at c_1. **(b)** 2-cell at c_2. **(c)** 2-cell at c_3.

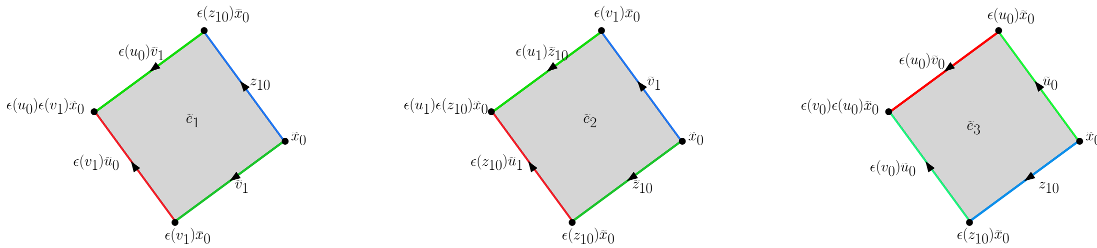
Figure 3.3: Cells of the deformation retract Y.



(a) 2-cell at c_1. **(b)** 2-cell at c_2. **(c)** 2-cell at c_3.

Figure 3.4: The cell structure at crossings c_1, c_1, c_3, viewed as a square.

Example 3.7 (Cohomology Gassner invariant)**.** Let L_1 be the string link in Figure 3.2; it has 3 crossing c_1, c_2 and c_3. Let $X = \left(D^2 \times [0,1]\right) - L_1$ and let Y be the deformation retract of the complement X with a 0-cell $q = x_0$, five 1-cells $u_0, u_1, z_{1,0}, v_0, v_1$, and three 2-cells e_1, e_2 and e_3 as shown in Figure 3.3 and Figure 3.4. Note that I_q also deformation retracts to q.



(a) 2-cell at c_1. **(b)** 2-cell at c_2. **(c)** 2-cell at c_3.

Figure 3.5: The lift of the cells at crossings c_1, c_1, c_3, viewed as a square.

Let $\widetilde{Y}$ be the covering space of Y determined by $\epsilon : \pi_1(X, q) \to \langle t_0, t_1 \rangle$, where $q = x_0$. By Proposition 2.6, the relative chain groups of the relative complex $0 \to C_2(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) \to 0$ are

$$\begin{aligned}
C_0(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) &= C_0(\widetilde{Y}; \mathbb{Z})/C_0(\widetilde{q}; \mathbb{Z}) = 0/0 = 0, \\
C_1(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) &= C_1(\widetilde{Y}; \mathbb{Z})/C_1(\widetilde{q}; \mathbb{Z}) = C_1(\widetilde{Y}; \mathbb{Z})/0 = \langle \widetilde{u}_0, \widetilde{u}_1, \widetilde{z}_{1,0}, \widetilde{v}_0, \widetilde{v}_1 \rangle, \\
C_2(\widetilde{Y}, \widetilde{q}; \mathbb{Z}) &= C_2(\widetilde{Y}; \mathbb{Z})/C_2(\widetilde{q}; \mathbb{Z}) = C_2(\widetilde{Y}; \mathbb{Z})/0 = \langle \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3 \rangle.
\end{aligned}$$

Up to isomorphism, the relative complex reduces to the chain complex

$$0 \to C_2(\widetilde{Y}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y}; \mathbb{Z}) \xrightarrow{\partial_1} 0.$$

Dualizing with $\mathrm{Hom}_{\mathbb{Z}[\pi_Y]}(-, \mathcal{F})$, gives the cochain complex

$$0 \xrightarrow{d^0} C^1(Y; \mathcal{F}) \xrightarrow{d^1} C^2(Y; \mathcal{F}) \to 0,$$

where the cochain groups are

$$C^1(Y; \mathcal{F}) = \langle \widetilde{U}_0, \widetilde{U}_1, \widetilde{Z}_{1,0}, \widetilde{V}_0, \widetilde{V}_1 \rangle, \quad C^2(Y; \mathcal{F}) = \langle \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3 \rangle.$$

For $\psi \in C^1(Y; \mathcal{F})$, $d^1(\psi) \in C^2(Y; \mathcal{F}) = \mathrm{Hom}_{\mathbb{Z}[\pi_Y]}(C_1(Y; \mathcal{F}), \mathcal{F})$. If $x \in C_2(Y; \mathcal{F})$, then $d^1(\psi)(x) = \psi(\partial_2(x))$. From Figure 3.5 and by Description 2.13, the image of the boundary map, ∂_2 on each generators $\widetilde{e}_i, i = 1, 2, 3$ is

$$\begin{aligned}
\partial_2(\widetilde{e}_1) &= (1 - t_0)\widetilde{v}_1 + t_1 \widetilde{u}_0 - \widetilde{z}_{1,0} \\
\partial_2(\widetilde{e}_2) &= (1 - t_1)\widetilde{z}_{1,0} + t_0 \widetilde{u}_1 - \widetilde{v}_1, \\
\partial_2(\widetilde{e}_3) &= (1 - t_0)\widetilde{u}_0 + t_0 \widetilde{v}_0 - \widetilde{z}_{1,0}
\end{aligned} \tag{3.1}$$

The matrix representation for d^1 is $\begin{pmatrix} \widetilde{U}_0(\partial_2(\widetilde{e}_1)) & \widetilde{U}_1(\partial_2(\widetilde{e}_1)) & \widetilde{Z}_{1,0}(\partial_2(\widetilde{e}_1)) & \widetilde{V}_0(\partial_2(\widetilde{e}_1)) & \widetilde{V}_1(\partial_2(\widetilde{e}_1)) \\ \widetilde{U}_0(\partial_2(\widetilde{e}_2)) & \widetilde{U}_1(\partial_2(\widetilde{e}_2)) & \widetilde{Z}_{1,0}(\partial_2(\widetilde{e}_2)) & \widetilde{V}_0(\partial_2(\widetilde{e}_2)) & \widetilde{V}_1(\partial_2(\widetilde{e}_2)) \\ \widetilde{U}_0(\partial_2(\widetilde{e}_3)) & \widetilde{U}_1(\partial_2(\widetilde{e}_3)) & \widetilde{Z}_{1,0}(\partial_2(\widetilde{e}_3)) & \widetilde{V}_0(\partial_2(\widetilde{e}_3)) & \widetilde{V}_1(\partial_2(\widetilde{e}_3)) \end{pmatrix}$, which is

$$d^1_{matrix} = \begin{pmatrix} t_1 & 0 & -1 & 0 & 1 - t_0 \\ 0 & t_0 & 1 - t_1 & 0 & -1 \\ 1 - t_0 & 0 & -1 & t_0 & 0 \end{pmatrix}.$$

The nullspace of d^1_{matrix} is $\left\langle \begin{pmatrix} -\frac{1-t_0}{t_0+t_1-1} \\ -\frac{t_1t_0-t_0-2t_1+1}{t_0+t_1-1} \\ -\frac{(t_0-1)^2}{t_0+t_1-1} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{t_0}{t_0+t_1-1} \\ \frac{(t_1-1)t_1}{t_0+t_1-1} \\ \frac{t_0t_1}{t_0+t_1-1} \\ 1 \\ 0 \end{pmatrix} \right\rangle$, from which we get the kernel of $d^1\colon Z^1(Y;p) =$ $\ker(d^1) = \left\langle \frac{t_0t_1\widetilde{Z}_{1,0}}{t_0+t_1-1} + \frac{t_0\widetilde{U}_0}{t_0+t_1-1} + \frac{(t_1-1)t_1\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_0, -\frac{(t_0-1)^2\widetilde{Z}_{1,0}}{t_0+t_1-1} - \frac{(1-t_0)\widetilde{U}_0}{t_0+t_1-1} - \frac{(t_1t_0-t_0-2t_1+1)\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_1 \right\rangle$, which is the relative cohomology $H^1(Y,q;\mathcal{F})$ of the relative complex of the pair (Y,q) by Lemma 3.5. By Lemma 3.5,

$$H^1(X,q;\mathcal{F}) \cong \ker(d^1).$$

According to Corollary 3.6, $H^1(X_0,q;\mathcal{F}) = C_1(X_0;\mathcal{F}) = \langle \widetilde{U}_0, \widetilde{U}_1 \rangle$ and $H^1(X_1,q;\mathcal{F}) = C_1(X_1;\mathcal{F}) = \langle \widetilde{V}_0, \widetilde{V}_1 \rangle$. By definition of the cohomology Gassner invariant, the map $\iota_0^* : H^1(X,q;\mathcal{F}) \to H^1(X_0,q;\mathcal{F})$ is given by

$$\iota_0^* : \begin{cases} \frac{t_0t_1\widetilde{Z}_{1,0}}{t_0+t_1-1} + \frac{t_0\widetilde{U}_0}{t_0+t_1-1} + \frac{(t_1-1)t_1\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_0 & \mapsto \quad \frac{t_0\widetilde{U}_0}{t_0+t_1-1} + \frac{(t_1-1)t_1\widetilde{U}_1}{t_0+t_1-1} \\ -\frac{(t_0-1)^2\widetilde{Z}_{1,0}}{t_0+t_1-1} - \frac{(1-t_0)\widetilde{U}_0}{t_0+t_1-1} - \frac{(t_1t_0-t_0-2t_1+1)\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_1 & \mapsto \quad -\frac{(1-t_0)\widetilde{U}_0}{t_0+t_1-1} - \frac{(t_1t_0-t_0-2t_1+1)\widetilde{U}_1}{t_0+t_1-1} \end{cases}$$

and the map $\iota_1^* : H^1(X,q;\mathcal{F}) \to H^1(X_1;\mathcal{F})$ is given by

$$\iota_1^* : \begin{cases} \frac{t_0t_1\widetilde{Z}_{1,0}}{t_0+t_1-1} + \frac{t_0\widetilde{U}_0}{t_0+t_1-1} + \frac{(t_1-1)t_1\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_0 & \mapsto \quad \widetilde{V}_0 \\ -\frac{(t_0-1)^2\widetilde{Z}_{1,0}}{t_0+t_1-1} - \frac{(1-t_0)\widetilde{U}_0}{t_0+t_1-1} - \frac{(t_1t_0-t_0-2t_1+1)\widetilde{U}_1}{t_0+t_1-1} + \widetilde{V}_1 & \mapsto \quad \widetilde{V}_1 \end{cases}.$$

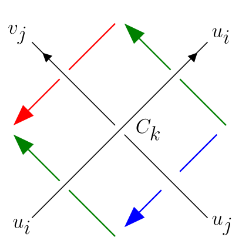
A matrix representations for ι_0^* and ι_1^* are

$$\iota_0^* = \begin{pmatrix} \frac{t_0}{t_0+t_1-1} & -\frac{(1-t_0)}{t_0+t_1-1} \\ \frac{(t_1-1)t_1}{t_0+t_1-1} & -\frac{(t_1t_0-t_0-2t_1+1)}{t_0+t_1-1} \end{pmatrix} \text{ and } \iota_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

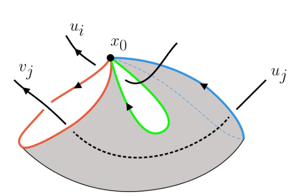
respectively. The cohomology Gassner invariant of the given string link is therefore $\mathcal{G}_c((L_1) = \iota_1^* \circ (\iota_0^*)^{-1} = (\iota_0^*)^{-1}$, so

$$\mathcal{G}_c((L_1) = (\iota_0^*)^{-1} = \begin{pmatrix} \frac{t_1t_0-t_0-2t_1+1}{t_1t_0-t_0-t_1} & \frac{t_0-1}{t_1t_0-t_0-t_1} \\ \frac{(t_1-1)t_1}{t_1t_0-t_0-t_1} & -\frac{t_0}{t_1t_0-t_0-t_1} \end{pmatrix}. \tag{3.2}$$

Example 3.8 (Cohomology Gassner invariant of over and under crossing)**.** Let $X = (D^2 \times [0,1]) - \nearrow\!\!\!\!\nwarrow$ be the complement of the over-crossing in Figure 3.6, with a deformation retract Y in Figure 3.6b.

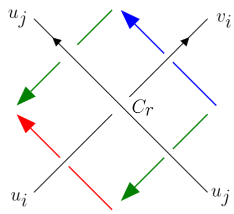


(a) 2-cell e at an over-crossing.

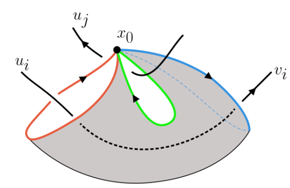


(b) Cell structure.

Figure 3.6: An over-crossing with cell structure for Y.



(a) 2-cell e at an under-crossing.



(b) Cell structure.

Figure 3.7: An under-crossing with cell structure.

The covering space $\widetilde{Y}$ of Y is determined by $\epsilon : \pi_1(X,q) \to \langle t_i, t_j \rangle$, where $q = x_0$. By Proposition 2.6, the relative chain groups of the complex

$$0 \to C_2(\widetilde{Y},\widetilde{q};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y},\widetilde{q};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{Y},\widetilde{q};\mathbb{Z}) \to 0$$

are

$$\begin{aligned} C_0(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_0(\widetilde{Y};\mathbb{Z})/C_0(\widetilde{q};\mathbb{Z}) = 0, \\ C_1(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_1(\widetilde{Y};\mathbb{Z})/C_1(\widetilde{q};\mathbb{Z}) = C_1(\widetilde{Y};\mathbb{Z})/0 = \langle \widetilde{u}_i, \widetilde{u}_j, \widetilde{v}_j \rangle, \\ C_2(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_2(\widetilde{Y};\mathbb{Z})/C_2(\widetilde{q};\mathbb{Z}) = C_2(\widetilde{Y};\mathbb{Z})/0 = \langle \widetilde{e} \rangle. \end{aligned}$$

By Lemma 3.5, the relative cohomology group $H^1(Y,q;\mathcal{F})$ is $\ker(C^1(Y;\mathcal{F}) \xrightarrow{d^1} C^2(Y;\mathcal{F}))$, where $C^1(Y;\mathcal{F}) = \langle \widetilde{U}_i, \widetilde{U}_j, \widetilde{V}_j \rangle$ and $C^2(Y;\mathcal{F}) = \langle \widetilde{E} \rangle$. By Description 2.13, the image of $\widetilde{e}$ under ∂_2 is $\partial_2(\widetilde{e}) = (1-t_j)\widetilde{u}_i + t_i\widetilde{v}_j - \widetilde{u}_j$. The matrix representation for d^1 is $\begin{pmatrix} \widetilde{U}_i(\partial_2(\widetilde{e})) & \widetilde{U}_j(\partial_2(\widetilde{e})) & \widetilde{V}_j(\partial_2(\widetilde{e})) \end{pmatrix}$, which evaluates to

$$d^1_{matrix} = \begin{pmatrix} 1-t_j & -1 & t_i \end{pmatrix}.$$

The nullspace of d^1_{matrix} is $\left\langle \begin{pmatrix} \frac{t_i}{t_j-1} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{t_j-1} \\ 1 \\ 0 \end{pmatrix} \right\rangle$. So, $\ker(d^1) = \left\langle \frac{t_i}{t_j-1}\widetilde{U}_i + \widetilde{V}_j, -\frac{1}{t_j-1}\widetilde{U}_i + \widetilde{U}_j \right\rangle$, which is the relative cohomology group $H^1(Y,q;\mathcal{F})$ of the relative complex of the pair (Y,q). By Lemma 3.5,

$$H^1(X,q;\mathcal{F}) \cong \ker(d^1).$$

According to Corollary 3.6, $H^1(X_0,q;\mathcal{F}) = C_1(X_0;\mathcal{F}) = \langle \widetilde{U}_i, \widetilde{U}_j \rangle$ and $H^1(X_1,q;\mathcal{F}) = C_1(X_1;\mathcal{F}) = \langle \widetilde{V}_j, \widetilde{U}_i \rangle$. By definition of the cohomology Gassner invariant, the map $\iota_0^* : H^1(X,q;\mathcal{F}) \to H^1(X_0,q;\mathcal{F})$ is given by

$$\iota_0^* : \begin{cases} \frac{t_i}{t_j-1}\widetilde{U}_i + \widetilde{V}_j & \mapsto \quad \frac{t_i}{t_j-1}\widetilde{U}_i \\ -\frac{1}{t_j-1}\widetilde{U}_i + \widetilde{U}_j & \mapsto \quad -\frac{1}{t_j-1}\widetilde{U}_i + \widetilde{U}_j \end{cases}$$

and the map $\iota_1^* : H^1(X,q;\mathcal{F}) \to H^1(X_1;\mathcal{F})$ is given by

$$\iota_1^* : \begin{cases} \frac{t_i}{t_j-1}\widetilde{U}_i + \widetilde{V}_j & \mapsto \quad \frac{t_i}{t_j-1}\widetilde{U}_i + \widetilde{V}_j \\ -\frac{1}{t_j-1}\widetilde{U}_i + \widetilde{U}_j & \mapsto \quad -\frac{1}{t_j-1}\widetilde{U}_i \end{cases}$$

A matrix representations for ι_0^* and ι_1^* are

$$\iota_0^* = \begin{pmatrix} \frac{t_i}{t_j-1} & -\frac{1}{t_j-1} \\ 0 & 1 \end{pmatrix} \text{ and } \iota_1^* = \begin{pmatrix} 1 & 0 \\ \frac{t_i}{t_j-1} & -\frac{1}{t_j-1} \end{pmatrix}$$

respectively. Thus, the cohomology Gassner invariant of the over-crossing is

$$\begin{aligned} \mathcal{G}_c(\nearrow\!\!\!\!\nwarrow) &= \iota_1^* \circ (\iota_0^*)^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{t_i}{t_j-1} & -\frac{1}{t_j-1} \end{pmatrix} \begin{pmatrix} \frac{t_i}{t_j-1} & -\frac{1}{t_j-1} \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where i is the label (colour) of the over strand and j is the label of the under strand. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

Taking the inverse of $\mathcal{G}_c(\nearrow\!\!\!\!\nwarrow)$ gives the cohomology Gassner invariant of the under-crossing $\nwarrow\!\!\!\!\nearrow$. That is $\mathcal{G}_c(\nearrow\!\!\!\!\nwarrow)^{-1} = \begin{pmatrix} 0 & 1 \\ t_i & 1-t_j \end{pmatrix}$

3.3 The Homology and the reduced homology Gassner invariant

In this section, we present the homological definition of the Gassner invariant and explore its relationship with the cohomology version. We provide several examples to illustrate these concepts and provide formulas for this invariant. Additionally, we demonstrate that the homology Gassner invariant is equivalent to the reduced homology Gassner invariant.

Lemma 3.9 (Homology version of Lemma 3.2)**.** *Let (X, W) be a pair of path connected cell complexes and $\epsilon : \pi_1(X, x_0) \to \mathbb{Z}^n$ a homomorphism. Consider the corresponding local coefficients $\mathcal{F}$ on the pair (X, W). Suppose the inclusion of W in X induces an isomorphism on homology with (*untwisted*) $\mathbb{Q}$ coefficients. Then $H_*(X, W; \mathcal{F}) = 0$.*

Proof. Let $(C_*(X, W), \partial)$ denote the cellular chain complex with coefficients in $\mathbb{Q}$ for the pair (X, W) and let $(C_*(\widetilde{X}, \widetilde{W}), \widetilde{\partial})$ denote the cellular chain complex also with $\mathbb{Q}$ of the covering space determined by the map $\epsilon : \pi_1(X, x_0) \to \mathbb{Z}^n$. Fix lifts of the cells of (X, W) to $(\widetilde{X}, \widetilde{W})$ to get a free $\mathcal{F}$-basis of $C_*(\widetilde{X}, \widetilde{W})$ by using Lemma 2.6.

Since the inclusion $W \hookrightarrow X$ induces an isomorphism on homology by then hypothesis, then $C_*(X, W)$ is acyclic. It follows that there exists a chain contraction $s : C_*(X, W) \to C_*(\widetilde{X}, \widetilde{W})$; s is a map of degree 1 satisfying $\partial_{n+1} s_n + s_{n-1} \partial_n = Id$.

Using the $\mathcal{F}$-free basis for $C_*(\widetilde{X}, \widetilde{W})$ and the formula for $\partial_{n+1} s_n + s_{n-1} \partial_n = Id$, define a chain homotopy $\widetilde{s} : C_*(\widetilde{X}, \widetilde{W}) \to C_*(\widetilde{X}, \widetilde{W})$; if $s(e) = \sum_i q_i y_i$ then define $\widetilde{s}(\widetilde{e}) = \sum_i q_i \widetilde{y_i}$ where $\widetilde{e}, \widetilde{y_i}$ are the chosen lifts of e, y_i.

$\Phi = \widetilde{\partial}_{n+1} \widetilde{s}_n + \widetilde{s}_{n-1} \widetilde{\partial}_n$ is a chain map whose matrix in the chosen basis augments to the identity map; if $a : \mathcal{F} \to \mathbb{Q}$ is the augmentation $t_i \mapsto 1$, then $a(\Phi) = Id$ is the identity map. The induced chain homotopy on the chain complex $C_*(\widetilde{X}, \widetilde{W}) \otimes_{\mathbb{Z}[\pi_1(X, x_0)]} \mathcal{F}$ is a chain homotopy from Φ_* to 0. Thus, Φ_* induces the zero map on the homology $H_*(X,W;\mathcal{F})$. Φ_* is an isomorphism since $a(\Phi_*) = Id$. This implies the zero map is also an isomorphism. It follows that the homology, $H_*(X,W;\mathcal{F})$, of $C_*(\widetilde{X},\widetilde{W})\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathcal{F}$ is 0. □

Lemma 3.10. *Let L be an $(n+1)$-string link and let $X = \big(D^2\times[0,1]\big)-L$. Also, let X_0 and X_1 be the subspaces of X described above. Then*

1. *$H_1(X_0;\mathcal{F}) \cong \mathcal{F}^n \cong H_1(X_1;\mathcal{F})$ and $H_1(X_0,q;\mathcal{F}) \cong \mathcal{F}^{n+1} \cong H_1(X_1,q;\mathcal{F})$, where $q = x_0$.*
2. *Let $\iota_j : X_j \hookrightarrow X$ be the inclusion maps for $j = 0,1$. Then $\iota_{j*} : H_1(X_j;\mathcal{F}) \to H_1(X;\mathcal{F})$ is an isomorphism for $j = 0,1$. Also, $\iota_{j*} : H_1(X_j,q;\mathcal{F}) \to H_1(X,I_q;\mathcal{F})$ for $j = 0,1$ is an isomorphism, where $I_q = \{q\}\times[0,1]$.*

Proof. 1. The subspace X_0 deformation retracts to the wedge product $W = \bigvee_{i=0}^{n} S^1$ of $n+1$ copies of S^1, which has a cell structure consisting of one 0-cell denoted q and $n+1$ 1-cells u_k, for $k = 0,1,\dots n$. The fundamental group $\pi_1(W,q)$ of W is the free group generated by the loops $\{u_k\}_{k=0}^n$ and the abelianisation map $\epsilon : \pi_1(W,q) \to \langle\, u_k\rangle_{k=0}^n$ sends each generator u_k to its corresponding t_k. Thus by Proposition 2.6, we have a cellular chain complex

$$C_*(\widetilde{X_0};\mathbb{Z}) = 0\xrightarrow{\partial_2}C_1(\widetilde{X_0};\mathbb{Z})\xrightarrow{\partial_1}C_0(\widetilde{X_0};\mathbb{Z})\xrightarrow{\partial_0}0.$$

Fixing $\widetilde{u}_0$, note that the kernel, $\ker\partial_1$, of ∂_1 is spanned by $\Big\{(t_0-1)\widetilde{u}_j-(t_j-1)\widetilde{u}_0\Big\}_{j=1}^n$ and image of ∂_2 is $\{0\}$. Hence $H_1(X_0;\mathcal{F}) = \Big\langle(t_0-1)\widetilde{u}_j-(t_j-1)\widetilde{u}_0\Big\rangle_{j=1}^n \cong \mathcal{F}^n$, where $\widetilde{u}_k$ is the lift of u_k.

Also, For the pair (X_0,q), there is an associated long exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & H_1(q;\mathcal{F}) \arrow[r] & H_1(X_0;\mathcal{F}) \arrow[r] & H_1(X_0,q;\mathcal{F}) \arrow[dll, rounded corners] \\ & H_0(q;\mathcal{F}) \arrow[r] & H_0(X_0;\mathcal{F}) \arrow[r] & H_0(X_0,q;\mathcal{F}) \arrow[r] & \dots, \end{tikzcd}$$

which reduces to $0\to 0\to H_1(X_0;\mathcal{F})\to H_1(X_0,q,\mathcal{F})\to H_0(q,\mathcal{F})\to 0\to H_1(X_0,q;\mathcal{F})\to\cdots$, since the homology groups $H_1(q;\mathcal{F})$ and $H_0(X_0;\mathcal{F})$ are isomorphic to 0. It follows that $H_1(X_0,q;\mathcal{F}) \cong H_1(X_0;\mathcal{F})\oplus H_0(q,\mathcal{F}) \cong \mathcal{F}^{n+1}$, since $0\to 0\to H_1(X_0;\mathcal{F})\to H_1(X_0,q,\mathcal{F})\to H_0(q,\mathcal{F})\to 0$ is an exact sequence of vector spaces.

2. The inclusions $\iota_j : X_j \hookrightarrow X$ $\iota_j : (X_j,q) \hookrightarrow (X,I_q)$ satisfy the hypothesis of Lemma 3.9, since the inclusion map of $\iota_j : X_j \to X$ induces the isomorphism $H_1(X_j;\mathbb{Q}) \cong H_1(X;\mathbb{Q}) \cong \mathbb{Q}^{n+1}$. Thus, the maps
$$H_1(X_j;\mathcal{F})\xrightarrow{\iota_{j*}}H_1(X;\mathcal{F}) \text{ and } H_1(X_j,q;\mathcal{F})\xrightarrow{\iota_{j*}}H_1(X,I_q;\mathcal{F})$$
are isomorphisms for $j = 0,1$. Here, we use the same argument in the proof of Part 2 of Lemma 3.3, replacing the long exact sequence with the homology version.

□

The proof of Lemma 3.10 leads to the following definitions.

Definition 3.11. To a string link L assign the map

$$\mathcal{G}_h(L) : H_1(X_0,p;\mathcal{F})\xrightarrow{\iota_{0*}}H_1(X,I_p;\mathcal{F})\xrightarrow{\iota_{1*}^{-1}}H_1(X_1,p;\mathcal{F}).$$

This is called the ***homology Gassner invariant*** of L.

Definition 3.12. To a string link L assign the map

$$\mathcal{G}_h^r(L): H_1(X_0;\mathcal{F}) \xrightarrow{\iota_{0*}} H_1(X;\mathcal{F}) \xrightarrow{\iota_{1*}^{-1}} H_1(X_1;\mathcal{F}).$$

This is called the ***reduced homology Gassner invariant*** of L.

Remark 3.13. Let L be an $n+1$ string link with the initial points of the strands indexed 0 through n from left to right. The endpoints are also, indexed likewise. Note that the permutation induced by the string link is not the same as the indexing at the endpoints. Let $X = (D^2 \times [0,1]) - L$ be the complement of L. Then the relative homology groups $H_1(X, I_q;\mathcal{F}), H_1(X_0, q;\mathcal{F})$ and $H_1(X_1, q;\mathcal{F})$ are vector spaces over the field $\mathcal{F}$. If we fix the set $\{x_k\}_{k=0}^n$ indexed from 0 through n and identify the basis of each of the vector spaces with this set, then the vector spaces are isomorphic to $\mathcal{F}\langle x_0, x_1, \cdots, x_n\rangle$. Likewise the homology groups $H_1(X;\mathcal{F}), H_1(X_0;\mathcal{F})$ and $H_1(X_1;\mathcal{F})$ are vector spaces over the field $\mathcal{F}$ isomorphic to $\mathcal{F}\langle x_1, \cdots, x_n\rangle$.

Fact 3.14. In homology groups $H_n(X) = \frac{\ker(\partial_n)}{im(\partial_{n+1})}$, elements in the image $im(\partial_{n+1})$ are taken to be the zero equivalence class $[0] \in H_n(X)$.

Example 3.15 (Homology Gassner invariant)**.** We compute the homology Gassner invariant of the string link, L_1 in Figure 3.2; it has 3 crossing c_1, c_2 and c_3. Let $X = \left(D^2 \times [0,1]\right) - L_1$ and let Y be the deformation retract of the complement X with cells in Figure 3.3 and Figure 3.4.

Let $\widetilde{Y}$ be the covering space of Y determined by $\epsilon : \pi_1(X, q) \to \langle t_0, t_1\rangle$, where $q = x_0$. The relative chain groups are given as follows. Notice that $C_i(\widetilde{q}) = 0$ for $i \neq 0$, $C_i(\widetilde{Y}) = 0$ for $i \neq 0,1,2$. The non-zero chain groups are $C_0(\widetilde{Y};\mathbb{Z}) = C_0(\widetilde{q};\mathbb{Z}) = \langle \widetilde{q}\rangle$, $C_1(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_1, \widetilde{z}_{1,0}, \widetilde{v}_0, \widetilde{v}_1\rangle$, $C_2(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\rangle$. So, the relative chain groups are $C_0(Y,p;\mathcal{F}) = C_0(\widetilde{Y};\mathbb{Z})/C_0(\widetilde{p};\mathbb{Z}) = 0, C_1(Y,p;\mathcal{F}) = C_1(\widetilde{Y};\mathbb{Z})/C_1(\widetilde{p};\mathbb{Z}) \cong C_1(\widetilde{Y};\mathbb{Z}), C_2(Y,p;\mathcal{F}) = C_2(\widetilde{Y};\mathbb{Z})/C_2(\widetilde{p};\mathbb{Z}) \cong C_2(\widetilde{Y};\mathbb{Z})$. Thus, it suffices to compute the relative homology groups using the chain complex

$$0 \to C_2(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_1} 0.$$

The kernel of ∂_1 is $\ker \partial_1 = C_1(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_1, \widetilde{z}_{1,0}, \widetilde{v}_0, \widetilde{v}_1\rangle$. The lift $\widetilde{e_i}$ of the 2-cell e_i can be view as a square (see Figure 3.5) whose boundary is of the form $(1-\epsilon(b_i))\widetilde{a_i} + \epsilon(a_i)\widetilde{b_i} - \widetilde{g_i}$. Thus, the image of ∂_2 is $\langle \partial_2(\widetilde{e}_1), \partial_2(\widetilde{e}_2), \partial_2(\widetilde{e}_3)\rangle$, where

$$\begin{aligned}
\partial_2(\widetilde{e}_1) &= (1-t_0)\widetilde{v}_1 + t_1\widetilde{u}_0 - \widetilde{z}_{1,0}, \\
\partial_2(\widetilde{e}_2) &= (1-t_1)\widetilde{z}_{1,0} + t_0\widetilde{u}_1 - \widetilde{v}_1, \\
\partial_2(\widetilde{e}_3) &= (1-t_0)\widetilde{u}_0 + t_0\widetilde{v}_0 - \widetilde{z}_{1,0}.
\end{aligned} \tag{3.3}$$

Since linear combinations of the generators of $\ker \partial_1$ can be formed, we can rewrite the kernel as $\ker \partial_1 = \langle \widetilde{u}_0, \partial_2(\widetilde{e}_2), \widetilde{z}_{1,0}, \partial_2(\widetilde{e}_3), \partial_2(\widetilde{e}_1)\rangle$. So, the quotient $H_1(Y,p;\mathcal{F})$ is given as

$$\begin{aligned}
H_1(Y,p;\mathcal{F}) &= \frac{\langle \widetilde{u}_0, \partial_2(\widetilde{e}_2), \widetilde{z}_{1,0}, \partial_2(\widetilde{e}_3), \partial_2(\widetilde{e}_1)\rangle}{\langle \partial_2(\widetilde{e}_1), \partial_2(\widetilde{e}_2), \partial_2(\widetilde{e}_3)\rangle} \\
&= \langle \widetilde{u}_0, \widetilde{z}_{1,0}\rangle.
\end{aligned}$$

Thus, $H_1(X, I_p; \mathcal{F}) = H_1(Y, p; \mathcal{F}) = \langle \widetilde{u}_0, \widetilde{z}_{1,0} \rangle$. The twice punctured disk X_0 deformation retracts to the wedge product of two circles. So, the relative chain groups are $C_i(X_0, p; \mathcal{F}) = 0$ for all i except for $i = 1$, where $C_1(X_0, p; \mathcal{F}) = C_1(\widetilde{X}_0; \mathbb{Z})/C_1(\widetilde{q}, \mathbb{Z}) = C_1(\widetilde{X}_0; \mathbb{Z})/0 = C_1(\widetilde{X}_0; \mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_1 \rangle$. The relative chain complex for $\widetilde{X_0}$ can therefore be taken as $0 \to C_1(\widetilde{X}_0; \mathbb{Z}) \xrightarrow{\partial_1} 0$. So, the relative homology $H_1(X_0, q; \mathcal{F})$ of the pair (X_0, q) is

$$H_1(X_0, q; \mathcal{F}) = \ker \partial_1 = \langle \widetilde{u}_0, \widetilde{u}_1 \rangle.$$

Similarly, the relative homology $H_1(X_1, q; \mathcal{F})$ of the pair (X_1, q) is

$$H_1(X_1, q; \mathcal{F}) = \ker \partial_1 = \langle \widetilde{v}_0, \widetilde{v}_1 \rangle.$$

According to the definition of the Gassner invariant in Definition 3.11, we need to find the isomorphisms $\iota_{0*} : H_1(X_0, q; \mathcal{F}) \to H_1(X, I_q; \mathcal{F})$ and $\iota_{1*}^{-1} : H_1(X, I_q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$. Define $\iota_{0*} : H_1(X_0, q; \mathcal{F}) \to H_1(X, I_q; \mathcal{F})$ as follows. Since $H_1(X, I_p; \mathcal{F}) = H_1(Y, p; \mathcal{F}) = \langle \widetilde{u}_0, \widetilde{z}_{1,0} \rangle$, map the generator $\widetilde{u}_0$ to $\widetilde{u}_0$. Next, we need to map the generator $\widetilde{u}_1$ to a linear combination of $\widetilde{u}_0$ and $\widetilde{z}_{1,0}$. Using Fact 3.14, and eliminating $\widetilde{v}_1$ from the first two equations of Equation 3.3, we obtain $\widetilde{u}_1 = \frac{t_1}{t_0(t_0-1)} \widetilde{u}_0 + \frac{t_0 t_1 - t_0 - t_1}{t_0(t_0-1)} \widetilde{z}_{1,0}$. Since ι_{0*} is an inclusion, then map $\widetilde{u}_1$ to $\frac{t_1}{t_0(t_0-1)} \widetilde{u}_0 + \frac{t_0 t_1 - t_0 - t_1}{t_0(t_0-1)} \widetilde{z}_{1,0}$. So, define $\iota_{0*} : H_1(X_0, q; \mathcal{F}) \to H_1(X, I_q; \mathcal{F})$ by

$$\iota_{0*} : \begin{cases} \widetilde{u}_0 & \mapsto & \widetilde{u}_0 \\ \widetilde{u}_1 & \mapsto & \frac{t_1}{t_0(t_0-1)} \widetilde{u}_0 + \frac{t_0 t_1 - t_0 - t_1}{t_0(t_0-1)} \widetilde{z}_{1,0} \end{cases}$$

Next, define $\iota_{1*}^{-1} : H_1(X, I_q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$ as follows. Using Fact 3.14, and eliminating $\widetilde{z}_{1,0}$ from the first and last equations of *Equation 3.3*, we get $\widetilde{u}_0 = \frac{t_0}{t_0+t_1-1} v_0 + \frac{t_0-1}{t_0+t_1-1} v_1$. Similarly, eliminating $\widetilde{u}_0$ from the same equations, we get $\widetilde{z}_{1,0} = \frac{t_0 t_1}{t_0+t_1-1} v_0 - \frac{(1-t_0)^2}{t_0+t_1-1} v_1$. Define $\iota_{1*}^{-1} : H_1(X, I_q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$ as

$$\iota_{1*}^{-1} : \begin{cases} \widetilde{u}_0 & \mapsto & \frac{t_0}{t_0+t_1-1} \widetilde{v}_0 + \frac{t_0-1}{t_0+t_1-1} \widetilde{v}_1 \\ \widetilde{z}_{1,0} & \mapsto & \frac{t_0 t_1}{t_0+t_1-1} \widetilde{v}_0 - \frac{(1-t_0)^2}{t_0+t_1-1} \widetilde{v}_1. \end{cases}$$

Then, the homology Gassner invariant is given by

$$\iota_{1*}^{-1} \circ \iota_{0*} : \begin{cases} \widetilde{u}_0 & \mapsto & \frac{t_0}{t_0+t_1-1} \widetilde{v}_0 + \frac{t_0-1}{t_0+t_1-1} \widetilde{v}_1 \\ \widetilde{u}_1 & \mapsto & \frac{t_1(t_1-1)}{t_0+t_1-1} \widetilde{v}_0 + \frac{2t_1+t_0-t_0 t_1-1}{t_0+t_1-1} \widetilde{v}_1 \end{cases}.$$

From computations above, the vector spaces are $H_1(X, I_p; \mathcal{F}) = \langle \widetilde{u}_0, \widetilde{z}_{1,0} \rangle, H_1(X_0, q; \mathcal{F}) = \langle \widetilde{u}_0, \widetilde{u}_1 \rangle$ and $H_1(X_1, q; \mathcal{F}) = \langle \widetilde{v}_0, \widetilde{v}_1 \rangle$. So, referring to Remark 3.13, we make the following identifications

$$\begin{matrix} \widetilde{u}_0 & \longleftrightarrow & x_0, \\ \widetilde{z}_{1,0} & \longleftrightarrow & x_1 \end{matrix}, \quad \begin{matrix} \widetilde{u}_0 & \longleftrightarrow & x_0, \\ \widetilde{u}_1 & \longleftrightarrow & x_1 \end{matrix}, \quad \begin{matrix} \widetilde{v}_0 & \longleftrightarrow & x_0, \\ \widetilde{v}_1 & \longleftrightarrow & x_1 \end{matrix}$$

respectively. With this identification, the homology Gassner invariant, $\mathcal{G}_h(L) : H_1(X_0, q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$ is given by

$$\mathcal{G}_h(L_1) : \begin{cases} x_0 & \mapsto & \frac{t_0}{t_0+t_1-1} x_0 + \frac{t_0-1}{t_0+t_1-1} x_1 \\ x_1 & \mapsto & \frac{t_1(t_1-1)}{t_0+t_1-1} x_0 + \frac{2t_1+t_0-t_0 t_1-1}{t_0+t_1-1} x_1 \end{cases},$$

with a matrix representation

$$\mathcal{G}_h(L) = \begin{pmatrix} \frac{t_0}{t_0+t_1-1} & \frac{t_1(t_1-1)}{t_0+t_1-1} \\ \frac{t_0-1}{t_0+t_1-1} & \frac{2t_1+t_0-t_0t_1-1}{t_0+t_1-1} \end{pmatrix}.$$

This is the end of the example.

Example 3.16 (Reduced homology Gassner invariant)**.** In this example, we compute the reduced homology Gassner invariant of the string link L_1'' in Figure 3.8. This string link is similar to the string link in Figure 3.2,

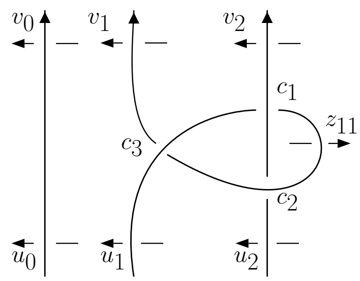
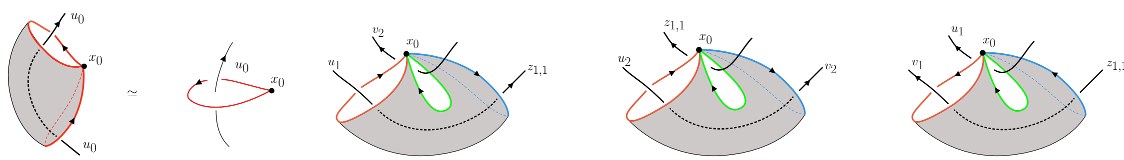


Figure 3.8: A 3-component string link L_1''.

except that it has a free strand to the left. The string link has 3 crossings c_1, c_2 and c_3. Let $X = \left(D^2 \times [0,1]\right) - L_1''$ and let Y be the deformation retract of the complement X with cells in Figure 3.9. These cells are similar to the cells in Figure 3.5.



(a) 1-cell of the free strand. **(b)** 2-cell at c_1. **(c)** 2-cell at c_2. **(d)** 2-cell at c_3.

Figure 3.9: Cells of the deformation retract Y.

Let $\widetilde{Y}$ be the covering space of Y determined by $\epsilon : \pi_1(X, q) \to \langle t_0, t_1, t_2 \rangle$, where $q = x_0$. The cellular chain complex of $\widetilde{Y}$ is

$$0 \to C_2(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{Y};\mathbb{Z}) \to 0,$$

where the chain groups are are

$$C_0(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{q} \rangle, \quad C_1(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{z}_{1,1}, \widetilde{v}_1, \widetilde{v}_2 \rangle \quad , C_2(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3 \rangle.$$

Fixing $\widetilde{u}_0$, the kernel of ∂_1 is

$$\ker \partial_1 = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\zeta}_1, \widetilde{\beta}_1, \widetilde{\beta}_2 \rangle,$$

where

$$
\begin{aligned}
\widetilde{\gamma_1} &= (t_0-1)\widetilde{u_1}-(t_1-1)\widetilde{u_0},\\
\widetilde{\gamma_2} &= (t_0-1)\widetilde{u_2}-(t_2-1)\widetilde{u_0},\\
\widetilde{\zeta_1} &= (t_0-1)\widetilde{z}_{1,1}-(t_1-1)\widetilde{u_0},\\
\widetilde{\beta_1} &= (t_0-1)\widetilde{v_1}-(t_1-1)\widetilde{u_0}\\
\widetilde{\beta_2} &= (t_0-1)\widetilde{v_2}-(t_2-1)\widetilde{u_0}
\end{aligned}
$$

Using Figure 3.5 with a some slight modification, the image of ∂_2 is $\langle\partial_2(\widetilde{e_1}),\partial_2(\widetilde{e_2}),\partial_2(\widetilde{e_3})\rangle$, where

$$
\begin{aligned}
\partial_2(\widetilde{e_1}) &= (1-t_1)\widetilde{v_2}+t_2\widetilde{u_1}-\widetilde{z}_{1,1} &&= \tfrac{t_2}{t_0-1}\widetilde{\gamma_1}-\tfrac{1}{t_0-1}\widetilde{\zeta_1}-\tfrac{t_1-1}{t_0-1}\widetilde{\beta_2}\\
\partial_2(\widetilde{e_2}) &= (1-t_2)\widetilde{z}_{1,1}+t_1\widetilde{u_2}-\widetilde{v_2} &&= \tfrac{t_1}{t_0-1}\widetilde{\gamma_2}-\tfrac{t_2-1}{t_0-1}\widetilde{\zeta_1}-\tfrac{1}{t_0-1}\widetilde{\beta_2}\\
\partial_2(\widetilde{e_3}) &= (1-t_1)\widetilde{u_1}+t_1\widetilde{v_1}-\widetilde{z}_{1,1} &&= -\tfrac{t_1-1}{t_0-1}\widetilde{\gamma_1}-\tfrac{1}{t_0-1}\widetilde{\zeta_1}-\tfrac{t_1}{t_0-1}\widetilde{\beta_1}.
\end{aligned}
\tag{3.4}
$$

Since the generators for the image of ∂_2 can be written as linear combinations of the generators $\ker\partial_1$, then we can rewrite the kernel as $\ker\partial_1=\langle\gamma_1,\partial_2(\widetilde{e_2}),\zeta_1,\partial_2(\widetilde{e_3}),\partial_2(\widetilde{e_1})\rangle$. So, the quotient $H_1(Y,\mathcal{F})$ is given as

$$
\begin{aligned}
H_1(Y,\mathcal{F}) &= \frac{\langle\widetilde{\gamma_1},\partial_2(\widetilde{e_2}),\widetilde{\zeta_1},\partial_2(\widetilde{e_3}),\partial_2(\widetilde{e_1})\rangle}{\langle\partial_2(\widetilde{e_1}),\partial_2(\widetilde{e_2}),\partial_2(\widetilde{e_3})\rangle}\\
&= \langle\widetilde{\gamma_1},\widetilde{\zeta_1}\rangle.
\end{aligned}
$$

Next, fixing u_0 and referring to the computations of Lemma 3.10, the first homology with local coefficients of X_0 is $H_1(X_0;\mathcal{F})=\langle\widetilde{\gamma_1},\widetilde{\gamma_2}\rangle$. Similarly, fixing v_0, $H_1(X_1;\mathcal{F})=\langle\widetilde{\beta_1},\widetilde{\beta_2}\rangle$, where $\widetilde{\beta_1}=(t_0-1)\widetilde{v_1}-(t_1-1)\widetilde{v_0}$ and $\widetilde{\beta_2}=(t_0-1)\widetilde{v_2}-(t_1-1)\widetilde{v_0}$.

Using Fact 3.14, and eliminating β_2 from the first two equations of Equation 3.4, we obtain $\gamma_2=\frac{t_2}{t_1(t_1-1)}\gamma_1+\frac{t_1t_2-t_1-t_2}{t_1(t_1-1)}\widetilde{\zeta_1}$. Since ι_{0*} is an inclusion, then map $\widetilde{\gamma_2}$ to $\frac{t_2}{t_1(t_1-1)}\widetilde{\gamma_1}+\frac{t_1t_2-t_1-t_2}{t_1(t_1-1)}\widetilde{\zeta_1}$. So, define ι_{0*} by

$$
\iota_{0*}:\begin{cases}\widetilde{\gamma_1} \mapsto \widetilde{\gamma_1}\\ \widetilde{\gamma_2} \mapsto \frac{t_2}{t_1(t_1-1)}\widetilde{\gamma_1}+\frac{t_1t_2-t_1-t_2}{t_1(t_1-1)}\widetilde{\zeta_1}\end{cases}
$$

Next, define $\iota_{1*}^{-1}:H_1(X;\mathcal{F})\to H_1(X_1;\mathcal{F})$ as follows. Using Fact 3.14, and eliminating $\widetilde{\zeta_1}$ from the first and last equations of *Equation* 3.4, we get $\widetilde{\gamma_1}=\frac{t_1}{t_1+t_2-1}\widetilde{\beta_1}+\frac{t_1-1}{t_1+t_2-1}\widetilde{\beta_2}$. Similarly, eliminating $\widetilde{\gamma_1}$ from the same equations, we get $\widetilde{\zeta_1}=\frac{t_1t_2}{t_1+t_2-1}\widetilde{\beta_1}-\frac{(t_1-1)^2}{t_1+t_2-1}\widetilde{\beta_2}$. Define $\iota_{1*}^{-1}:H_1(X;\mathcal{F})\to H_1(X_1,\mathcal{F})$ by

$$
\iota_{1*}^{-1}:\begin{cases}\widetilde{\gamma_1} \mapsto \frac{t_1}{t_1+t_2-1}\widetilde{\beta_1}+\frac{t_1-1}{t_1+t_2-1}\widetilde{\beta_2}\\ \widetilde{\zeta_1} \mapsto \frac{t_1t_2}{t_1+t_2-1}\widetilde{\beta_1}-\frac{(t_1-1)^2}{t_1+t_2-1}\widetilde{\beta_2}.\end{cases}
$$

Then, the reduced homology Gassner invariant is given by

$$
\iota_{1*}^{-1}\circ\iota_{0*}:\begin{cases}\widetilde{\gamma_1} \mapsto \frac{t_1}{t_1+t_2-1}\widetilde{\beta_1}+\frac{t_1-1}{t_1+t_2-1}\widetilde{\beta_2}\\ \widetilde{\gamma_2} \mapsto \frac{t_2(t_2-1)}{t_1+t_2-1}\widetilde{\beta_1}+\frac{2t_2+t_1-t_1t_2-1}{t_1+t_2-1}\widetilde{\beta_2}\end{cases}.
$$

From computations above, the vector spaces are $H_1(X;\mathcal{F})=\langle\gamma_1,\zeta_1\rangle$, $H_1(X_0;\mathcal{F})=\langle\widetilde{\gamma_1},\widetilde{\gamma_2}\rangle$ and $H_1(X_1;\mathcal{F})=$

$\langle\widetilde{\beta}_1,\widetilde{\beta}_2\rangle$. So, referring to Remark 3.13, we make the following identifications

$$\begin{array}{ccc}\widetilde{\gamma}_1 & \longleftrightarrow & x_1, \\ \widetilde{\zeta}_1 & \longleftrightarrow & x_2\end{array}, \quad \begin{array}{ccc}\widetilde{\gamma}_1 & \longleftrightarrow & x_1, \\ \widetilde{\gamma}_2 & \longleftrightarrow & x_2\end{array}, \quad \begin{array}{ccc}\widetilde{\beta}_1 & \longleftrightarrow & x_1, \\ \widetilde{\beta}_2 & \longleftrightarrow & x_2\end{array}$$

respectively. The reduced homology Gassner invariant, $\mathcal{G}_h^r(L_1'') : H_1(X_0;\mathcal{F}) \to H_1(X_1;\mathcal{F})$ is given by

$$\mathcal{G}_h^r(L_1'') : \begin{cases} x_1 & \mapsto \quad \frac{t_1}{t_1+t_2-1}x_1 + \frac{t_1-1}{t_1+t_2-1}x_2 \\ x_2 & \mapsto \quad \frac{t_2(t_2-1)}{t_1+t_2-1}x_1 + \frac{2t_2+t_1-t_1t_2-1}{t_1+t_2-1}x_2 \end{cases},$$

with a matrix representation

$$\mathcal{G}_h^r(L_1'') = \begin{pmatrix} \frac{t_1}{t_1+t_2-1} & \frac{t_2(t_2-1)}{t_1+t_2-1} \\ \frac{t_1-1}{t_1+t_2-1} & \frac{2t_2+t_1-t_1t_2-1}{t_1+t_2-1} \end{pmatrix}.$$

This is the end of the example.

Example 3.17 (Homology Gassner invariant for an over-crossing)**.** Let $X = (D^2 \times [0,1]) - \diagup\!\!\!\!\diagdown$ be the complement of the over-crossing in Figure 3.6, with a deformation retract Y in Figure 3.6b. The covering space $\widetilde{Y}$ of Y is determined by $\epsilon : \pi_1(X,q) \to \langle t_i, t_j\rangle$, where $q = x_0$. The relative chain groups of the pair (Y,q) are

$$\begin{aligned} C_0(Y,q;\mathcal{F}) &= C_0(\widetilde{Y};\mathbb{Z})/C_0(\widetilde{p};\mathbb{Z}) \cong 0 \\ C_1(Y,q;\mathcal{F}) &= C_1(\widetilde{Y};\mathbb{Z})/C_1(\widetilde{p};\mathbb{Z}) \cong C_1(\widetilde{Y};\mathbb{Z}) \\ C_2(Y,q;\mathcal{F}) &= C_2(\widetilde{Y};\mathbb{Z})/C_2(\widetilde{p};\mathbb{Z}) \cong C_2(\widetilde{Y};\mathbb{Z}), \end{aligned}$$

since $C_1(\widetilde{p};\mathbb{Z}) \cong C_2(\widetilde{p};\mathbb{Z}) \cong 0$, and $C_i(Y,q;\mathcal{F}) = 0$ for $i > 2$. So, it suffices to consider the chain complex $0 \to C_2(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y};\mathbb{Z}) \xrightarrow{\partial_1} 0$, where $C_2(\widetilde{Y};\mathbb{Z}) = \langle\widetilde{e}\rangle$ and $C_1(\widetilde{Y};\mathbb{Z}) = \langle\widetilde{u}_i,\widetilde{u}_j,\widetilde{v}_j\rangle$. By Description 2.13, the image of $\widetilde{e}$ under ∂_2 is $(1-t_j)\widetilde{u}_i + t_i\widetilde{v}_j - \widetilde{u}_j$. The kernel of ∂_1 is $\ker\partial_1 = C_1(\widetilde{Y};\mathcal{F})$. So, the quotient of the kernel by the image is $H_1(Y,q;\mathcal{F}) = \frac{\ker\partial_1}{\mathrm{im}\partial_2} = \langle\widetilde{u}_i,\widetilde{u}_j\rangle$. Hence $H_1(X,I_q;\mathcal{F}) = \langle\widetilde{u}_i,\widetilde{u}_j\rangle$.

X_0 deformation retracts to the wedge product of two circles. So, the relative chain groups are $C_i(X_0,p;\mathcal{F}) = 0$ for all i except for $i = 1$, where $C_1(X_0,p;\mathcal{F}) = C_1(\widetilde{X}_0;\mathbb{Z})/C_1(\widetilde{q},\mathbb{Z}) \cong C_1(\widetilde{X}_0;\mathbb{Z}) = \langle\widetilde{u}_i,\widetilde{u}_j\rangle$. The relative chain complex for $\widetilde{X_0}$ can therefore be taken as $0 \to C_1(\widetilde{X}_0;\mathbb{Z}) \xrightarrow{\partial_1} 0$. So, the relative homology $H_1(X_0,q;\mathcal{F})$ of the pair (X_0,q) is

$$H_1(X_0,q;\mathcal{F}) = \ker\partial_1 = \langle\widetilde{u}_i,\widetilde{u}_j\rangle.$$

Similarly, the relative homology $H_1(X_1,q;\mathcal{F})$ of the pair (X_1,q) is

$$H_1(X_1,q;\mathcal{F}) = \ker\partial_1 = \langle\widetilde{v}_j,\widetilde{u}_i\rangle.$$

Next, define the isomorphism $\iota_{0*} : H_1(X_0,q;\mathcal{F}) \to H_1(X,q;\mathcal{F})$ by

$$\iota_{0*} : \begin{cases} \widetilde{u}_i & \mapsto \quad \widetilde{u}_i \\ \widetilde{u}_j & \mapsto \quad \widetilde{u}_j \end{cases}.$$

By Fact 3.14, note that $0 \equiv \partial_2(\widetilde{e})$. So, define the isomorphism $\iota_{1*}^{-1} : H_1(X, I_q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$ by

$$\iota_{1*}^{-1} : \begin{cases} \widetilde{u}_i & \mapsto \quad \widetilde{u}_i \\ \widetilde{u}_j & \mapsto \quad t_i\widetilde{v}_j + (1-t_j)\widetilde{u}_i \end{cases}.$$

From computations above, the vector spaces are $H_1(X, I_p; \mathcal{F}) = \langle \widetilde{u}_i, \widetilde{u}_j \rangle$, $H_1(X_0, q; \mathcal{F}) = \langle \widetilde{u}_i, \widetilde{u}_j \rangle$ and $H_1(X_1, q; \mathcal{F}) = \langle \widetilde{v}_j, \widetilde{v}_i \rangle$. The homology Gassner invariant $\mathcal{G}_h(\nearrow\!\!\!\!\nwarrow) = \iota_{1*}^{-1} \circ \iota_{0*} : H_1(X_0, q; \mathcal{F}) \to H_1(X_1, q; \mathcal{F})$ of the over-crossing is therefore given by

$$\mathcal{G}_h(\nearrow\!\!\!\!\nwarrow) : \begin{cases} \widetilde{u}_i & \mapsto \quad \widetilde{u}_i \\ \widetilde{u}_j & \mapsto \quad t_i\widetilde{v}_j + (1-t_j)\widetilde{u}_i \end{cases},$$

which is represented by a matrix as

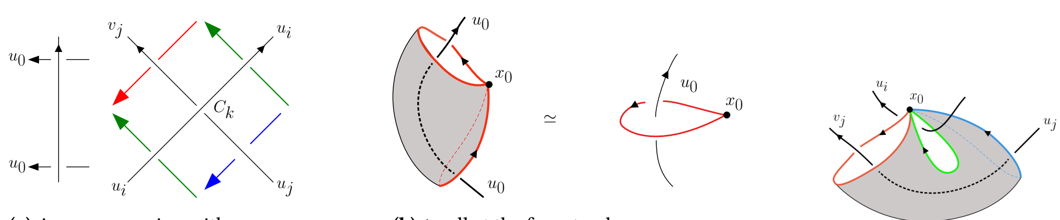
$$\mathcal{G}_h(\nearrow\!\!\!\!\nwarrow) = \begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}, \tag{3.5}$$

where i is the label of the over strand and j is the label of the under strand.

The homology Gassner invariant of the under crossing $\nwarrow\!\!\!\!\nearrow$ is computed similarly. It has a matrix representation

$$\mathcal{G}_h(\nwarrow\!\!\!\!\nearrow) = \begin{pmatrix} \frac{t_j-1}{t_i} & 1 \\ \frac{1}{t_i} & 0 \end{pmatrix}. \tag{3.6}$$

Example 3.18 (Reduced homology Gassner invariant). In the previous example, we computed the homology Gassner invariant of an over-crossing. In this example, we compute the reduced homology Gassner invariant of the over-crossing with a free strand on the left. Let L_c denote this over-crossing. The deformation retract



(a) An over-crossing with a free stand on the left.

(b) 1-cell at the free stand on the left.

(c) 2-cell structure.

Figure 3.10: An over-crossing with cell structure for Y.

Y of the complement $X = (D^2 \times [0,1]) - L_c$ has cell structure in Figure 3.10b and Figure 3.10. The covering space $\widetilde{Y}$ of Y is determined by $\epsilon : (\pi_X, q) \to \langle t_0, t_i, t_j \rangle$. The cellular chain complex of $\widetilde{Y}$ is

$$0 \to C_2(\widetilde{Y}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{Y}; \mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{Y}; \mathbb{Z}) \to 0,$$

where $C_0(\widetilde{Y}; \mathbb{Z}) = \langle \widetilde{q} \rangle$, $\quad C_1(\widetilde{Y}; \mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_i, \widetilde{u}_j, \widetilde{v}_j \rangle \quad$, $C_1(\widetilde{Y}; \mathbb{Z}) = \langle \widetilde{e} \rangle$. Fixing $\widetilde{u}_0$, the kernel of ∂_1 is

$$\ker \partial_1 = \langle \widetilde{\gamma}_i, \widetilde{\gamma}_j, \widetilde{\beta}_j \rangle,$$

where

$$\begin{aligned}\widetilde{\gamma}_i &= (t_0-1)\widetilde{u}_i-(t_i-1)\widetilde{u}_0,\\ \widetilde{\gamma}_j &= (t_0-1)\widetilde{u}_j-(t_j-1)\widetilde{u}_0,\\ \widetilde{\beta}_j &= (t_0-1)\widetilde{v}_j-(t_j-1)\widetilde{u}_0\end{aligned}$$

By Description 2.13, the image of $\widetilde{e}$ under ∂_2 is $(1-t_j)\widetilde{u}_i+t_i\widetilde{v}_j-\widetilde{u}_j$. Note also that $\partial_2(\widetilde{e})=\frac{1-t_j}{t_0-1}\widetilde{\gamma}_i+\frac{t_i}{t_0-1}\widetilde{\beta}_j-\frac{1}{t_0-1}\widetilde{\gamma}_j$. So, the quotient of the kernel by the image is $H_1(X;\mathcal{F})=H_1(Y;\mathcal{F})=\langle\widetilde{\gamma}_i,\widetilde{\gamma}_j\rangle$.

Next, fixing u_0 and referring to the computations of Lemma 3.10, the first homology with local coefficients of X_0 is $H_1(X_0;\mathcal{F})=\langle\widetilde{\gamma}_i,\widetilde{\gamma}_j\rangle$. Similarly, fixing $v_0=u_0$, $H_1(X_1;\mathcal{F})=\langle\widetilde{\beta}_j,\widetilde{\beta}_i\rangle$, where $\widetilde{\beta}_j=(t_0-1)\widetilde{v}_j-(t_j-1)\widetilde{u}_0$ and $\widetilde{\beta}_i=(t_0-1)\widetilde{u}_i-(t_i-1)\widetilde{u}_0=\widetilde{\gamma}_i$.

By Fact 3.14,

$$0\equiv\partial_2(\widetilde{e})=\frac{1-t_j}{t_0-1}\widetilde{\gamma}_i+\frac{t_i}{t_0-1}\widetilde{\beta}_j-\frac{1}{t_0-1}\widetilde{\gamma}_j,$$

which implies $\widetilde{\gamma}_j=t_i\widetilde{\beta}_j+(1-t_j)\widetilde{\gamma}_i$.

Finally, from the above computations, define the isomorphism $\iota_{0*}:H_1(X_0;\mathcal{F})\to H_1(X;\mathcal{F})$ by

$$\iota_{0*}:\begin{cases}\widetilde{\gamma}_i & \mapsto & \widetilde{\gamma}_i\\ \widetilde{\gamma}_j & \mapsto & \widetilde{\gamma}_j\end{cases},$$

and the isomorphism $\iota_{1*}^{-1}:H_1(X;\mathcal{F})\to H_1(X_1;\mathcal{F})$ by

$$\iota_{1*}^{-1}:\begin{cases}\widetilde{\gamma}_i & \mapsto & \widetilde{\gamma}_i\\ \widetilde{\gamma}_j & \mapsto & t_i\beta_j+(1-t_j)\widetilde{\gamma}_i\end{cases}.$$

From computations above, the vector spaces with ordered bases are $H_1(X;\mathcal{F})=\langle\widetilde{\gamma}_i,\widetilde{\gamma}_j\rangle, H_1(X_0;\mathcal{F})=\langle\widetilde{\gamma}_i,\widetilde{\gamma}_j\rangle$ and $H_1(X_1;\mathcal{F})=\langle\widetilde{\beta}_j,\widetilde{\beta}_i\rangle$. The reduced homology Gassner invariant $\mathcal{G}_h^r=\iota_{1*}^{-1}\circ\iota_{0*}:H_1(X_0;\mathcal{F})\to H_1(X_1;\mathcal{F})$ of the over-crossing is therefore given by

$$\mathcal{G}_h^r\left(\,|\times\,\right):\begin{cases}\widetilde{\gamma}_i & \mapsto & \widetilde{\gamma}_i\\ \widetilde{\gamma}_j & \mapsto & t_i\widetilde{\beta}_j+(1-t_j)\widetilde{\gamma}_i\end{cases},$$

with a matrix representation given by

$$\mathcal{G}_h^r\left(\,|\times\,\right)=\begin{pmatrix}0 & t_i\\ 1 & 1-t_j\end{pmatrix}.$$

Example 3.19 (Reduced homology Gassner invariant)**.** Let L_1''' be the string link in Figure 3.11; it has 3 crossing c_1, c_2 and c_3. Let $X=\left(D^2\times[0,1]\right)-L_1'''$ and let Y be the deformation retract of the complement X with cells in Figure 3.12.

Let $\widetilde{Y}$ be the covering space of Y determined by $\epsilon:\pi_1(X,q)\to\langle t_0,t_1,t_2\rangle$, where $q=x_0$. The cellular chain complex of $\widetilde{Y}$ is

$$0\to C_2(\widetilde{Y};\mathbb{Z})\xrightarrow{\partial_2}C_1(\widetilde{Y};\mathbb{Z})\xrightarrow{\partial_1}C_0(\widetilde{Y};\mathbb{Z})\to 0,$$

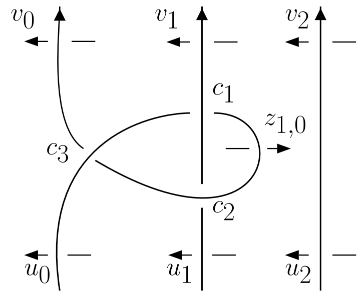
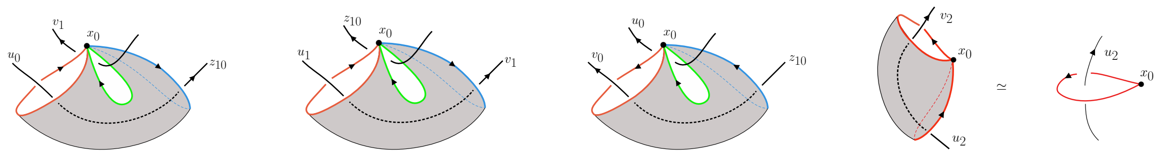


Figure 3.11: A 3-component string link.



(a) 2-cell at c_1. **(b)** 2-cell at c_2. **(c)** 2-cell at c_3. **(d)** 1-cell of the free strand.

Figure 3.12: Cells of the deformation retract Y.

where the chain groups are are

$$C_0(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{q} \rangle, \quad C_1(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{z}_{1,0}, \widetilde{v}_0, \widetilde{v}_1 \rangle \quad , C_2(\widetilde{Y};\mathbb{Z}) = \langle \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3 \rangle.$$

We are interested in the first homology with local coefficients and the computation is as follows. The image of $w_r \in \{\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{z}_{1,0}, \widetilde{v}_0, \widetilde{v}_1\}$ under the map ∂_1 is $\partial_1(w_r) = (t_r - 1)\widetilde{q}$. Thus the kernel of ∂_1 is

$$\ker \partial_1 = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\zeta}_0, \widetilde{\beta}_0, \widetilde{\beta}_1 \rangle,$$

where

$$\begin{aligned}
\widetilde{\gamma}_1 &= (t_0 - 1)\widetilde{u}_1 - (t_1 - 1)\widetilde{u}_0, \\
\widetilde{\gamma}_2 &= (t_0 - 1)\widetilde{u}_2 - (t_2 - 1)\widetilde{u}_0, \\
\widetilde{\zeta}_0 &= \widetilde{z}_{1,0} - \widetilde{u}_0, \\
\widetilde{\beta}_0 &= \widetilde{v}_0 - \widetilde{u}_0 \\
\widetilde{\beta}_1 &= (t_0 - 1)\widetilde{v}_1 - (t_1 - 1)\widetilde{u}_0
\end{aligned}$$

Using Figure 3.5, the image of ∂_2 is $\langle \partial_2(\widetilde{e}_1), \partial_2(\widetilde{e}_2), \partial_2(\widetilde{e}_3) \rangle$, where

$$\begin{aligned}
\partial_2(\widetilde{e}_1) &= (1 - t_0)\widetilde{v}_1 + t_1\widetilde{u}_0 - \widetilde{z}_{1,0} &&= \zeta_0 - \beta_1 \\
\partial_2(\widetilde{e}_2) &= (1 - t_1)\widetilde{z}_{1,0} + t_0\widetilde{u}_1 - \widetilde{v}_1 &&= \tfrac{t_0}{t_0 - 1}\gamma_1 + (1 - t_1)\zeta_0 - \tfrac{1}{t_0 - 1}\beta_1, \\
\partial_2(\widetilde{e}_3) &= (1 - t_0)\widetilde{u}_0 + t_0\widetilde{v}_0 - \widetilde{z}_{1,0} &&= \zeta_0 + t_0\beta_0
\end{aligned} \tag{3.7}$$

Using Equation 3.7, we can rewrite the kernel of ∂_1 as $\ker \partial_1 = \langle\ \partial_2(\widetilde{e}_2), \gamma_2, \zeta_0, \partial_2(\widetilde{e}_3), \partial_2(\widetilde{e}_1) \rangle$. It follows that $H_1(X;\mathcal{F}) = H_1(Y;\mathcal{F}) = \langle \gamma_2,\ \zeta_0 \rangle$

Next, fixing u_0 and referring to the computations of Lemma 3.10, the first homology with local coefficients of X_0 is $H_1(X_0;\mathcal{F}) = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2 \rangle$. Similarly, fixing $\widetilde{v}_0$, $H_1(X_1;\mathcal{F}) = \langle \widetilde{l}_1, \widetilde{l}_2 \rangle$, where $\widetilde{l}_1 = (t_0 - 1)\widetilde{v}_1 - (t_1 - 1)\widetilde{v}_0$

and $\widetilde{l}_2 = (t_0 - 1)\widetilde{u}_2 - (t_1 - 1)\widetilde{v}_0$, noting that $u_2 = v_2$.

By Fact 3.14, the equivalence classes $[0], [\partial_2(\widetilde{e}_1)], [\partial_2(\widetilde{e}_2)], [\partial_2(\widetilde{e}_3)]$ are equal. Using the first two equations of *Equation* 3.7, we have $\widetilde{\gamma}_1 = \frac{2+t_0t_1-t_0-t_1}{t_0}\widetilde{\zeta}_0$. Recall from Lemma 3.10 that the induced maps

$$H_1(X_0;\mathcal{F}) \xrightarrow{\iota_{0*}} H_1(X;\mathcal{F}) \xleftarrow{\iota_{1*}} H_1(X_1;\mathcal{F})$$

are isomorphisms. Let $f_0 = \iota_{0*}$ and $f_1 = \iota_{1*}$. Define ι_{0*} by

$$\iota_{0*} : \begin{cases} \widetilde{\gamma}_1 & \mapsto & \frac{2+t_0t_1-t_0-t_1}{t_0}\widetilde{\zeta}_1 \\ \widetilde{\gamma}_2 & \mapsto & \widetilde{\gamma}_2. \end{cases}$$

Again, using Fact 3.14, we have $l_1 = \widetilde{\beta}_1 - (t_1 - 1)\widetilde{\beta}_0 = \frac{t_0+t_1-1}{t_0}\widetilde{\zeta}_0$ and $\widetilde{l}_2 = \widetilde{\gamma}_2 - (t_2 - 1)\widetilde{\beta}_0 = \widetilde{\gamma}_2 + \frac{t_2-1}{t_0}\widetilde{\zeta}_0$. Define f_1^{-1} as

$$\iota_{1*}^{-1} : \begin{cases} \widetilde{\gamma}_2 & \mapsto & \widetilde{l}_2 - \frac{t_2-1}{t_0+t_1-1}\widetilde{l}_1 \\ \widetilde{\zeta}_0 & \mapsto & \frac{t_0}{t_0+t_1-1}\widetilde{l}_1. \end{cases}$$

The composition $\iota_{1*}^{-1} \circ \iota_{0*} : H_1(X_0;\mathcal{F}) \to H_1(X_1;\mathcal{F})$ is

$$\iota_{1*}^{-1} \circ \iota_{0*} : \begin{cases} \widetilde{\gamma}_1 & \mapsto & \frac{2+t_0t_1-t_0-t_1}{t_0+t_1-1}\widetilde{l}_1 \\ \widetilde{\gamma}_2 & \mapsto & \widetilde{l}_2 - \frac{t_2-1}{t_0+t_1-1}\widetilde{l}_1. \end{cases}$$

Finally, referring to Remark 3.13, we make the following identifications: $\begin{matrix} \widetilde{\gamma}_1 & \longleftrightarrow & x_1 & \longleftrightarrow & \widetilde{l}_1, \\ \widetilde{\gamma}_2 & \longleftrightarrow & x_2 & \longleftrightarrow & \widetilde{\zeta}_0 \end{matrix}$. The reduced homology Gassner invariant of L_1''' is therefore given as

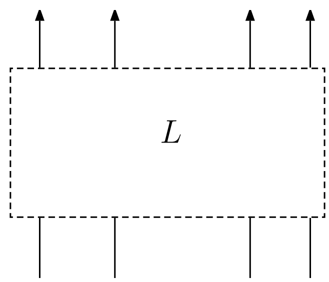
$$\mathcal{G}_h^r(L_1''') = \iota_{1*}^{-1} \circ \iota_{0*} : \begin{cases} x_1 & \mapsto & \frac{2+t_0t_1-t_0-t_1}{t_0+t_1-1}x_1 \\ & & \\ x_2 & \mapsto & x_2 - \frac{t_2-1}{t_0+t_1-1}x_1. \end{cases} \tag{3.8}$$

This is the end of the example.

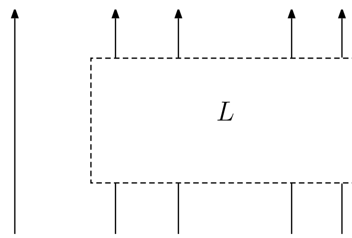
We have seen some examples of how to compute the Gassner and reduced Gassner invariant using homology with local coefficients in $\mathcal{F}$. The homology Gassner invariant of the string link L_1 in Example 3.15 and the reduced homology Gassner invariant of the string link L_1'' in Example 3.16 are the equal. Likewise, the homology Gassner invariant of the over-crossing ⤫ in Example 3.17 and the reduced homology Gassner invariant of the over-crossing |⤫ in Example 3.18 are the equal. However, the reduced homology Gassner invariant of the string link L_1''' in Example 3.19 and the reduced homology Gassner invariant of the string link L_1''' in Example 3.19 are not equal, and certainly not equal to the homology Gassner invariant of the string link L_1. Let L_m represent the m-component string link in Figure 3.13a and L_{m+1} be the string link L_m with a free strand added to the left as in Figure 3.13b. Then, one can observe from the examples that the homology Gassner invariant $\mathcal{G}_h(L_m)$ of L_m and the reduced homology Gassner invariant $\mathcal{G}_h^r(L_{m+1})$ of L_{m+1} are equivalent in appropriate basis.

Lemma 3.20. *The homology and reduced homology Gassner invariants $\mathcal{G}_h(L_m)$ and $\mathcal{G}_h^r(L_{m+1})$ are equivalent in appropriate bases.*

Proof. The addition of a strand to the leftmost side of the string link L_m results in the string link L_{m+1}.



(a) m-component string link, L_m.



(b) $m+1$-component string link, L_{m+1}.

Figure 3.13: An m-component string link with a free strand on the left.

Consequently, the homology Gassner invariant of L_{m+1} (see Figure 3.13b) is the $(m+1)\times(m+1)$ matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\mathcal{G}_h(L_m)} \\ 0 & & \end{pmatrix}$$

But the reduced homology Gassner invariant $\mathcal{G}_h^r(L_{m+1})$ of L_{m+1} is the $m\times m$ matrix corresponding to the block matrix $\mathcal{G}_h(L_m)$ of the matrix above in appropriate basis. The lemma follows. □

Based on the lemma above, the reduced homology Gassner invariant of a string link or braid with a free strand on the left is equivalent to the homology Gassner invariant of the string link or braid without the free strand in appropriate basis. From now on, all string links will have a free strand to the left, as shown in Figure 3.13b. If there is no free strand, it will be assumed. The reduced homology Gassner invariant $\mathcal{G}_h^r$ of such string links or braids will be referred to as the homology Gassner invariant.

Remark 3.21. We have computed the cohomology and homology Gassner invariants $\mathcal{G}_c$ and $\mathcal{G}_h$ of some string links including the over-crossing and under-crossing. In Example 3.8 and Example 3.17 we computed

$$\mathcal{G}_c(\nwarrow\!\!\!\!\nearrow) = \begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix} \text{ and } \mathcal{G}_h\left(\nwarrow\!\!\!\!\nearrow\right) = \begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}$$

for the over-crossing and under-crossing. But note that $\mathcal{G}_h(\nwarrow\!\!\!\!\nearrow)$ is the inverse transpose of $\mathcal{G}_c(\nwarrow\!\!\!\!\nearrow)$. That is, $\mathcal{G}_h(\nwarrow\!\!\!\!\nearrow) = (\mathcal{G}_c(\nwarrow\!\!\!\!\nearrow)^{-1})\ //\ m^t$. We deduce that $\mathcal{G}_h = (\mathcal{G}_c^{-1})\ //\ m^t$, meaning that the homology Gassner invariant and the cohomology Gassner invariant are inverse transpose of each other in appropriate basis.

We have already seen from Definition 2.8(3) that n-braids form the braid group B_n with generators $\sigma_i, i = 1, 2, \cdots, n-1$; where σ_i is the over-crossing in Figure 3.14 below. Recall that induces a permutation on the set of labels $T = \{0, 1, \cdots, n\}$. Let $T[k]$ be the label (colour) at position k in T. From Example 3.18,

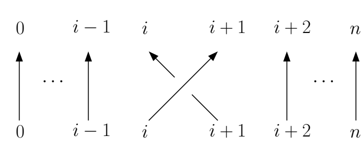


Figure 3.14: Braid generator σ_i.

one can deduce that the homology Gassner invariant $\mathcal{G}_h(\sigma_i) : \mathcal{F}\langle x_1, \cdots, x_n\rangle \to \mathcal{F}\langle x_1, \cdots, x_n\rangle$ of σ_i, is given by

$$\mathcal{G}_h(\sigma_i):\begin{cases} x_i & \mapsto & x_{i+1} \\ x_{i+1} & \mapsto & t_{T[i]}x_i+\left(1-t_{T[i+1]}\right)x_{i+1} \\ x_k & \mapsto & x_k, \quad k\neq i,i+1 \end{cases},$$

The matrix representation of $\mathcal{G}_h(\sigma_i)$ is the $n\times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 0 & t_{T[i]} \\ 1 & 1-t_{T[i+1]} \end{pmatrix}$. Taking the inverse transpose of $\mathcal{G}_h(\sigma_i)=\begin{pmatrix} 0 & t_{T[i]} \\ 1 & 1-t_{T[i+1]} \end{pmatrix}$ gives the cohomology Gassner invariant of σ_i :

$$\mathcal{G}_c(\sigma_i)=\begin{pmatrix} \frac{t_{T[i+1]}-1}{t_{T[i]}} & \frac{1}{t_{T[i]}} \\ 1 & 0 \end{pmatrix}.$$

Recall from Remark 2.10 that there is a multiplication on coloured string links. The following lemma shows that the homology Gassner invariant is multiplicative considering when coloured string links.

Lemma 3.22. *Let L be a string link. The assignment $\mathcal{G}_h : L\mapsto \mathcal{G}_h(L)$ is multiplicative under the multiplication of labeled string links obtained by stacking one above the other: $\mathcal{G}_h(L_1L_2)=\mathcal{G}_h(L_1)\mathcal{G}_h(L_2)$.*

Proof. Let $L=L_1L_2$ be the product of two n coloured string links L_1 and L_2 such that L_2 stacks appropriately on L_1. We have $\mathcal{G}_h(L)=\iota_1^{-1}\iota_0(L)$. Also, $\mathcal{G}_h(L_1)=\kappa^{-1}\iota_0(L_1)$ and $\mathcal{G}_h(L_2)=\iota_1^{-1}\kappa(L_2)$. It follow that

$$\begin{aligned} \mathcal{G}_h(L) &= \iota_1^{-1}\iota_0(L) \\ &= \iota_1^{-1}\kappa\kappa^{-1}\iota_0(L) \\ &= \iota_1^{-1}\kappa(L_2)\kappa^{-1}\iota_0(L_1) \\ &= \mathcal{G}_h(L_2)\mathcal{G}_h(L_1). \end{aligned}$$

□

The spaces X_0 and X_1 are canonically identified via the homeomorphism $(x,0)\mapsto(x,1)$. This homeomorphism induces an isomorphism $H_1(X_0;\mathcal{F})\cong H_1(X_1;\mathcal{F})$. By Remark 2.10, it follows that the reduced homology Gassner invariant restricts to a homomorphism

$$\mathcal{G}_h^r : PSL_n\to GL(H_1(X_0;\mathcal{F}))\cong GL_{n-1}(\mathcal{F})$$

on the semi group, PSL_n, of pure string links on $n+1$ strands called the ***reduced homology Gassner representation***.

3.4 (Co)homology Gassner invariant and the Gassner representation

Recall the Gassner representations discussed in [BN14] and [Knu05] and the relation between them. In this section we establish the connection between the (co)homology Gassner invariant and the Gassner representation in [Knu05]. The relation is as follows.

By Remark 3.21, we have $\mathcal{G}_c(\times)=(\mathcal{G}_h(\times)^{-1})//m^t$. That is,

$$\begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix}=\left(\begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}^{-1}\right)//m^t.$$

Also, notice that

$$\begin{pmatrix} 1-t_j & t_i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t_i & 0 \\ 0 & t_j \end{pmatrix} \begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_j^{-1} & 0 \\ 0 & t_i^{-1} \end{pmatrix} // \left(t_k \mapsto \frac{1}{t_k} \right). \tag{3.9}$$

Recall the construction of the Gassner representation in [BN14]: Let t be a formal variable and let $U_i(t) = U_{n:i}(t)$ denote the $n \times n$ identity matrix with its 2×2 block at rows i and $i+1$ and column i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$. Transposing $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$ and replacing $1-t$ and t with $1-t_j$ and t_i respectively yields the matrix in Equation 3.9: $\begin{pmatrix} 1-t_j & t_i \\ 1 & 0 \end{pmatrix}$.

Let $\Gamma_{ch}(\sigma_i)$ be the $n \times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 1-t_{T[i+1]} & t_{T[i]} \\ 1 & 0 \end{pmatrix}$. Taking the inverse transpose of $\Gamma_{ch}(\sigma_i)$ gives the $n\times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 0 & 1 \\ \frac{1}{t_{T[i]}} & \frac{1-t_{T[i+1]}}{t_{T[i]}} \end{pmatrix}$.

Previously mentioned, each braid can be expressed as a product of braid generators. In contrast, string links do not possess this property and require more time and space to calculate their Gassner invariant. However, in Chapter 4, we will establish a relationship between braids and string links. This will simplify the computation of the Gassner invariant for string links

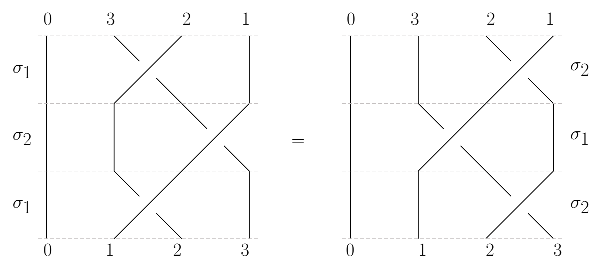


Figure 3.15: Reidemeister 3: $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

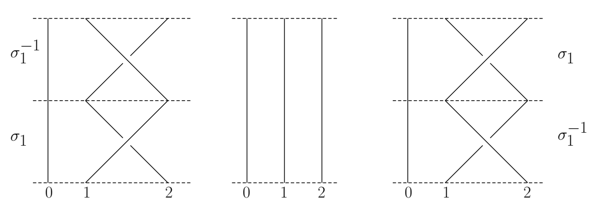


Figure 3.16: Reidemeister 2 moves on a string link $\sigma_1^{-1}\sigma_1 = \sigma_1\sigma_1^{-1} = id$.

Theorem 3.23. *Let β a braid. Then the assignment $\mathcal{G}_h^r : \beta \mapsto \mathcal{G}_h^r(\beta)$ defines an invariant of labeled $(n+1)$ braids with values in $GL_n(\mathcal{F})$. It is multiplicative under the multiplication of labeled braids obtained by stacking one above the other. Restricting to pure braids yields the reduced homology Gassner representation.*

Proof. It suffices to show that the braid group relations, $\sigma_i\sigma_j = \sigma_j\sigma_i$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, are satisfied under the mapping $\mathcal{G}_h^r : \beta \mapsto \mathcal{G}_h^r(\beta)$. As $\mathcal{G}_h^r : \beta \mapsto \mathcal{G}_h^r(\beta)$ is multiplicative by Lemma 3.22, it is sufficient to show that $\mathcal{G}_h^r(\sigma_i)\mathcal{G}_h^r(\sigma_{i+1})\mathcal{G}_h^r(\sigma_i) = \mathcal{G}_h^r(\sigma_{i+1})\mathcal{G}_h^r(\sigma_i)\mathcal{G}_h^r(\sigma_{i+1})$ and $\mathcal{G}_h^r(\sigma_i)\mathcal{G}_h^r(\sigma_j) = \mathcal{G}_h^r(\sigma_j)\mathcal{G}_h^r(\sigma_i)$. But, these are straightforward computations given below.

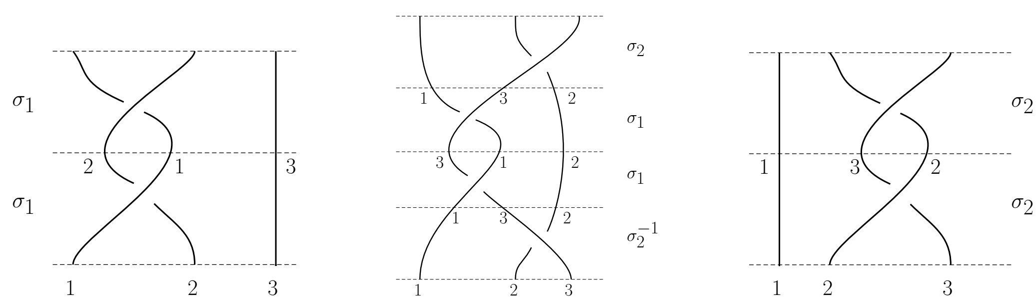
$$
\begin{aligned}
\mathcal{G}_h^r(\sigma_1)\mathcal{G}_h^r(\sigma_2)\mathcal{G}_h^r(\sigma_1) &= \begin{pmatrix} 0 & t_2 & 0 \\ 1 & 1-t_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_1 \\ 0 & 1 & 1-t_3 \end{pmatrix}\begin{pmatrix} 0 & t_1 & 0 \\ 1 & 1-t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & t_1 t_2 \\ 0 & t_1 & t_1(1-t_3) \\ 1 & 1-t_2 & 1-t_3 \end{pmatrix} \\
\mathcal{G}_h^r(\sigma_2)\mathcal{G}_h^r(\sigma_1)\mathcal{G}_h^r(\sigma_2) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_1 \\ 0 & 1 & 1-t_2 \end{pmatrix}\begin{pmatrix} 0 & t_1 & 0 \\ 1 & 1-t_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_2 \\ 0 & 1 & 1-t_3 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{G}_h^r(\sigma_1^{-1})\mathcal{G}_h^r(\sigma_1) &= \begin{pmatrix} \frac{t_2-1}{t_1} & 1 \\ \frac{1}{t_1} & 0 \end{pmatrix}\begin{pmatrix} 0 & t_1 \\ 1 & 1-t_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathcal{G}_h^r(\sigma_1)\mathcal{G}_h^r(\sigma_1^{-1}) &= \begin{pmatrix} 0 & t_2 \\ 1 & 1-t_1 \end{pmatrix}\begin{pmatrix} \frac{t_1-1}{t_2} & 1 \\ \frac{1}{t_2} & 0 \end{pmatrix}.
\end{aligned}
$$

□

3.4.1 The Gassner representation for the pure braid group PB_3



(a) The generator $A_{1,2}$. **(b)** The generator $A_{1,3}$. **(c)** The generator $A_{2,3}$.

Figure 3.17: Generators for the pure braid group PB_3.

The pure braid group is generated by the generators $\{A_{1,2}, A_{1,3}, A_{2,3}\}$ which satisfy the pure braid relations in Equation 1.7 of [KT08]. Recall the Gassner representation for $A_{i,j}$ in Equation 1.1 (see [Knu05]. The group PB_3 has the the following representations for the generators:

$$
[A_{1,2}]^{\phi} = \begin{pmatrix} 1-t_1+t_1 t_2 & t_1(1-t_1) & 0 \\ 1-t_2 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$[A_{1,3}]^{\phi} = \begin{pmatrix} 1-t_1+t_1t_3 & 0 & t_1(1-t_1) \\ (1-t_2)(1-t_3) & 1 & (1-t_2)(t_1-1) \\ 1-t_3 & 0 & t_1 \end{pmatrix}$$

and

$$[A_{2,3}]^{\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_2+t_2t_3 & t_2(1-t_2) \\ 0 & 1-t_3 & t_2 \end{pmatrix}.$$

Next, let us consider Equation 3.9:

$$\begin{pmatrix} 1-t_j & t_i \\ 1 & 0 \end{pmatrix} = \left(\begin{pmatrix} t_i & 0 \\ 0 & t_j \end{pmatrix} \begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_j^{-1} & 0 \\ 0 & t_i^{-1} \end{pmatrix} \right) // \left(t_k \mapsto \frac{1}{t_k} \right),$$

which comes from the (co)homology Gassner invariant. Let $\Gamma_{ch}(\nwarrow\!\!\!\!\nearrow) = \begin{pmatrix} 1-t_j & t_i \\ 1 & 0 \end{pmatrix}$. Then Γ_{ch} is an invariant since the $\mathcal{G}_c$ is an invariant, and it assigns the same matrices to the generators of PB_3 as $[-]^{\phi}$ does. That is,

$$\Gamma_{ch}(A_{1,2}) = \Gamma_{ch}(\sigma_1)\Gamma_{ch}(\sigma_1) = \begin{pmatrix} 1-t_1+t_1t_2 & t_1(1-t_1) & 0 \\ 1-t_2 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_{ch}(A_{1,3}) = \Gamma_{ch}(\sigma_2)\Gamma_{ch}(\sigma_1)\Gamma_{ch}(\sigma_1)\Gamma_{ch}(\sigma_2^{-1}) = \begin{pmatrix} 1-t_1+t_1t_3 & 0 & t_1(1-t_1) \\ (1-t_2)(1-t_3) & 1 & (1-t_2)(t_1-1) \\ 1-t_3 & 0 & t_1 \end{pmatrix}$$

and

$$\Gamma_{ch}(A_{2,3}) = \Gamma_{ch}(\sigma_2)\Gamma_{ch}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_2+t_2t_3 & t_2(1-t_2) \\ 0 & 1-t_3 & t_2 \end{pmatrix}.$$

The above shows that $\Gamma_{ch}(A_{i,j}) = [A_{i,j}]^{\phi}$ for $1 \le i < j \le 3$. Thus the (co)homology Gassner invariant, when restricted to the pure braid group PB_3 yields the Gassner representation for PB_3. The following proposition shows that the Gassner invariant yields the Gassner representation.

Proposition 3.24. *Let*

$$\Gamma_{ch}(\sigma_i) = \left(\begin{pmatrix} t_{T[i]} & 0 \\ 0 & t_{T[i+1]} \end{pmatrix} \mathcal{G}_c(\sigma_i) \begin{pmatrix} \frac{1}{t_{T[i+1]}} & 0 \\ 0 & \frac{1}{t_{T[i]}} \end{pmatrix} \right) // \left(t_k \mapsto \frac{1}{t_k} \right),$$

where $\mathcal{G}_c(\sigma_i) = (\mathcal{G}_h(\sigma_i)^{-1}) \,//\, m^T$. *Then* $\Gamma_{ch}(\sigma_i)$ *is an invariant of braids. In particular, restricting* Γ_{ch} *to the pure braids yields the Gassner representation. That is* $\Gamma_{ch}(A_{r,s}) = [A_{r,s}]^{\phi}$.

Proof. The invariance of Γ_{ch} follows immediately from the invariance of the cohomology Gassner invariant. Recall that the generator $A_{r,s}$ can be expressed as

$$A_{r,s} = \sigma_{s-1}\sigma_{s-2}\cdots\sigma_{r+1}\sigma_r^2\sigma_{r+1}^{-1}\cdots\sigma_{s-2}^{-1}\sigma_{s-1}^{-1}.$$

According to Lemma 3.22, we have

$$\Gamma_{ch}(A_{r,s}) = \Gamma_{ch}(\sigma_{s-1})\Gamma_{ch}(\sigma_{s-2})\cdots\Gamma_{ch}(\sigma_{r+1})\Gamma_{ch}(\sigma_r)\Gamma_{ch}(\sigma_r)\Gamma_{ch}(\sigma_{r+1}^{-1})\cdots\Gamma_{ch}(\sigma_{s-2}^{-1})\Gamma_{ch}(\sigma_{s-1}^{-1}).$$

But this is exactly $[A_{r,s}]^{\phi}$ in Equation 1.1. The proposition follows. □

3.5 A Mathematica implementation of the homology Gassner invariant

In this section, we perform computations of the homology Gassner invariant of braids using Mathematica. We define a Mathematica function for the homology Gassner invariant of the generator σ_i and its inverse. This function is then used to compute the invariant for a given braid β. Finally, we test the second and third Reidemeister moves. A reader with Mathematica can get the notebook by clicking the following link: *GassnerInvariantMathematicaNotebook.nb*

Notations:

- σ_i is the Mathematica function representing Gassner invariant, where β_i represents the generator x_i of $H_1(X_0,\mathcal{F})$ and $H_1(X_1,\mathcal{F})$. Here i is the position of the over strand below the horizontal level of the crossing σ_i. In the case of σ_i^{-1}, i is the position of the under strand instead. for an over-crossing (positive crossing).

  The function takes an argument $h[T,L_1]$ and outputs $h[\rho(T),L_2]$.

- Here, the argument $h[T,L_1]$ has two parameters:

  i. T is the set of labels of the strands of an $(n+1)$ braid.

  ii. L is an element of $\mathcal{F}\langle\beta_1,\ldots,\beta_n\rangle \cong H_1(X_0;\mathcal{F})$.

- The output $h[\rho(T),L_1]$ also has two parameters where

  i. $\rho(T)$ is a permutation of T induced by the braid.

  ii. L_2 is an element of $\mathcal{F}\langle\beta_1,\ldots,\beta_n\rangle \cong H_1(X_1;\mathcal{F})$.

- M_i evaluates the matrix corresponding to σ_i and $\overline{M}_i$ is the inverse of M_i.

The code for the over-crossing: σ_i

```
In[1]:= σ_i_[h[T_,L_]]/;i>0:=h[
    Permute[T,Cycles[{{i,i+1}}]],
    Expand[L/.{β_i→β_(i+1),β_(i+1)→t_T[[i]] β_i+(1-t_T[[i+1]]) β_(i+1)}]
    ]
In[2]:= M_i_[T_]:=Table[
    Coefficient[σ_i[h[T,β_j]][[2]],β_k],
    {j,1,Length[T]},{k,1,Length[T]}
    ]//Transpose
```

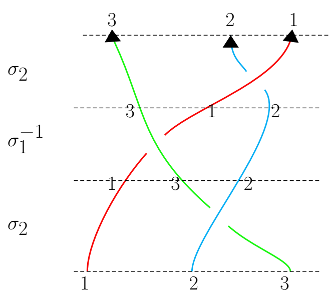
The code for under-crossing: σ_i^{-1}

```
In[3]:= σ̄i_[h[T_,L_]]/;i>0:=h[
        Permute[T,Cycles[{{i,i+1}}]],
        Expand[L/.{β_i→β_{i+1}/t_{T[[i+1]]}+(t_{T[[i]]}-1) β_i/t_{T[[i+1]]},β_{i+1}→β_i}]//Simplify
        ]
```

```
In[4]:= M̄i_[T_]:=Table[
        Coefficient[σ̄i[h[T,β_j]][[2]],β_k],
         {j,1,Length[T]},{k,1,Length[T]}]//Transpose
```

Applying the code on the braid β below

Note that the free strand labeled is not involved.



$\beta = \sigma_2\sigma_1^{-1}\sigma_2$

Here, we compute the homology Gassner invariants of the generators σ_2, σ_1^{-1} and σ_2, and multiply them to get the homology Gassner invariant of tβ.

```
In[260]:= {M2[{3, 1, 2}] // MatrixForm, M̄1[{1, 3, 2}] // MatrixForm, M2[{1, 2, 3}] // MatrixForm}
```

Out[260]= $\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_1 \\ 0 & 1 & 1-t_2 \end{pmatrix}, \begin{pmatrix} \frac{-1+t_1}{t_3} & 1 & 0 \\ \frac{1}{t_3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_2 \\ 0 & 1 & 1-t_3 \end{pmatrix}\right\}$

```
In[261]:= M2[{3, 1, 2}].M̄1[{1, 3, 2}].M2[{1, 2, 3}] // Simplify // MatrixForm
```

Out[261]//MatrixForm=

$$\begin{pmatrix} \frac{-1+t_1}{t_3} & 0 & t_2 \\ 0 & t_1 & -t_1(-1+t_3) \\ \frac{1}{t_3} & 1-t_2 & (-1+t_2)\times(-1+t_3) \end{pmatrix}$$

In Theorem 3.23, we verified that the homology Gassner invariant is indeed and invairiat of braids. We test this result using Mathematica below.

Testing Reidemeister 2 move

Here, we are verifying $\sigma_1^{-1}\sigma_1 = \sigma_1\sigma_1^{-1}$ (see Figure 3.16). Note: we ignore the free strand labeled 0.

```
In[5]:= {h[{1,2}, xβ1 + yβ2]//σ̄1//σ1}=={h[{1,2}, xβ1 + yβ2]//σ1//σ̄1}

Out[5]= True
```

Testing Reidemeister 3 move

Finally, we test $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ (see Figure 3.15). Again, we ignore the free strand labeled 0.

In[6]:=
```
{h[{1,2,3}, xβ1 + yβ2+ zβ3]//σ1//σ2//σ1}=={h[{1,2,3}, xβ1+ yβ2+ zβ3]//σ2//σ1//σ2}
```

Out[6]=
```
True
```

Chapter 4

Flying Cars and The Gassner invariant

4.1 Summary of Chapter

In [BNa], the author explores the concept of cars, which involves assigning a $(2n+1)\times(2n+1)$ matrix to a long knot with n crossings. This chapter introduces the concept of flying cars (see Definition 4.1), a modification of the one discussed in the cited work above. Flying cars involve assigning an $n \times n$ matrix to an $n+1$ string link or braid, where the leftmost strand is always free. It is demonstrated that this assignment serves as an invariant of string links and is connected to the homology Gassner invariant. Furthermore, the stitching operation is defined to establish a relationship between string links and braids. Finally, examples are provided to illustrate this concept.

4.2 String Links and Flying Cars, Bridges and Traffic Counters

In this section, we give the definition of ***flying cars***, which a modified version of the car concept in [BNa]. This is similar to the "probabilistic" interpretation of the Burau representation for string discussed in [LTW98], which is further extended to give a similar interpretation of the Gassner representation in Section 8 of [KLW01]. We also discuss an invariant flying cars assign to string links.

A flying car always moves forward along a path which has a start point i and an endpoint j. There are bridges along the path. A traffic counter is placed at j to measure the probability $P_{i,j}$ of the flying car exiting at j. A flying car flies under a bridge with an (algebraic) probability t^s and, it flies up a bridge with probability $1-t^s$, where t is a variable and $s \in \{-1,+1\}$. Let χ be the set of all possible paths from i to j. Then the probability of starting at i and ending at j is

$$P_{i,j} = \sum_{p\in\chi}\prod_{b} prob(b),$$

where

$$prob(b) = \begin{cases} t^s & \text{flying car flies under at bridge } b \\ 1-t^s & \text{flying car flies up at bridge } b \\ 1 & \text{flying car flies over at bridge } b \\ 0 & \text{flying car flies down at bridge } b \end{cases}.$$

Relating this to labeled string links, the oriented strands are the path along which flying cars move and the crossings represent the bridges. Here we keep track of the labeled strand a flying car is on. Let L be an $(n+1)$-string link, $T = \{0, 1, \cdots, n\}$ be the labels of the strands of L, and $T' = T - \{0\}$. Let $\mathcal{F}'$ be $\mathbb{Q}(\{t_k \,:\, k \in T'\})$, the field of rational functions in the variables t_k, where $k \in T'$.

Definition 4.1 (Flying cars on string links)**.** A flying car $\mathcal{C}$ on a string link (or braid) diagram is a continuous map $\mathcal{C} : [0,1] \to \mathbb{R}^2$ whose image lies in the projection of the string link (or braid) to the plane, and which starts at one of the bottom endpoints labeled i and ends at one of the top endpoints labeled j with algebraic probability

$$P_{i,j} = \sum_{p \in \chi} \prod_{c} prob(c),$$

where $prob(c)$ is the probability measured after moving past a crossing c and χ is the set of all possible paths of $\mathcal{C}$ from i to j. The probability $prob(c)$ is defined as

$$prob(c) = \begin{cases} t_k^s & \text{flying car flies under at bridge } c \\ 1 - t_k^s & \text{flying car flies up at bridge } c \\ 1 & \text{flying car flies over at bridge } c \\ 0 & \text{flying car flies down at bridge } c \end{cases},$$

where $k \in T'$ is the label of the over strand at crossing c and $s \in \{-1, +1\}$.

Let $Mat_{T' \times T'}(\mathcal{F}')$ be the collection of $n \times n$ ***labeled matrices*** with rows and columns labeled by T'. Let $\mathcal{C}$ denote a flying car. Define the map $\mathcal{C} : SL_n \to M_{T' \times T'}(\mathcal{F}')$ by assigning a labeled $(n+1)$-string link $L \in SL_n$ the $n \times n$ matrix

$$\mathcal{C}(L) = \left(\begin{array}{c|c} & T' \\ \hline T' & P_{i,j} \end{array} \right)_{i,j \in T'},$$

where $P_{i,j}$ is the probability of the flying car $\mathcal{C}$ starting at $q_i \times \{0\}$ and exiting at $q_j \times \{1\}$. Notice that we ignore the free strand labeled 0, so the matrix $\mathcal{C}(L)$ does not include t_0. This allows us to relate flying cars to the reduced homology Gassner invariant (see Section 4.3) since the matrices of the latter do not have t_0 when we take the labeling into consideration.

Let $R_{i,j}$ and $R_{j,i}$ denote an over-crossing and an under-crossing respectively where i is the label of the over strand and j is the label the under strand. Then flying-cars assign the following matrices to $R_{i,j}$ and $R_{j,i}$

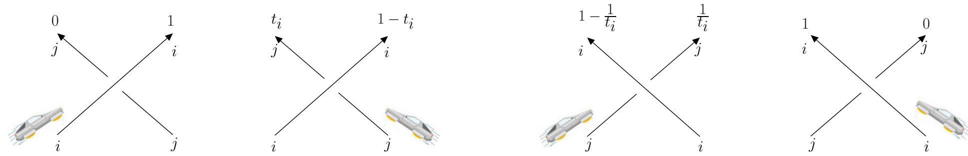
$$\mathcal{C}(R_{i,j}) = \left(\begin{array}{c|cc} & i & j \\ \hline i & 1 & 0 \\ j & 1 - t_i & t_i \end{array} \right) \quad \text{and} \quad \mathcal{C}(R_{j,i}) = \left(\begin{array}{c|cc} & i & j \\ \hline i & 1 & 0 \\ j & 1 - \frac{1}{t_i} & \frac{1}{t_i} \end{array} \right).$$

Figure 4.1 shows flying cars moving along the strands of an over and under-crossing.

It turns out that this assignment is an invariant of braids and string links. It is sufficient to show that $\mathcal{C}$ remains unchanged under the three Reidemeister moves.

RM 1 Verifying invariance under Reidemeister 1 move.

(a) In Figure 4.2a, for a kink with a negative crossing, a car starts from the initial point at i. It can either move under the bridge with probability t_i^{-1}, then move over the bridge with probability 1, and exit with probability $t_i^{-1} \times 1 = t_i^{-1}$, or move up the bridge at the crossing and exit with probability $1 - t_i^{-1}$. The total probability is $t_i^{-1} + 1 - t_i^{-1} = 1$, which is the same probability for no kink in Figure 4.2c.



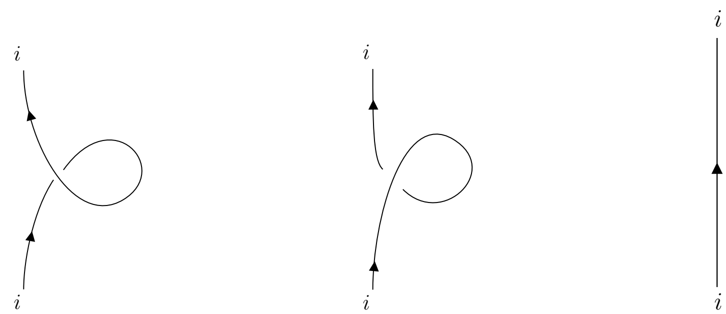
(a) Positive crossing: Car starts at i

(b) Positive crossing: Car starts at j

(c) Negative crossing: Car starts at i

(d) Negative crossing: Car starts at j

Figure 4.1: Flying car starting positions

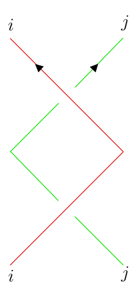


(a) Kink with a negative crossing

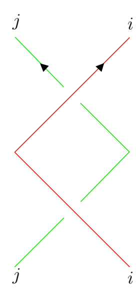
(b) Kink with a positive crossing.

(c) Undoing the kink.

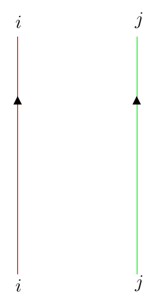
Figure 4.2: Reidemeister 1 move: (a) and (b) are the equivalent to (c).



(a) Type one Reidemeister 2 move

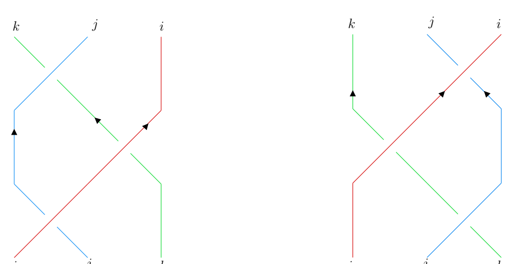


(b) Type two Reidemeister 2 move.



(c) Free strand

Figure 4.3: Reidemeister 2 move: (a) and (b) are the equivalent to (c).



(a) I

(b) II.

Figure 4.4: Reidemeister 3 move: (a) and (b) are the equivalent.

(b) In Figure 4.2b, consider the kink with a positive crossing, where a car starts at the initial point i.

The car has two options: it can descend to the lower strand and exit with a probability of 0, or it can move over the bridge with a probability of 1 and return to the crossing. At the crossing, the car can move up the bridge with a probability of $1 - t_i$ and return to the crossing. This process can repeat with probability $\sum_{k=1}^{\infty}(1-t_i)^k$ until it eventually moves under the bridge with a probability of t_i. The car finally exits with probability $t_i \sum_{k=1}^{\infty}(1-t_i)^k = \frac{1}{t_i}t_i = 1$. Therefore, the total probability is $1 + 0 = 1$. Again, the probability is the same for no kink. This proves invariance under Reidemeister 1 move.

RM 2 Verifying invariance under Reidemeister 2 move.

(a) In Figure 4.3a, there are four paths to consider: the paths from i to i, i to j, j to i, and j to j. In each case, there are two options to consider. For instance, a car moving from i to i starts from i, moves over strand j at the positive crossing with probability 1, then over strand j again at the negative crossing with probability 1, and finally exits with probability $1 \cdot 1 = 1$. Alternatively, the car starts at i, descends onto strand j at the positive crossing with probability 0, then moves up strand i again with probability $1 - t_i^{-1}$, and finally exits with probability $0 \cdot (1 - t_i^{-1}) = 0$. The total probability is $1 + 0 = 1$. Checking the other situations similarly, the corresponding probabilities for the paths i to j, j to i and j to j are 0, 0 and 1 respectively. This is the same as the probabilities for the paths in Figure 4.3c.

(b) In Figure 4.3b, a similar situation is reached as in the case of Figure 4.3a. This proves invariance under Reidemeister 2 move.

RM 3 Verifying invariance under Reidemeister 3 move.

In both Figure 4.4a and Figure 4.4b there are nine paths to consider: the paths from i to i, i to j, i to k, j to i, j to k, j to j, k to i, k to j and k to k.

For instance, the path from k to j in Figure 4.4a is as follows: a car starts at k and moves under strand i at the positive crossing with probability t_i, then moves up to strand j with probability $1 - t_j$, and finally exits at j with probability $t_i(1 - t_j)$.

In the case of Figure 4.4b, there are two options: a car starts at k moves under strand j at the first positive crossing with probability t_j, move up onto strand i at the next positive crossing with probability $(1 - t_i)$, descends onto strand j with probability 0, then finally exits with probability $t_j(1 - t_i) \cdot 0 = 0$. Alternatively, the car starts at k, moves up onto strand j with probability $1 - t_j$, then moves under strand i with probability t_i, and exits with probability $t_i(1 - t_j)$. So, the total probability is $0 + t_i(1 - t_j) = t_i(1 - t_j)$.

In both figures, the final probability for the path k to j is $t_i(1 - t_j)$. Similarly, all the other eight paths are the same. This proves invariance under Reidemeister 3 move.

From the above demonstration of the invariance of $\mathcal{C}$ under the Reidemeister moves, we have established the following proposition.

Proposition 4.2. *Let L be an oriented $(n + 1)$-component string link. Then the assigned matrix $\mathcal{C}(L) = (P_{i,j})$ is an invariant of string links.*

4.3 Flying cars and the Homology Gassner invariant.

In this section, we discuss the relation between $\mathcal{C}(L)$ and $\mathcal{G}_h^r(L)$ of a labeled string link L. Here, L is of the form as shown in Figure 3.13b. So, both $\mathcal{C}(L)$ and $\mathcal{G}_h^r(L)$ do not have t_0 in all their entries. In view of this, we have $\mathcal{C}(L)$, $\mathcal{G}_h^r(L) \in M_{T' \times T'}(\mathcal{F}')$. In Example 3.18, we computed the homology Gassner invariant of $R_{i,j}$, and it is given by

$$\mathcal{G}_h^r\left(R_{i,j}\right) = \begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}.$$

We have also seen above that flying-cars assign a matrix to $R_{i,j}$ which is given by

$$\mathcal{C}\left(R_{i,j}\right) = \begin{pmatrix} 1 & 0 \\ 1-t_i & t_i \end{pmatrix}.$$

Let ρ_{col} denote the permutation of the over-crossing and denote by m^t the transposition of a matrix. The subscript col of ρ_{col} denotes a column permutation. Let D_n be the $n \times n$ matrix with the diagonal entries given by $d_{i,i} = 1 - t_i, 1 \leq i \leq n$. When $n = 2$, then

$$D_2 = \begin{pmatrix} 1-t_i & 0 \\ 0 & 1-t_j \end{pmatrix}.$$

Then, the two matrices $\mathcal{G}_h^r\left(R_{i,j}\right)$ and $\mathcal{C}\left(R_{i,j}\right)$ are related by the formula

$$\mathcal{G}_h^r\left(R_{i,j}\right) = \left(D_2 \cdot \mathcal{C}\left(R_{i,j}\right) \cdot D_2^{-1}\right) // \rho_{col} // m^t. \tag{4.1}$$

Thus, the homology Gassner invariant is given by first conjugating the matrix $\mathcal{C}(L)$ with D_n, followed by a permutation of the columns and finally transposing the resulting matrix. That is

$$\mathcal{G}_h^r(L) = \left(D_n \cdot \mathcal{C}(L) \cdot D_n^{-1}\right) // \rho_{col} // m^t \tag{4.2}$$

From equations, Equation 4.2 the commutative diagram below is obtained. This is useful because, given the Gassner invariant Γ, one can produce the homology Gassner invariant and vice versa.

$$\begin{tikzcd} & SL_n \arrow[dl, "\mathcal{C}"'] \arrow[dr, "\mathcal{G}_h^r"] & \\ Mat_n(\mathcal{F}') \arrow[rr, "D_n(-)D_n^{-1}//\rho_{col}//m^t"'] & & Mat_n(\mathcal{F}') \end{tikzcd}$$

Figure 4.5: Relation between the homology Gassner and flying cars

At the end of Chapter 3, it was verified that the homology Gassner is an invariant of braids. In this section, we have seen that the assignment $\mathcal{C} : L \mapsto \mathcal{C}(L)$ defines an invariant of labeled $(n+1)$ string links with values in $GL_n(\mathcal{F})$; it is multiplicative under the multiplication of labeled string links obtained by stacking one above the other. Thus, by the relation in Equation 4.2 we have verified Theorem 3.23 for string links. This is the statement of Theorem 2.4 in [KLW01].

4.4 Stitching operation, Braids and String links.

In this section, we define the stitching operation on string links an on the collection $Mat_{T'\times T'}(\mathcal{F}')$, of $n\times n$ labeled matrices. Given an $(n+1)$-string link L labeled by $T=\{0,1,\cdots,n\}$, let $S_k^{i,j}: SL_{n+1}\to SL_n$ be the ***stitching operation*** defined as follows: connect the head of a strand labeled $i\neq 0$ to the tail of another strand labeled $j\neq 0$ and relabel the resulting strand $k=\min\{i,j\}$ as shown in Figure 4.6. Here, we require that $i\neq j$ to avoid circle components. We will see that this operation is well defined after Definition 4.3.

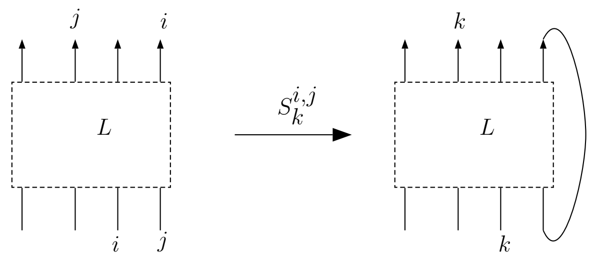
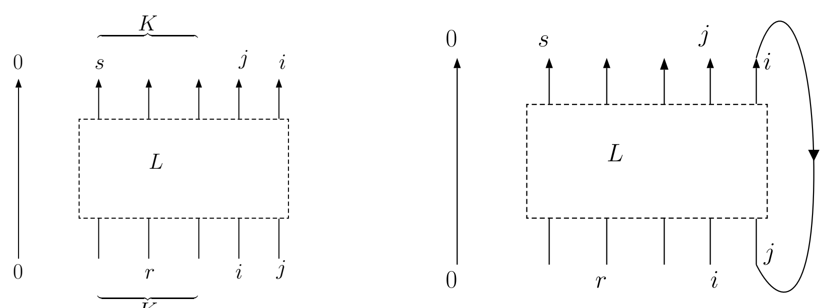


Figure 4.6: The stitching operation.

4.4.1 The stitching operation and the matrix from by flying cars.



(a) An $(n+1)$-string link

(b) An n-string link after stitching.

Figure 4.7: The stitching operation.

The stitching operation defined above induces an operation on the collection $Mat_{T'\times T'}(\mathcal{F}')$, of $n\times n$ labeled matrices as follows: We denote the induced operation also by $S_k^{i,j}$. Let $\mathcal{C}(L)\in Mat_{T'\times T'}(\mathcal{F}')$ be the labeled $n\times n$ matrix assigned the $(n+1)$-string link L in Figure 4.7a by the flying car $\mathcal{C}$:

$$\mathcal{C}(L)=\left(\begin{array}{c|ccc} & K & i & j \\ \hline K & a & b & c \\ i & d & e & f \\ j & g & h & k \end{array}\right), \tag{4.3}$$

and let

$$\begin{aligned} \mathcal{G}_h^r(L) &= \left(D_n \cdot \mathcal{C}(L) \cdot D_n^{-1}\right)//\rho_{col}//m^t \\ &= \left(\begin{array}{c|ccc} & \rho(K) & j & i \\ \hline K & \Xi & \psi & \phi \\ i & \theta & \beta & \alpha \\ j & \epsilon & \delta & \gamma \end{array}\right), \end{aligned} \tag{4.4}$$

where $T' = K \cup \{i, j\}$. Suppose strand i is stitched to strand j, and the resulting strand is relabeled $k = \min\{i, j\}$. Then, the stitching operation $S_k^{i,j}$ on L yields a matrix $L//S_k^{i,k}$ with entries described below.

1. When starting at position i, there are two possible paths for the car to exit at position j. The first option is to directly exit at j with a probability of $P_{i,j}$. The second option is to exit j by first going through a loop. In this case, the car exits i at the top with probability $P_{i,i}$. However, as strand i is stitched to strand j, the car needs to fly back to the bottom at j along the stitched strands. From j, the car exits the top at i again with probability $P_{i,i}P_{j,i}$. This process may repeat with a probability of $P_{i,i}P_{j,i}^*$, where $P_{j,i}^* = \sum_{n=1}^{\infty} P_{j,i}^n = \frac{1}{1-P_{j,i}}$. However, the loop terminates when the car exits the top at j. The resulting probability is $P_{i,i}P_{j,i}^*P_{j,j}$. Taking into account both paths, the final probability of a car exiting the top at j, when starting at i, is $\left(L//m_k^{i,j}\right)_{i,j} = L_{i,j} + P_{i,i}P_{j,i}^*P_{j,j}$. Referring to the matrix $\mathcal{C}(L)$, $P_{i,j} = \beta$, $P_{i,i} = \alpha$, $P_{j,i} = \gamma$, and $P_{j,j} = \delta$. Thus, $\left(L//m_k^{i,j}\right)_{i,j} = \beta + \alpha\frac{1}{1-\gamma}\delta$.
2. When starting at position r, there are two possible paths for the car to exit at position s. Using the same explanation above, the (r, s)-th entry of the matrix $L//m_k^{i,k}$ is $\left(L//m_k^{i,j}\right)_{r,s} = P_{r,s} + P_{r,i}P_{j,i}^*P_{j,s}$. Referring to the matrix $\mathcal{C}(L)$, $P_{r,s} = \Xi_{r,s}, P_{r,i} = \phi_r, P_{j,i} = \gamma$ and $P_{j,s} = \epsilon_s$, so that $\left(L//m_k^{i,j}\right)_{r,s} = \Xi_{r,s} + \phi_r\frac{1}{1-\gamma}\epsilon_s$.
3. Again, when starting at position r, there are two possible paths for the car to exit at position j, the same explanation above applies. The (r, j)-th entry of the matrix $\mathcal{C}(L)//m_k^{i,k}$ is $\left(L//m_k^{i,j}\right)_{r,j} = P_{r,j} + P_{r,i}P_{j,i}^*P_{j,j} = \psi_r + \phi_r\frac{1}{1-\gamma}\delta$.
4. Finally, when starting at position i, there are two possible paths for the car to exit at position s. Referring to the above explanation $\left(L//m_k^{i,j}\right)_{i,s} = P_{i,s} + P_{i,i}P_{j,i}^*P_{j,s} = \theta_s + \alpha\frac{1}{1-\gamma}\epsilon_s$.

Let $S(P_{j,i})$ denote the resulting matrix after stitching strand i to strand j, where $P_{j,i}$ is (j, i)th entry γ of the matrix $\mathcal{G}_h^r(L)$ in Equation 4.4. The dimension of $S(P_{j,i})$ is $(n-1)\times(n-1)$ and it is given by

$$S(\gamma) = \left(\begin{array}{c|cc} & \rho(K) & k \\ \hline K & \Xi_{r,s} + \phi_r\frac{1}{1-\gamma}\epsilon_s & \psi_r + \phi_r\frac{1}{1-\gamma}\delta \\ k & \theta_s + \alpha\frac{1}{1-\gamma}\epsilon_s & \beta + \alpha\frac{1}{1-\gamma}\delta \end{array}\right) // \, t_i, t_j \mapsto t_k. \tag{4.5}$$

Definition 4.3. Given $\mathcal{G}_h^r(L) \in Mat_{T'\times T'}(\mathcal{F}')$. Define the ***stitching operation***, $S_k^{i,j} : Mat_{T'\times T'}(\mathcal{F}') \to$

$Mat_{T''\times T''}(\mathcal{F}'')$, on the collection of labeled $n\times n$ matrices by

$$\left(\begin{array}{c|ccc} & \rho(K) & j & i \\ \hline K & \Xi & \psi & \phi \\ i & \theta & \beta & \alpha \\ j & \epsilon & \delta & \gamma \end{array}\right) //S_k^{i,j} := \left(\begin{array}{c|cc} & \rho(K) & k \\ \hline K & \Xi_{r,s}+\phi_r\frac{1}{1-\gamma}\epsilon_s & \psi_r+\phi_r\frac{1}{1-\gamma}\delta \\ k & \theta_s+\alpha\frac{1}{1-\gamma}\epsilon_s & \beta+\alpha\frac{1}{1-\gamma}\delta \end{array}\right) //\ t_i, t_j \mapsto t_k,\ k=\min\{i,j\}.$$

The stitching operation $S_k^{i,j} : Mat_{T'\times T'}(\mathcal{F}') \to Mat_{T''\times T''}(\mathcal{F}'')$ is well-defined as it maps the $n\times n$ identity matrix to the $(n-1)\times(n-1)$ identity matrix. In this process, we replace the block matrix Ξ with the corresponding identity matrix, setting $\alpha=\delta=1$ and all other entries to zero. Note that this is done on the corresponding matrix without the permutation operation.

The diagram below describes the induced operation. Here, $T''=T'-\{\max\{i,j\}\}$ and $\mathcal{F}''=\mathbb{Q}(\{t_k : k\in T''\})$. Also, $Mat_{T''\times T''}(\mathcal{F}'')$ is the collection of labeled $(n-1)\times(n-1)$ matrices.

$$\begin{tikzcd} SL_{n+1} \arrow[r, "S_k^{i,j}"] \arrow[d, "\mathcal{C}"'] & SL_n \arrow[d, "\mathcal{C}"] \\ Mat_{T'\times T'}(\mathcal{F}') \arrow[r, "S_k^{i,j}"] \arrow[d, "D_n(-)D_n^{-1}//\rho_{col}//m^t"'] & Mat_{T''\times T''}(\mathcal{F}'') \arrow[d, "D_{n-1}(-)D_{n-1}^{-1}//\rho_{col}//m^t"] \\ Mat_{T'\times T'}(\mathcal{F}') \arrow[r, "S_k^{i,j}"] & Mat_{T''\times T''}(\mathcal{F}'') \end{tikzcd}.$$

Thus, given a string link L and the corresponding matrix $\mathcal{C}(L)$, the matrix $S(P_{j,i})$ associated with stitching strand i to strand j is computed as follows: Pick the (j,i)th entry $P_{j,i}=\gamma$ of

$$\mathcal{G}_h^r(L) = \left(D_n\cdot\mathcal{C}(L)\cdot D_n^{-1}\right)//\rho_{col}//tr.$$

Find the minor $Min(P_{j,i})$, the row $Row(\widehat{P_{j,i}})$ and the column $Col(\widehat{P_{j,i}})$ of $P_{i,j}$, where $\widehat{\cdot}$ means omit $P_{j,i}$ from the row and column. Then, after stitching, $S(\gamma)$ in Equation 4.5 can be written as

$$S(P_{j,i}) = Min(P_{j,i}) + \frac{1}{1-\gamma}Col(\widehat{P_{j,i}})\cdot Row(\widehat{P_{j,i}}). \tag{4.6}$$

We have seen from Lemma 3.22 that the assignment $\mathcal{G}_h : L\to\mathcal{G}_h(L)$ is multiplicative under the multiplication of labeled string links. So, to calculate the homology Gassner invariant $\mathcal{G}_h^r(\beta)$ of a coloured braid β, we only need to compute the homology Gassner invariant of the braid generators and then combine them appropriately to obtain $\mathcal{G}_h^r(\beta)$. However, this is not the case for string links, as $\mathcal{G}_h^r$ is not generally multiplicative. Fortunately, there is a solution to this issue: string links can be derived from braids using the stitching operation, and this relationship is described by the following lemma.

Lemma 4.4 ([Vo18], Lemma 6.1.)**.** *Let L be a string link. Then L can be obtained as a partial closure of some braid β.*

Proof. In a string link L, some strands may have downward arcs that connect cups and caps (see Figure 4.8), since the strands are not required to be monotonic. We want to transform each downward arc into a stitching of the right-most outgoing strand with the right-most incoming strand. If there are no downward arcs, then L is a braid and there is nothing to prove. Let us assume L has strands that are not monotonic. In this case, L can be decomposed into its basic components, consisting of over-crossings ⤲, under-crossings ⤱, and cups and caps.

Figure 4.8: A downward arc connecting a cup and a cap.

Consider an arc connecting a cap and cup like the one on the right of Figure 4.8. Upward strands may go over or under this piece like the one Figure 4.9, which can be transformed so that all downward arcs

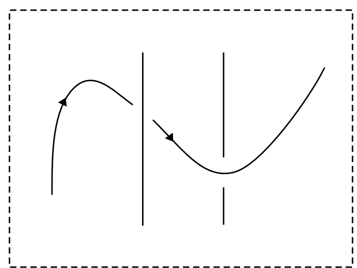


Figure 4.9: A downward arc connecting a cup and a cap with over and under strands.

have one crossing as in Figure 4.10.

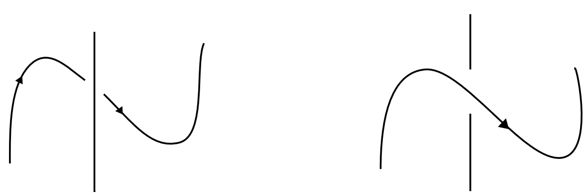


Figure 4.10: Simplified downward arc passing under or over a single strand.

Now, consider a downward arc with a negative crossing, similar to the picture on the left of Figure 4.11 and perform the following steps.

1. Perform a Reidemeister 1 move to create a kink with a negative crossing.
2. Pull the kink to the right, passing under all other strands by performing a sequence of Reidemeister 2 and 3 moves until the picture on the right is reached. Notice that the downward arc stitches the right-most outgoing strand to the right-most incoming strand.
3. Repeat the two steps above until all downward strands inside the dashed rectangle as shown in Figure 4.11 are exhausted.

If there are downward arcs with a positive crossing, similar to the picture on the left of Figure 4.12, follow the same steps as mentioned earlier. However, this time the kink should have a positive crossing and the kink passes over all other strands when pulling to the right. Once all downward arcs have been exhausted, you will arrive at a diagram with all upward strands inside a dashed box and all downward arcs

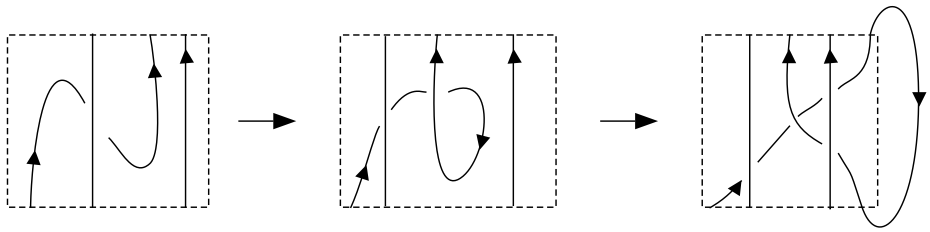


Figure 4.11: Moving a downward arc passing under a strand to the right.

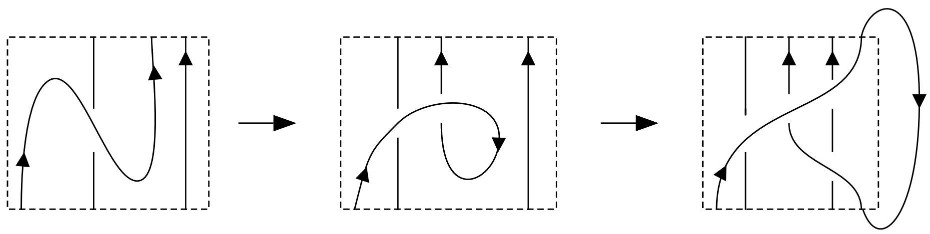


Figure 4.12: Moving a downward arc passing over a strand to the right.

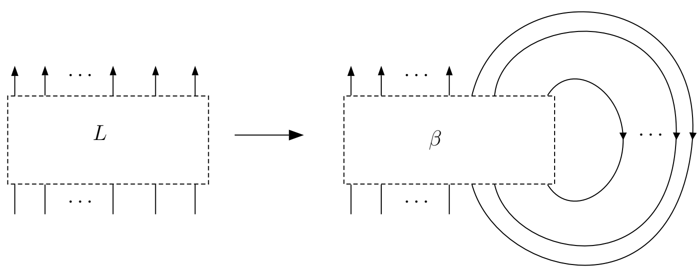
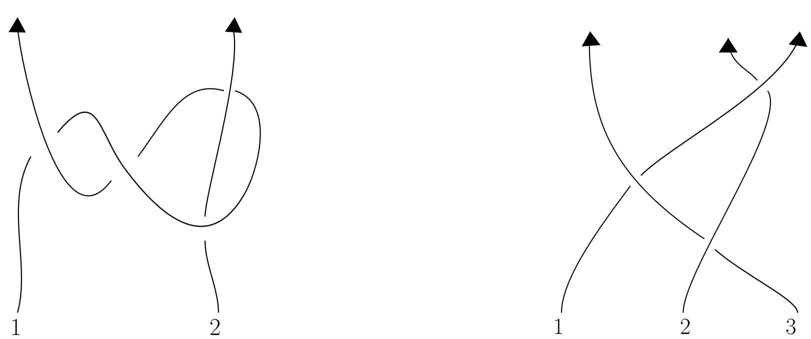


Figure 4.13: Transforming a string link L to the partial closure of a β.

outside the box. This represents a partial closure of a braid as shown in Figure 4.13. Therefore, L is given as the partial closure of a braid, which concludes the proof.

□



(a) String link L_4

(b) Braid β.

Figure 4.14: A 2-string link and a 3-braid

Example 4.5. Consider the string link L_4 in Figure 4.14a. On the leftmost side of Figure 4.15, the downward arc of L_4 is coloured and it goes over a strand. Referring to Figure 4.12, transform the red arc into a stitching of the right-most outgoing strand and the right-most incoming strand, as described visually in Figure 4.15.

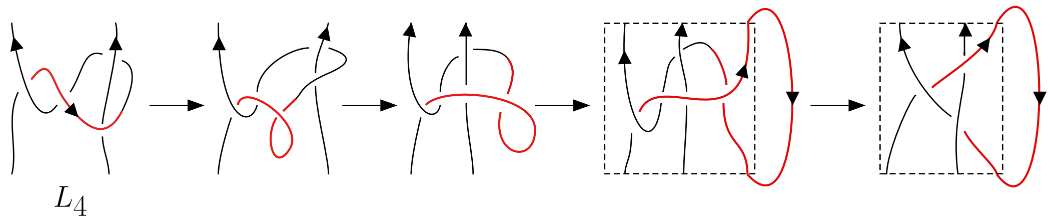


Figure 4.15: Transforming the 2-string link L_4 to the partial closure of the braid β.

This will result in a partially closed braid. Specifically, when strands 1 and 3 of the braid β in Figure 4.14b are stitched together, a different projection of L_4 is obtained.

In the next example, we will show that the homology Gassner invariant of L_4 is the same as the homology Gassner invariant of partially closed braid β. We will first compute the cohomology Gassner invariant whose transpose is the homology Gassner invariant. Then, we will compute that invariant $\mathcal{C}(\beta)$ of the braid β, using flying cars. Next, we will stitch strand 1 to strand 3 and rename it 1. Finally, applying the formula in Equation 4.1 gives the homology Gassner invariant of the partially closed braid, the same for the string link L_4.

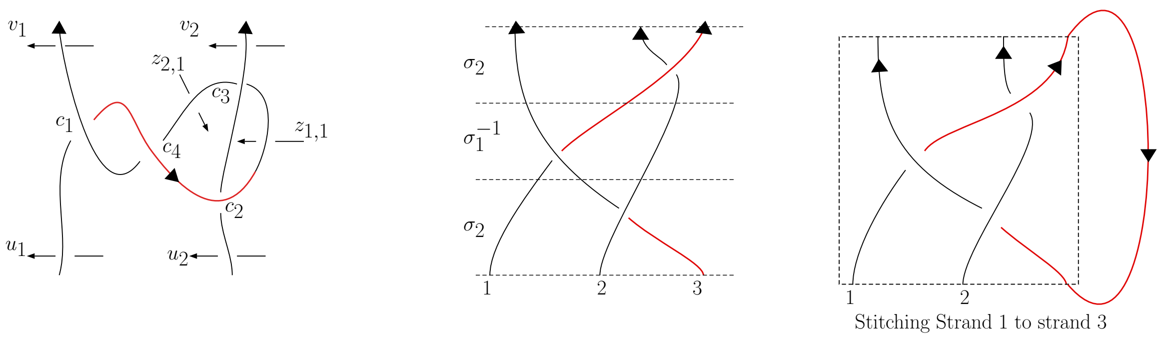


Figure 4.16: Stitching strand 1 to strand 3.

The following example is help to explain the relation between braids and string links.

Example 4.6. 1. **The cohomology Gassner invariant:** The string link L_4 has four crossings, labeled c_1, c_2, c_3 and c_4 ordered according to Ordering 3.1. Let Y be the deformation retract of the complement $X = (D^2 \times [0,1] - L_4)$. Using the cell structure given to the complement of a string described in Section 2.4.1, Y has one 0-cell $q = x_0$, six 1-cells $u_1, u_2, z_{1,1}, z_{2,1}, v_1, v_2$, and four 2-cells e_1, e_2, e_3, e_4. The covering space $\widetilde{Y}$ of Y is determined by $\epsilon : \pi_1(X,q) \to \langle t_0, t_1, t_2 \rangle$ By Proposition 2.6, the relative chain groups of $\widetilde{Y}$ are

$$\begin{aligned}
C_0(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_0(\widetilde{Y};\mathbb{Z})/C_0(\widetilde{q};\mathbb{Z}) = 0,\\
C_1(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_1(\widetilde{Y};\mathbb{Z})/C_1(\widetilde{q};\mathbb{Z}) = C_1(\widetilde{Y};\mathbb{Z})/0 = \langle\, \widetilde{u}_1, \widetilde{u}_2, \widetilde{z}_{1,1}, \widetilde{z}_{2,1}, \widetilde{v}_1, \widetilde{v}_2 \,\rangle,\\
C_2(\widetilde{Y},\widetilde{q};\mathbb{Z}) &= C_2(\widetilde{Y};\mathbb{Z})/C_2(\widetilde{q};\mathbb{Z}) = C_2(\widetilde{Y};\mathbb{Z})/0 = \langle\, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4 \,\rangle.
\end{aligned}$$

By Lemma 3.5, the relative cohomology group $H^1(Y,q;\mathcal{F})$ is $\ker(C^1(Y;\mathcal{F}) \xrightarrow{d^1} C^2(Y;\mathcal{F}))$, where $C^1(Y;\mathcal{F}) = \langle \widetilde{U}_1, \widetilde{U}_2, \widetilde{Z}_{1,1}, \widetilde{Z}_{2,1}, \widetilde{V}_1, \widetilde{V}_2 \rangle$ and $C^2(Y;\mathcal{F}) = \langle \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3, \widetilde{E}_4 \rangle$. By Description 2.13, the boundaries of the 2-cells are

$$\begin{aligned}\partial_2(\widetilde{e}_1) &= (1-t_1)\widetilde{v}_1 + t_1\widetilde{u}_1 - \widetilde{z}_{1,1}\\ \partial_2(\widetilde{e}_2) &= (1-t_2)\widetilde{z}_{1,1} + t_1\widetilde{v}_2 - \widetilde{u}_2\\ \partial_2(\widetilde{e}_3) &= (1-t_1)\widetilde{v}_2 + t_2\widetilde{z}_{2,1} - \widetilde{z}_{1,1}\\ \partial_2(\widetilde{e}_4) &= (1-t_1)\widetilde{z}_{1,1} + t_1\widetilde{z}_{2,1} - \widetilde{v}_1.\end{aligned}$$

So the matrix of d^1 is

$$\begin{aligned} d^1_{matrix} &= \begin{pmatrix} \widetilde{U}_1(\partial_2(\widetilde{e}_1)) & \widetilde{U}_2(\partial_2(\widetilde{e}_1)) & \widetilde{Z}_{1,1}(\partial_2(\widetilde{e}_1)) & \widetilde{Z}_{2,1}(\partial_2(\widetilde{e}_1)) & \widetilde{V}_1(\partial_2(\widetilde{e}_1)) & \widetilde{V}_2(\partial_2(\widetilde{e}_1))\\ \widetilde{U}_1(\partial_2(\widetilde{e}_2)) & \widetilde{U}_2(\partial_2(\widetilde{e}_2)) & \widetilde{Z}_{1,1}(\partial_2(\widetilde{e}_2)) & \widetilde{Z}_{2,1}(\partial_2(\widetilde{e}_2)) & \widetilde{V}_1(\partial_2(\widetilde{e}_2)) & \widetilde{V}_2(\partial_2(\widetilde{e}_2))\\ \widetilde{U}_1(\partial_2(\widetilde{e}_3)) & \widetilde{U}_2(\partial_2(\widetilde{e}_3)) & \widetilde{Z}_{1,1}(\partial_2(\widetilde{e}_3)) & \widetilde{Z}_{2,1}(\partial_2(\widetilde{e}_3)) & \widetilde{V}_1(\partial_2(\widetilde{e}_3)) & \widetilde{V}_2(\partial_2(\widetilde{e}_3))\\ \widetilde{U}_1(\partial_2(\widetilde{e}_4)) & \widetilde{U}_2(\partial_2(\widetilde{e}_4)) & \widetilde{Z}_{1,1}(\partial_2(\widetilde{e}_4)) & \widetilde{Z}_{2,1}(\partial_2(\widetilde{e}_4)) & \widetilde{V}_1(\partial_2(\widetilde{e}_4)) & \widetilde{V}_2(\partial_2(\widetilde{e}_4)) \end{pmatrix}\\ &= \begin{pmatrix} t_1 & 0 & -1 & 0 & 1-t_1 & 0\\ 0 & -1 & 1-t_2 & 0 & 0 & t_1\\ 0 & 0 & -1 & t_2 & 0 & 1-t_1\\ 0 & 0 & 1-t_1 & t_1 & -1 & 0 \end{pmatrix}\end{aligned}$$

The nullspace of d^1_{matrix} is

$$\left\langle \begin{pmatrix} -\frac{1-t_1}{t_2t_1-t_1-t_2}\\ -\frac{t_1}{t_2t_1-t_1-t_2}\\ \frac{(t_1-1)t_1}{t_2t_1-t_1-t_2}\\ \frac{(t_1-1)^2}{t_2t_1-t_1-t_2}\\ 0\\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{-t_2t_1+t_1+2t_2-1}{t_2t_1-t_1-t_2}\\ -\frac{t_2-t_2^2}{t_2t_1-t_1-t_2}\\ -\frac{t_2}{t_2t_1-t_1-t_2}\\ -\frac{1}{t_2t_1-t_1-t_2}\\ 1\\ 0 \end{pmatrix} \right\rangle$$

So,

$$\begin{aligned}\ker(d^1) = \Bigg\langle & -\frac{t_2Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{Z_{2,1}}{t_2t_1-t_1-t_2} - \frac{(-t_2t_1+t_1+2t_2-1)\,U_1}{t_2t_1-t_1-t_2} - \frac{\left(t_2-t_2^2\right)U_2}{t_2t_1-t_1-t_2} + V_1\,,\\ & \frac{(t_1-1)^2Z_{2,1}}{t_2t_1-t_1-t_2} + \frac{t_1\,(t_1-1)\,Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{(1-t_1)\,U_1}{t_2t_1-t_1-t_2} - \frac{t_1U_2}{t_2t_1-t_1-t_2} + V_2\Bigg\rangle,\end{aligned}$$

which is $H^1(Y,q;\mathcal{F})$ by Lemma 3.5. Corollary 3.6 implies $H^1(X_0,q;\mathcal{F}) = C_1(X_0;\mathcal{F}) = \langle\widetilde{U}_1,\widetilde{U}_2\rangle$ and $H^1(X_1,q;\mathcal{F}) = C_1(X_1;\mathcal{F}) = \langle\widetilde{V}_1,\widetilde{V}_2\rangle$. By definition of the cohomology Gassner invariant, the map $\iota_0^*: H^1(X,q;\mathcal{F}) \to H^1(X_0,q;\mathcal{F})$ is given by

$$\iota_0^*:\begin{cases} -\frac{t_2Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{Z_{2,1}}{t_2t_1-t_1-t_2} - \frac{(-t_2t_1+t_1+2t_2-1)U_1}{t_2t_1-t_1-t_2} - \frac{(t_2-t_2^2)U_2}{t_2t_1-t_1-t_2} + V_1 & \mapsto\quad -\frac{(-t_2t_1+t_1+2t_2-1)U_1}{t_2t_1-t_1-t_2} - \frac{(t_2-t_2^2)U_2}{t_2t_1-t_1-t_2}\\ \frac{(t_1-1)^2Z_{2,1}}{t_2t_1-t_1-t_2} + \frac{t_1(t_1-1)Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{(1-t_1)U_1}{t_2t_1-t_1-t_2} - \frac{t_1U_2}{t_2t_1-t_1-t_2} + V_2 & \mapsto\quad -\frac{(1-t_1)U_1}{t_2t_1-t_1-t_2} - \frac{t_1U_2}{t_2t_1-t_1-t_2} \end{cases},$$

an it represented by the matrix $\begin{pmatrix} \frac{t_1(t_2-1)-2t_2+1}{t_1(t_2-1)-t_2} & \frac{t_1-1}{t_1(t_2-1)-t_2}\\ \frac{(t_2-1)t_2}{t_1(t_2-1)-t_2} & \frac{t_1}{t_2-t_1(t_2-1)} \end{pmatrix}$. Also, the map $\iota_1^*: H^1(X,q;\mathcal{F}) \to H^1(X_1;\mathcal{F})$

is given by

$$\iota_1^* : \begin{cases} -\frac{t_2 Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{Z_{2,1}}{t_2t_1-t_1-t_2} - \frac{(-t_2t_1+t_1+2t_2-1)U_1}{t_2t_1-t_1-t_2} - \frac{(t_2-t_2^2)U_2}{t_2t_1-t_1-t_2} + V_1 & \mapsto & \widetilde{V}_1 \\ \frac{(t_1-1)^2 Z_{2,1}}{t_2t_1-t_1-t_2} + \frac{t_1(t_1-1)Z_{1,1}}{t_2t_1-t_1-t_2} - \frac{(1-t_1)U_1}{t_2t_1-t_1-t_2} - \frac{t_1U_2}{t_2t_1-t_1-t_2} + V_2 & \mapsto & \widetilde{V}_2 \end{cases};$$

it is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The cohomology Gassner invariant of L_4 is therefore

$$\mathcal{G}_c(L_4) = (\iota_0^*)^{-1} = \begin{pmatrix} \frac{t_1}{t_1+t_2-1} & \frac{t_1-1}{t_1+t_2-1} \\ \frac{(t_2-1)t_2}{t_1+t_2-1} & \frac{-t_1(t_2-1)+2t_2-1}{t_1+t_2-1} \end{pmatrix}.$$

By Remark 3.21, inverting and transposing $\mathcal{G}_c(L_4)$ gives the homology Gassner invariant of L_4:

$$\mathcal{G}_h^r(L_4) = \begin{pmatrix} \frac{t_2t_1-t_1-2t_2+1}{t_2t_1-t_1-t_2} & \frac{(t_2-1)t_2}{t_2t_1-t_1-t_2} \\ \frac{t_1-1}{t_2t_1-t_1-t_2} & -\frac{t_1}{t_2t_1-t_1-t_2} \end{pmatrix}.$$

2. **The invariant $\mathcal{C}(\beta)$ of the braid β:** Let us analyze the car's options starting from each initial point, which are 1, 2, and 3. If we consider the initial point 1, the car can exit at the endpoint 1, 2, or 3. To exit at 1, the car first passes under strand 3 at a negative crossing with probability $\frac{1}{t_3}$, then passes over strand 2 at a positive crossing with a probability of 1. This gives us a probability of $\frac{1}{t_3}$ for exiting at 1. To exit at 2, the car first passes under strand 3 at a negative crossing with probability $\frac{1}{t_3}$, then moves down onto strand 2 at the positive crossing with a probability of 0, resulting in a probability of 0 for exiting at 2. Finally, to exit at 3, the car moves up onto strand 3 at the negative crossing, giving us a probability of $1-\frac{1}{t_3}$ for exiting at 3.

   Repeating the analysis for the initial points 2 and 3, we find that the invariant $\mathcal{C}(\beta)$ is

$$\mathcal{C}(\beta) = \left(\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \frac{1}{t_3} & 0 & 1-\frac{1}{t_3} \\ 2 & 1-t_1 & t_1 & 0 \\ 3 & (1-t_2)(1-t_1) & (1-t_2)t_1 & t_2 \end{array}\right) \tag{4.7}$$

   Applying Equation 4.2, the homology Gassner invariant of β is

$$\begin{aligned} \mathcal{G}_h^r(\beta) &= \left(D_3 \cdot \mathcal{C}(\beta) \cdot D_3^{-1}\right) // \rho_{col} // tr \\ &= \left(\begin{array}{c|ccc} & 3 & 2 & 1 \\ \hline 1 & \frac{t_1-1}{t_3} & 0 & t_2 \\ 2 & 0 & t_1 & -t_1(t_3-1) \\ 3 & \frac{1}{t_3} & 1-t_2 & (t_2-1)(t_3-1) \end{array}\right) \\ &= \mathcal{G}_h^r(\sigma_2) \cdot \mathcal{G}_h^r(\sigma_1^{-1}) \cdot \mathcal{G}_h^r(\sigma_2) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_1 \\ 0 & 1 & 1-t_2 \end{pmatrix} \begin{pmatrix} \frac{t_1-1}{t_3} & 1 & 0 \\ \frac{1}{t_3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_2 \\ 0 & 1 & 1-t_3 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{t_1-1}{t_3} & 0 & t_2 \\ 0 & t_1 & -t_1(t_3-1) \\ \frac{1}{t_3} & 1-t_2 & (t_2-1)(t_3-1) \end{pmatrix}, \tag{4.8}$$

where $\rho_{col} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is the permutation matrix given by the permutation $\rho : 1 \mapsto 3, 2 \mapsto 2$ and $3 \mapsto 1$ induced by the braid β, and $D_3 = \begin{pmatrix} 1-t_1 & 0 & 0 \\ 0 & 1-t_2 & 0 \\ 0 & 0 & 1-t_3 \end{pmatrix}$.

3. **Stitching strand** 1 **to strand** 3 **of** β**:** Stitching strand 1 to strand 3 corresponds to finding the entry $P_{3,1}$ of $\mathcal{G}_h^r(\beta)$ in Equation 4.8 and applying the formula in Equation 4.6. The $P_{3,1}$ entry of the matrix $\mathcal{G}_h^r(\beta)$ is $P_{3,1} = (t_2-1)(t_3-1)$. After applying the stitching operation, $\mathcal{G}_h^r(\beta)$ becomes the matrix $S(P_{3,1})$:

$$\begin{aligned} S(P_{3,1}) &= Min(P_{3,1}) + \frac{1}{1-P_{3,1}} Col(\widehat{P_{3,1}}) Row(\widehat{P_{3,1}}) \\ &= \begin{pmatrix} \frac{t_1-1}{t_3} & 0 \\ 0 & t_1 \end{pmatrix} + \frac{1}{1-(t_2-1)(t_3-1)} \begin{pmatrix} t_2 \\ -t_1(t_3-1) \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{t_3} & 1-t_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{t_1}{t_3} - \frac{t_2-1}{t_2(t_3-1)-t_3} & \frac{(t_2-1)t_2}{t_2(t_3-1)-t_3} \\ \frac{t_1(t_3-1)}{(t_2(t_3-1)-t_3)t_3} & \frac{t_1}{t_3-t_2(t_3-1)} \end{pmatrix}_{t_3 \mapsto t_1} \\ \mathcal{G}_h^r(L_4) &= \begin{pmatrix} \frac{t_2t_1-t_1-2t_2+1}{t_2t_1-t_1-t_2} & \frac{(t_2-1)t_2}{t_2t_1-t_1-t_2} \\ \frac{t_1-1}{t_2t_1-t_1-t_2} & -\frac{t_1}{t_2t_1-t_1-t_2} \end{pmatrix}, \end{aligned}$$

which is the same as the homology Gassner invariant of the string link L_4.

In the previous example, we have demonstrated that the homology Gassner invariant of L_4 can be obtained from that of β through the stitching operation. In general, this is true. We state this as a theorem:

Theorem 4.7. *Let L be an $n-1$ string link. Then the homology Gassner invariant of L can be obtained from the homology Gassner invariant of an n braid β labeled by T such that L is a partial closure of β.*

Proof. According to Lemma 4.4, every string link L can be realized as a partial closure of some braid β. This implies that there exists a braid β such that performing a partial closure on β yields the string link L.

Let L be an $n-1$ string link, and let β be an (n) braid labeled by $T\prime = \{1, \cdots, n\}$ such that L is a partial closure of β. Assume the permutation induced by β is $\rho = \begin{pmatrix} 1 & \cdots & n-1 & n \\ \rho(1) & \cdots & \rho(n-1) & \rho(n) \end{pmatrix}$. Without loss of generality, we assume that L is obtained from β by a single stitching. To show that the homology Gassner invariant of L can be obtained from the homology Gassner invariant of β such that L is a partial closure of β, we need to show that the diagram below is commutative:

$$
\begin{tikzcd}
SL_n \arrow[r, "\mathcal{G}_h"] \arrow[ddd, bend right, "S^{\rho(n),n}_{\rho(n)}//n\equiv\rho(n)"'] & M_n(\mathcal{F}) \arrow[ddd, bend left, "S^{\rho(n),n}_{\rho(n)}//t_n\mapsto t_{\rho(n)}"] \\
\beta \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{G}_h(\beta) \arrow[d, mapsto] \\
L \arrow[r, mapsto] & \mathcal{G}_h(L) \\
SL_{n-1} \arrow[r, "\mathcal{G}_h"'] & M_{n-1}(\mathcal{F})
\end{tikzcd},
$$

where $n \equiv \rho(n)$ mean relabel $L = \beta \,//\, S^{\rho(n),n}_{\rho(n)}$ after the stitching operation with $1, \cdots n-1$.

1. Let X^β be the complement of β, and let X_0^β and X_1^β be the corresponding punctured disks. Compute the homology Gassner invariant

$$\mathcal{G}_h(\beta) : H_1(X_0^\beta, q; \mathcal{F}) \to H_1(X^\beta, I_q; \mathcal{F}) \to H_1(X_1^\beta, q; \mathcal{F})$$

where $H_1(X_0^\beta, q; \mathcal{F}) \cong H_1(X^\beta, I_q; \mathcal{F}) \cong H_1(X_1^\beta, q; \mathcal{F}) \cong \mathcal{F}\langle x_1, \cdots x_n \rangle$. Let $\mathcal{G}_h(\beta) = \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix}$ be the $n \times n$ matrix representation of the homology Gassner invariant of β. In terms of labeled matrices, recall that

$$\mathcal{G}_h(\beta) = \left(D_n \cdot \mathcal{C}(\beta) \cdot D_n^{-1} \right) // \rho_{col} // m^t,$$

so

$$\mathcal{G}_h(\beta) = \begin{array}{c|c} & \begin{array}{ccc} \rho(K) & \rho(n-1) & \rho(n) \end{array} \\ \hline \begin{array}{c} K \\ n-1 \\ n \end{array} & \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} \end{array}.$$

Next, apply the stitching operation to $\mathcal{G}_h^r(\beta)$ to obtain the $(n-1) \times (n-1)$ matrix

$$S(P_{n,\rho(n)}) = \beta \,//\, \mathcal{G}_h \,//\, S^{\rho(n),n}_{\rho(n)} \,//\, t_n \mapsto t_{\rho(n)}. \tag{4.9}$$

2. Perform the stitching operation on β to obtain L. This involves connecting the head of the strand labeled $\rho(n)$ to the tail of the strand labeled n and renaming the strands appropriately. Note that each stitching operation on the braid reduces the number of strands by one.

Let X^L be the complement of L, and let X_0^L and X_1^L be the corresponding punctured disks. Now, compute the homology Gassner invariant

$$\mathcal{G}_h(L) : H_1(X_0^L, q; \mathcal{F}) \to H_1(X^L, I_q; \mathcal{F}) \to H_1(X_1^L, q; \mathcal{F})$$

of L. Here, $H_1(X_0^L; \mathcal{F}) \cong H_1(X^L; \mathcal{F}) \cong H_1(X_1^L; \mathcal{F}) \cong \mathcal{F}\langle x_1, \cdots x_{n-1} \rangle$, since the number of strands is reduced by one after stitching. The homology Gassner invariant $\mathcal{G}_h(L)$ can be derived from $\mathcal{G}_h(\beta)$:

$H_1(X_0^\beta, q; \mathcal{F}) \to H_1(X^\beta, I_q; \mathcal{F}) \to H_1(X_1^\beta, q; \mathcal{F})$ as follows:

The stitching operation on β does not affect the number and types of crossings, meaning that the former and latter remain the same in L. Moreover, the cell structure of L is similar to that of β with the exception that two meridians are identified. This implies that the cycles and boundaries in the homology groups involved in the computation of $\mathcal{G}_h(\beta)$ are not affected. However, the cycles u_n and $v_{\rho(n)}$ in $H_1(X^\beta, I_q, \mathcal{F})$ are identified. It follows that

$$H_1(X^L, I_q, \mathcal{F}) = H_1(X^\beta, I_q, \mathcal{F}) \,//\, u_n \equiv v_{\rho(n)}. \tag{4.10}$$

As a linear map, $\mathcal{G}_h(\beta) : H_1(X_0^\beta, q; \mathcal{F}) \to H_1(X^\beta, I_q; \mathcal{F}) \to H_1(X_1^\beta, q; \mathcal{F})$ is given by

$$\mathcal{G}_h(\beta) : \begin{array}{ccc} \mathbf{x}_K & \mapsto & \Xi\mathbf{x}_K + \psi x_{n-1} + \phi x_n \\ x_{n-1} & \mapsto & \theta\mathbf{x}_K + \beta x_{n-1} + \alpha x_n \\ x_n & \mapsto & \epsilon\mathbf{x}_K + \delta x_{n-1} + \gamma x_n \end{array},$$

where $\mathbf{x}_K$ is the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-2} \end{pmatrix}$ and $K = \{1, \cdots n-2\}$. Since we are stitching the strand labeled $\rho(n)$ to the strand labeled n, the cycles u_n and $v_{\rho(n)}$ are identified. Note that u_n is identified with x_n. So, by Equation 4.10, the last equation in the system becomes $x_n \mapsto \frac{\epsilon}{1-\gamma}\mathbf{x}_K + \frac{\delta}{1-\gamma}x_{n-1}$. The implication is that the linear system becomes

$$\mathcal{G}_h(\beta \,//\, S_{\rho(n)}^{\rho(n),n}) : \left\{ \begin{array}{ccc} \mathbf{x}_K & \mapsto & \left(\Xi + \frac{\phi\epsilon}{1-\gamma}\right)\mathbf{x}_K + \left(\psi + \frac{\phi\delta}{1-\gamma}\right)x_{n-1} \\ x_{n-1} & \mapsto & \left(\theta + \frac{\alpha\epsilon}{1-\gamma}\right)\mathbf{x}_K + \left(\beta + \frac{\alpha\delta}{1-\gamma}\right)x_{n-1} \end{array} \right. .$$

But this is the linear system $\mathcal{G}_h(L)$ with a matrix representation $\begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\phi\delta}{1-\gamma} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} \end{pmatrix}$, which is exactly the matrix in Equation 4.9. We have shown that

$$\beta \,//\, S_{\rho(n)}^{\rho(n),n} \,//\, n \equiv \rho(n) \,//\, \mathcal{G}_h = \beta \,//\, \mathcal{G}_h \,//\, S_{\rho(n)}^{\rho(n),n} \,//\, t_n \mapsto t_{\rho(n)}.$$

It follows that the diagram is commutative, hence the theorem.

□

Chapter 5

Unitarity of the Gassner invariant

5.1 Summary of Chapter

In this chapter, we discuss the unitarity of the homology Gassner invariant with respect to a skew hermitian product given by an intersection product defined on the cycles of the first homology groups $H_1(X_0; \mathcal{F})$ and $H_1(X_1; \mathcal{F})$. We provide details on the computation of the intersection product defined on the elements of $H_1(X_0; \mathcal{F})$. Furthermore, we provide a detailed proof of the unitary statement in Theorem 3.2 of [KLW01] (see Theorem 5.18). We also provide an alternative proof of Theorem 5.18 (see Theorem 5.20). Finally, we present a Mathematica implementation of the unitarity of the homology Gassner invariant.

5.2 The intersection product on $H_1(P; \mathcal{F})$, $P = X_j$ for $j = 0, 1$

It is natural to obtain a skew-Hermitian matrix using the cup product on cohomology. However, achieving this for the cohomology Gassner invariant is quite challenging. In contrast, obtaining the Hermitian matrix via the intersection product defined on the cycles is easier in the context of the homology Gassner invariant. We begin by discussing the intersection product presented in Section 3.2.1 of [KT08]..

Let α and β be two oriented closed loops on an oriented surface Σ. Deform α and β slightly, and assume that they intersect transversely in a finite set of points that are not self-crossings of α or β. The ***algebraic intersection number*** of α and β is the sum $\alpha \cdot \beta = \sum\limits_{p \in \alpha \cap \beta} \varepsilon_p$, where $\varepsilon_p = +1$ if the tangent vectors of α and β at p form a positively oriented basis and $\varepsilon_p = -1$ otherwise. Let α, β represent the homology classes $[\alpha], [\beta] \in H_1(\Sigma; \mathbb{Z})$ respectively. Then the algebraic intersection number on $[\alpha]$ and $[\beta]$ is defined as

$$[\alpha] \cdot [\beta] = \alpha \cdot \beta = \sum_{p \in \alpha \cap \beta} \varepsilon_p \tag{5.1}$$

More on intersection number can be found in [GP74].

Let P represent the $(n+1)$-punctured disk, $X_j, j = 0, 1$, which are subspaces of the complement X of an $(n+1)$ string link. Let $\widetilde{P} \to P$ be the covering space of P determined by the map $\epsilon : \pi_1(X, x_0) \to \langle t_k \rangle_{k=1}^{n}$. Also, recall the local efficient system $\mathcal{F} = \mathbb{Q}(\langle t_0, \cdots, t_n \rangle)$. The $\mathcal{F}$-module $H_1(P; \mathcal{F})$ carries a natural $\mathcal{F}$-valued skew-hermitian form defined as follows. Consider the associated intersection product $H_1(\widetilde{P}; \mathbb{Z}) \times H_1(\widetilde{P}; \mathbb{Z}) \to \mathbb{Z}$ defined in Equation 5.1 whose value $[\widetilde{\alpha}] \cdot [\widetilde{\beta}]$ on the homology classes $[\widetilde{\alpha}], [\widetilde{\beta}] \in H_1(\widetilde{P}; \mathbb{Z})$

represented by transversal oriented loops $\widetilde{\alpha}$, $\widetilde{\beta}$ on $\widetilde{P}$ is the algebraic intersection number of these loops obtained by counting their intersections with signs ± 1 determined by the orientations on $\widetilde{P}$. Note that the orientation on P lifts to the covering space $\widetilde{P} \to P$.

Now, tensor $H_1(\widetilde{P};\mathbb{Z})$ with $\mathcal{F}$ and define a pairing

$$\langle \cdot, \cdot \rangle : \left(H_1(\widetilde{P};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{F}\right) \times \left(H_1(\widetilde{P};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \overline{\mathcal{F}}\right) \to \mathcal{F}$$

by

$$\left\langle \widetilde{\alpha}, \widetilde{\beta} \right\rangle = \sum_{(k_0,\ldots,k_n)\in\mathbb{Z}^{n+1}} \left(\widetilde{\alpha} \cdot (t_0^{k_0} \ldots t_n^{k_n} \widetilde{\beta})\right) t_0^{k_0} \ldots t_n^{k_n}, \tag{5.2}$$

where $\mathcal{F} \to \overline{\mathcal{F}}$[1] is the automorphism sending t_i to t_i^{-1}. Since $\mathcal{F}$ is a field, then $\mathbb{Z} \otimes \mathcal{F} \cong \mathcal{F}$. So, $H_1(\widetilde{P};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{F} \cong H_1(\widetilde{P};\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{F}) \cong H_1(\widetilde{P};\mathcal{F})$. To simplify notations, let $\mathbf{k} = (k_0, \ldots, k_n) \in \mathbb{Z}^{n+1}$ and $\mathbf{t}^{\mathbf{k}} = t_0^{k_0} \ldots t_n^{k_n}$. In this notation, Equation 5.2 becomes

$$\left\langle \widetilde{\alpha}, \widetilde{\beta} \right\rangle = \sum_{\mathbf{k}\in\mathbb{Z}^{n+1}} \left(\widetilde{\alpha} \cdot (\mathbf{t}^{\mathbf{k}} \widetilde{\beta})\right) \mathbf{t}^{\mathbf{k}} \tag{5.3}$$

Lemma 5.1. *The intersection product in Equation 5.3 is well-defined.*

Proof. The product $\widetilde{\alpha} \cdot (\mathbf{t}^{\mathbf{k}} \widetilde{\beta})$ represents the algebraic intersection number of the lifts $\widetilde{\alpha}$ and $\mathbf{t}^{\mathbf{k}} \widetilde{\beta}$. This number is finite because the covering map $\widetilde{X_0} \to X_0$ maps $\widetilde{\alpha}$ bijectively onto α and maps the set $\alpha \cap \left(\bigcup_{\mathbf{k}\in\mathbb{Z}^{n+1}} \mathbf{t}^{\mathbf{k}} \widetilde{\beta}\right)$ bijectively onto the finite set $\alpha \cap \beta$. Consequently, the sum in Equation 5.2 is finite. Let $\widetilde{\phi} = \partial(\widetilde{\beta})$ be the boundary of some chain $\widetilde{\beta}$. Then if $\widetilde{\alpha}$ does not intersect the boundary of ϕ, there is nothing to show. If $\widetilde{\alpha}$ intersect the boundary of ϕ, it does so an even number of times. Half of the intersection points have -1 signs and the other half have $+1$ signs. Thus, $\left\langle \widetilde{\alpha}, \widetilde{\beta} \right\rangle = 0$. This shows that the pairing is well defined. ☐

Lemma 5.2. *The intersection product in Equation 5.2 is skew hermitian. That is, let $\gamma, \eta \in H_1(\widetilde{X_0};\mathcal{F})$, then*

1. *$\langle \gamma, \eta \rangle = -\overline{\langle \eta, \gamma \rangle}$.*
2. *$\langle f\gamma, \eta \rangle = f\langle \gamma, \eta \rangle$ and $\langle \gamma, f\eta \rangle = \overline{f}\langle \gamma, \eta \rangle$, where $f \mapsto \overline{f}$ is the automorphism of $\mathcal{F}$, sending t_i to t_i^{-1}.*

Proof. Let $\gamma, \eta \in H_1(P;\mathcal{F})$ and $f \in \mathcal{F}$. Then

1. We show that $\langle \gamma, \eta \rangle = -\overline{\langle \eta, \gamma \rangle}$.

$$\begin{aligned} \langle \gamma, \eta \rangle &= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\gamma \cdot (\mathbf{t}^{\mathbf{k}} \eta)\right) \mathbf{t}^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\mathbf{t}^{-\mathbf{k}} \gamma \cdot \eta\right) \mathbf{t}^{\mathbf{k}} \\ &= -\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\eta \cdot (\mathbf{t}^{-\mathbf{k}} \gamma)\right) \mathbf{t}^{\mathbf{k}} \\ &= -\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\eta \cdot (\mathbf{t}^{\mathbf{k}} \gamma)\right) \mathbf{t}^{-\mathbf{k}} \\ &= -\overline{\langle \eta, \gamma \rangle}. \end{aligned}$$

[1] $\mathcal{F}$ and $\overline{\mathcal{F}}$ are the same.

2. We show that $\langle f\gamma,\, \eta\rangle = f\langle \gamma,\, \eta\rangle$ and $\langle \gamma,\, f\eta\rangle = \overline{f}\langle \gamma,\, \eta\rangle$. It suffices to show the case when $f = \mathbf{t}^{\mathbf{r}}$, for $\mathbf{r} \in \mathbb{Z}^{n+1}$.

$$\begin{aligned}
\langle \gamma,\, \mathbf{t}^{\mathbf{r}}\eta\rangle &= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\gamma\cdot(\mathbf{t}^{\mathbf{r}+\mathbf{k}}\eta)\right)\mathbf{t}^{\mathbf{k}} \\
&= \mathbf{t}^{-\mathbf{r}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\gamma\cdot(\mathbf{t}^{\mathbf{r}+\mathbf{k}}\eta)\right)\mathbf{t}^{\mathbf{r}+\mathbf{k}} \\
&= \mathbf{t}^{-\mathbf{r}}\langle \gamma,\, \eta\rangle.
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathbf{t}^{\mathbf{r}}\gamma,\, \eta\rangle &= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\mathbf{t}^{\mathbf{r}}\gamma\cdot(\mathbf{t}^{\mathbf{k}}\eta)\right)\mathbf{t}^{\mathbf{k}} \\
&= \mathbf{t}^{\mathbf{r}} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left(\gamma\cdot(\mathbf{t}^{\mathbf{k}-\mathbf{r}}\eta)\right)\mathbf{t}^{\mathbf{k}-\mathbf{r}} \\
&= \mathbf{t}^{\mathbf{r}}\langle \gamma,\, \eta\rangle.
\end{aligned}$$

□

5.2.1 Understanding the intersection product on $\widetilde{X}_j, j = 0, 1$

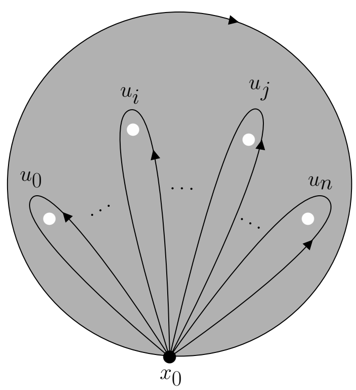


Figure 5.1: The space X_0 with loops $u_0, u_1, \cdots, u_n$.

In this subsection we will understand how to compute the intersection pairing on covering space of the punctured disk shown in Figure 5.1.

Description 5.3 (Intersection pairing of two cycles)**.** Let $q = x_0 \in P$ be the basepoint and let $\widetilde{q}$ be a fixed lift of q to $\widetilde{P}$ (namely, fix $\widetilde{q} \in \widetilde{P}$ is such that the projection of $\widetilde{q}$ onto the base space P is q.) Given a curve α in P, which begins and ends at p, let $\widetilde{\alpha}$ be the unique lift of α to $\widetilde{P}$ for such that $\widetilde{\alpha}(0) = \widetilde{q}$. Then we will have a unique tuple $w(\alpha) \in \mathbb{Z}^{n+1}$ for which $\widetilde{\alpha}(1) = \mathbf{t}^{w(\alpha)}\widetilde{q}$.

From Lemma 3.10 the homology group $H_1(X_0;\mathcal{F})$ is generated by the set $\{\widetilde{\beta}_k\}_{k=1}^n$ over the field $\mathcal{F}$, where $\widetilde{\beta}_k = (t_0 - 1)\widetilde{u}_k - (t_k - 1)\widetilde{u}_0$ (represented as a closed curve which start and end at $\widetilde{x}_0$ as shown in Figure 5.2, also denoted $\widetilde{\beta}_k, k = 1, \cdots, n$) and $\widetilde{u}_i$ is the lift of the meridian u_i (which are closed curves that generate the fundamental group of X_0, see Figure 5.1) such that $\widetilde{u}_i(0) = \widetilde{x}_0$ and $\widetilde{u}_i(1) = t_i\widetilde{x}_0$ for $i = 0, 1, \cdots, n$.

Every element in $H_1(\widetilde{P};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{F} \cong H_1(X_0;\mathcal{F})$ can be written as a finite linear combination of

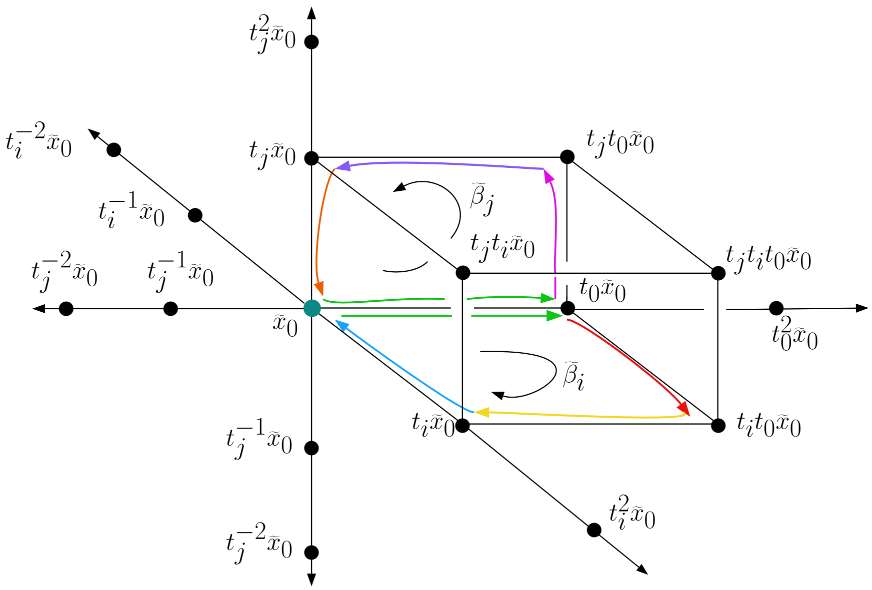


Figure 5.2: A schematic 3-dimensional view of the covering space $\widetilde{X}_0 \to X_0$.

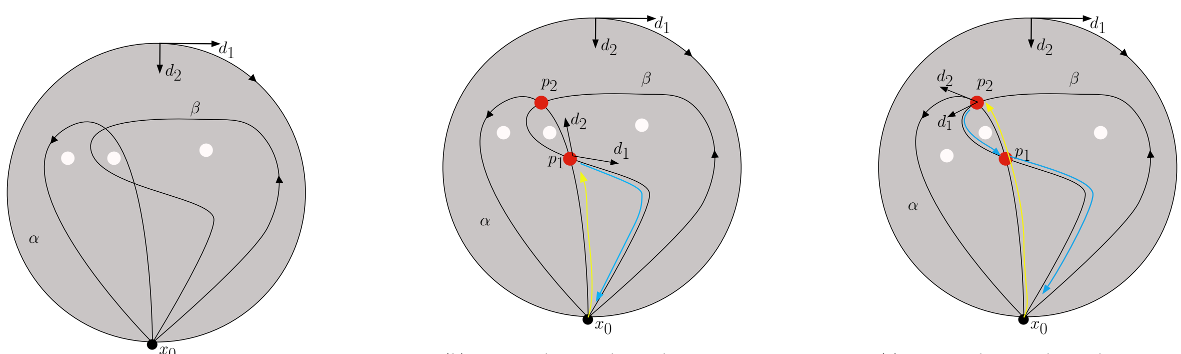
the elements of the set $\{\widetilde{\beta}_k\}_{k=1}^n$. Thus, for $\widetilde{\alpha}, \widetilde{\beta} \in H_1(\widetilde{P};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{F}$ write $\widetilde{\alpha} = \sum_{k=1} \theta_k \widetilde{\beta}_k$, $\theta_k \in \mathcal{F}$ and $\widetilde{\beta} = \sum_{r=1} \lambda_r \widetilde{\beta}_r$, $\lambda_r \in \mathcal{F}$. So,

$$\langle \widetilde{\alpha}, \widetilde{\beta} \rangle = \sum_{k,r} \theta_k \overline{\lambda}_r \langle \widetilde{\beta}_k, \widetilde{\beta}_r \rangle,$$

where

$$\langle \widetilde{\beta}_k, \widetilde{\beta}_r \rangle = \sum_{p \in \beta_k \cap \beta_r} \varepsilon_p \mathbf{t}^{-w(\beta_k \#_p \beta_r)}.$$

and $\lambda \mapsto \overline{\lambda}$ is $\mathcal{F}$-linear automorphism of $\mathcal{F}$ which extends $\mathbf{t^k} \mapsto \mathbf{t^{-k}} := t_1^{-k_1} \dots t_n^{-k_n}$. See pages 100 and 101 of [KT08]. Here, $\beta_k \#_p \beta_r$ is the curve in X_0 that follows β_k from its beginning up to the intersection point p, and follows β_r backward from p to the beginning of β_r.



(a) Intersection of two loops.

(b) Path through intersection point p_1.

(c) Path through intersection point p_2.

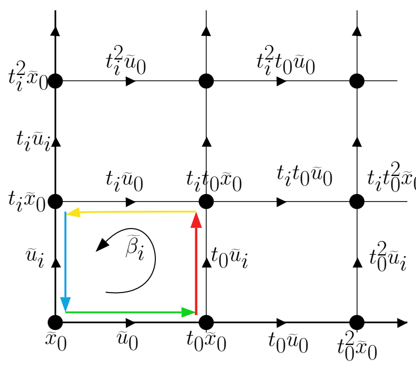
Figure 5.3: A schematic diagram showing the intersection of α and β, and their intersection points.

For example in Figure 5.3, the algebraic intersection pairing of α and β, $\langle\alpha,\beta\rangle$[2], is computed as follows. Fix a positive orientation $\{d_1, d_2\}$ on X_0. Referring to Figure 5.3b, start at x_0 and move along α following the yellow path until the first intersection point p_1 is encountered. Now, move along β, starting from p_1, following the blue path back to x_0. This forms a path ξ_{p_1} whose lift $\widetilde{\xi}_{p_1}$ to the covering space of P is such that $\widetilde{\xi}_{p_1}(0) = \widetilde{x}_0$ and $\widetilde{\xi}_{p_1}(1) = \epsilon(\xi_{p_1})\widetilde{\xi}_{p_1}(0) = \epsilon(\xi_{p_1})\widetilde{x}_0$.

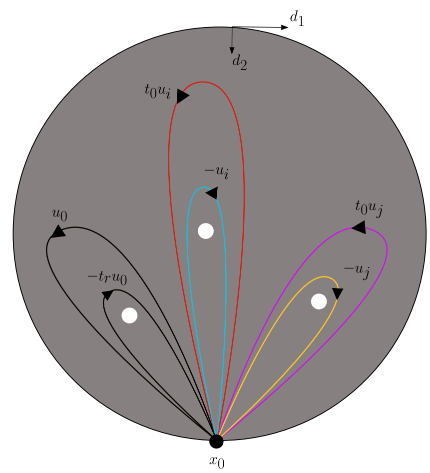
The next intersection point is p_2. Referring to Figure 5.3b, move along α following the yellow path until p_2 is encountered, and then along β following the blue path back to x_0. Let ξ_{p_2} be the completed path; it is such that $\widetilde{\xi}_{p_2}(0) = \widetilde{x}_0$ and $\widetilde{\xi}_{p_2}(1) = \epsilon(\xi_{p_2})\widetilde{\xi}_{p_2}(0) = \epsilon(\xi_{p_2})\widetilde{x}_0$. Repeat the process for all the other $p_i \in \alpha\cap\beta, p_i \neq p_1, p_2$. Then, the intersection pairing of α and β is given as

$$\langle\alpha,\beta\rangle = \sum_{p_i\in\alpha\cap\beta} \varepsilon_{p_i}\epsilon\left(\alpha'_{p_i}\beta''_{p_i}\right), \tag{5.4}$$

where $\epsilon\left(\alpha'_{p_i}\beta''_{p_i}\right)$ is the product of local coefficient assigned to the intersection point p_i and $\varepsilon_{p_i} \in \{-1,+1\}$ is the sign $\mathbf{sgn}(p_i)$ of the intersection point p_i, determined by the given orientation. This concludes the description.



(a) A 2 dimensional view of the covering space $\widetilde{X}_0$ showing the generator $\widetilde{\beta}_i$.



(b) A projection of the generators $\widetilde{\beta}_i, \widetilde{\beta}_j$ onto the punctured disk X_0 under the covering map $\widetilde{X}_0 \to X_0$, where $r = i, j$.

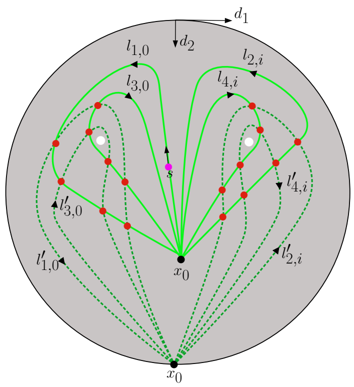
Figure 5.4: The generator $\widetilde{\beta}_i = (t_0 - 1)\widetilde{u}_i - (t_i - 1)\widetilde{u}_0$.

Example 5.4 (Intersection pairing $\langle\widetilde{\beta}_i, \widetilde{\beta}_j\rangle$).

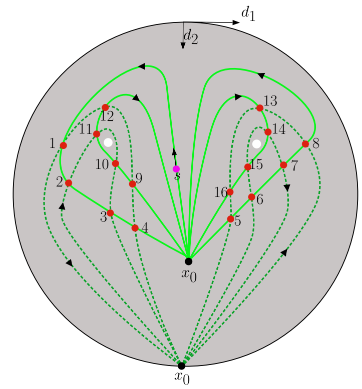
1. (Self-intersection, that is when $i = j$): Consider the $(n+1)$-punctured disk X_0 in Figure 5.1 and $\widetilde{X}$ the covering space determined by $\epsilon : \pi_1(X_0, x_0) \to \langle t_k\rangle_{k=0}^n$. Fixing $\widetilde{\gamma}_0$, recall from Lemma 3.10 that the homology group $H_1(X_0;\mathcal{F})$ is isomorphic to $\mathcal{F}^n$ with basis $\{\widetilde{\beta}_i\}_{i=1}^n$, where $\widetilde{\beta}_i = (t_0 - 1)\widetilde{u}_i - (t_i - 1)\widetilde{u}_0$. (See Figure 5.4)

   Before computing $\langle\widetilde{\beta}_i, \widetilde{\beta}_i\rangle$, let us first understand the diagram in Figure 5.5. In Figure 5.5a the solid loop is β_i and the dashed loop is a small perturbation of β_i. The path followed by β_i is $l_{1,0}l_{2,i}l_{3,0}l_{4,i}$, and the path followed by the perturbation is $l'_{1,0}l'_{2,i}l'_{3,0}l'_{4,i}$. The loops $l_{k,i}$ and $l'_{k,i}$ are such that $\epsilon(l_{k,j}) = \epsilon(l'_{k,j}) = t_j$. In Figure 5.5b, the intersection points are numbered 1 through 16 [3].

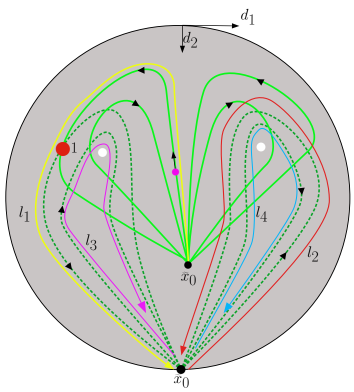
[2] The expression $\langle\alpha,\beta\rangle$ is interpreted as tracing the path α first, followed by β.
[3] We follow this order for simplicity. Any order can be chosen.



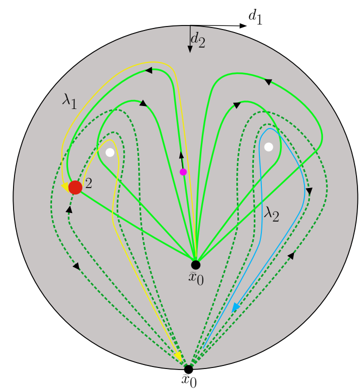
(a) Self intersection of $\widetilde{\beta}_i$.



(b) Numbered intersection points.



(c) The path through the intersection point 1 is $l_1 l_2 l_3 l_4$.



(d) The path through the intersection point 2 is $\lambda_1 \lambda_2$.

Figure 5.5: Self intersection points.

The self-intersection product $\langle \widetilde{\beta}_i, \widetilde{\beta}_i \rangle$ is computed by using Description 5.3 as follows. In Figure 5.5c, the path through the intersection point 1 is $\zeta_1 = l_1 l_2 l_3 l_4$, which is formed by the yellow path followed by the red, pink and blue paths in that order. This path is such that its lift $\widetilde{\zeta}_1$ satisfies $\widetilde{\zeta}_1(0) = \widetilde{x}_0$ and $\widetilde{\zeta}_1(1) = \epsilon(\zeta_1)\widetilde{x}_0 = \epsilon(l_1 l_2 l_3 l_4)\widetilde{x}_0$. Thus $\epsilon(\zeta_1) = \epsilon(l_1 l_2 l_3 l_4) = t_0 t_j t_0^{-1} t_j^{-1} = 1$ and the sign at 1 is -1.

Next, in Figure 5.5d the path through the intersection point 1 is $\zeta_2 = \lambda_1 \lambda_2$, which is formed by the yellow path followed by the blue path. Again, this path is such that its lift $\widetilde{\zeta}_2$ satisfies $\widetilde{\zeta}_2(0) = \widetilde{x}_0$ and $\widetilde{\zeta}_2(1) = \epsilon(\zeta_2)\widetilde{x}_0 = \epsilon(\lambda_1 \lambda_2)\widetilde{x}_0$. Thus $\epsilon(\zeta_2) = \epsilon(\lambda_1 \lambda_2) = t_i^{-1}$. The sign at 2 is $+1$.

Now, let $\zeta_3, \zeta_4, \cdots \zeta_{16}$ be the corresponding paths passing through the intersection points $3, 4, \cdots, 16$ respectively. Then repeating the above process for these points, we have the following table.

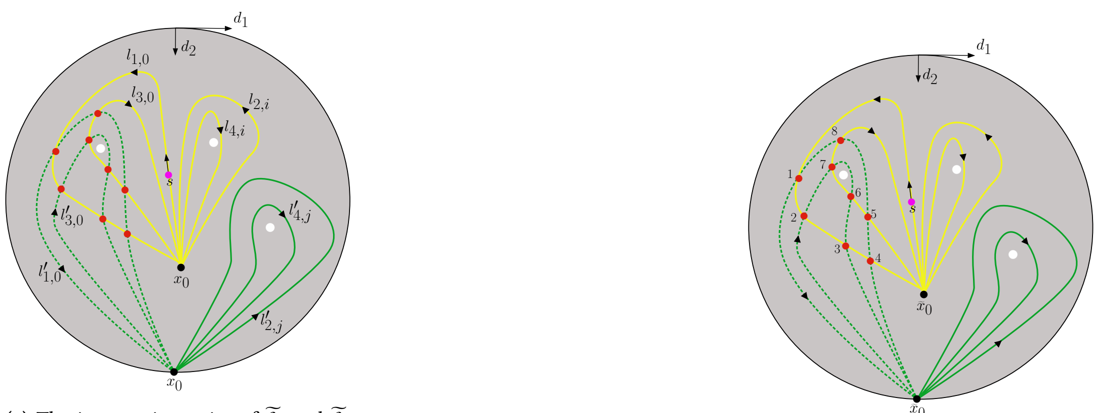
Int. pt. k	$\mathbf{sgn}(k)$	$\epsilon(\zeta_k)$	$\mathbf{sgn}(k)\epsilon(\zeta_k)$
1	-1	1	-1
2	+1	t_i^{-1}	t_i^{-1}
3	-1	$t_0 t_i^{-1}$	$-t_0 t_i^{-1}$
4	+1	t_0	t_0
5	-1	t_i^{-1}	$-t_i^{-1}$
6	+1	$t_0 t_i^{-1}$	$t_0 t_i^{-1}$
7	-1	t_0	$-t_0$
8	+1	1	1

Int. pt. k	$\mathbf{sgn}(k)$	$\epsilon(\zeta_k)$	$\mathbf{sgn}(k)\epsilon(\zeta_k)$
9	-1	$t_0 t_i$	$-t_0 t_i$
10	+1	t_0	t_0
11	-1	1	-1
12	+1	t_i	t_i
13	-1	t_0^{-1}	$-t_0^{-1}$
14	+1	1	1
15	-1	t_i^{-1}	$-t_i^{-1}$
16	+1	$t_0^{-1} t_i^{-1}$	$t_0^{-1} t_i^{-1}$

Summing up the entries of the last column, we have

$$\langle \widetilde{\beta_i}, \widetilde{\beta_i} \rangle = \sum_{k=1}^{16} \mathbf{sgn}(k)\epsilon(\zeta_k) = \frac{(t_0 - 1)(t_i - 1)(1 - t_0 t_i)}{t_0 t_i}.$$

2. (Intersection when $i \neq j$) The intersection product $\langle \widetilde{\beta_i}, \widetilde{\beta_j} \rangle$ of $\widetilde{\beta_i}$ and $\widetilde{\beta_j}$ is also computed in a similar way using Figure 5.6 together with Description 5.3, where the yellow path is the generator β_i and the green path is the generator β_j. The intersection product is



(a) The intersection point of $\widetilde{\beta_i}$ and $\widetilde{\beta_j}$ shown on the disk X_0.

(b) Numbered intersection points

Figure 5.6: Intersection points of β_i and β_j.

$$\langle \widetilde{\beta_i}, \widetilde{\beta_j} \rangle = -\frac{(t_0 - 1)(t_i - 1)(t_j - 1)}{t_j}, \quad \text{for } i < j.$$

This completes the example. Thus, we have the following lemma.

Lemma 5.5 (Intersection product on X_0). *Consider the n-punctured disk X_0 in Figure 5.1 and $\widetilde{X_0}$ the covering space determined by $\epsilon : \pi_1(X, x_0) \to \langle t_k \rangle_{k=0}^{n}$. Fixing $\widetilde{u}_0$, the intersection pairing on the generators $\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n$*

of $H_1(X_0;\mathcal{F})$ *is given by the following formulas:*

$$\langle\widetilde{\beta}_i,\widetilde{\beta}_j\rangle = \begin{cases} \frac{(t_0-1)(t_i-1)(1-t_1t_i)}{t_0t_i}, & i=j \quad (self-intersection) \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j}, & i<j \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_j}, & i>j \end{cases} \tag{5.5}$$

where $\widetilde{\beta}_m = (t_0-1)\widetilde{u}_m - (t_m-1)\widetilde{u}_0$, $m \neq 1$. .

Proof. From Example 5.4 and property 1 of Lemma 5.2, we have proved that

$$\langle\widetilde{\beta}_i,\widetilde{\beta}_j\rangle = \begin{cases} \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i}, & i=j \quad (self-intersection) \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j}, & i<j \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_j}, & i>j \end{cases}$$

□

The intersection product on X_1 is similar to the intersection product on X_0. The points and loops on X_0 and X_1 differ by a permutation induced by the associated string link. Thus, fixing $\widetilde{u}_0$, the formulas for the intersection pairing on X_1 is the same as Equation 5.5, with appropriate t_i's. So, in general, the intersection product is given by

$$\langle\widetilde{\beta}_i,\widetilde{\beta}_j\rangle = \begin{cases} \frac{(t_0-1)(t_{T[i]}-1)(1-t_0t_{T[i]})}{t_0t_{T[i]}}, & i=j \quad (self-intersection) \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_{T[j]}}, & i<j \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_0t_{T[j]}}, & i>j \end{cases}, \tag{5.6}$$

where $T=\{0,1,\cdots,n\}$ is the set of labels and $T[k]$ is the label at position k in T. Here, note that the order of set T corresponding to X_0 may differ from the order of the set T corresponding to X_1 due to the permutation induced by the string link (see the conventions in Chapter 1).

5.3 Relation between intersection product and cup product

In this section, we discuss an abstract relation between the intersection product and the cup product. Details can be found in [Hat02].

Definition 5.6 (Cap product: [Hat02])**.** Let R be a ring. For an arbitrary space Y, define an R-bilinear cap product

$$\cap : C_k(Y;R)\times C^l(Y;R) \to C_{k-l}(Y;R)$$

for $k \geq l$ by setting $\sigma\cap\varphi = \varphi(\sigma|_{[v_0,\dots,v_l]})\sigma|_{[v_l,\dots,v_k]}$ for a k-simplex $\sigma : \Delta^k \to Y$ and a cochain $\varphi \in C^l(X;R)$.

The definition of cap product can be extended to homology and cohomology by using representatives of the homology and cohomology classes: $\cap : H_k(Y;R) \times H^l(Y;R) \to H_{k-l}(Y;R)$ for $k \geq l$.

Definition 5.7 (Cup product: [Hat02]). Let $\phi \in C^k(Y;R)$ and $\psi \in C^l(Y;R)$. The cup product $\phi \smile \psi \in C^{k+l}(Y;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \to X$ is given by the formula

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+l}]}),$$

where the right-hand side is the product in R. The cup product is defined on cocycles by using representatives of the cohomology classes: $\smile : H^k(Y;R) \times H^l(Y;R) \to H^{k+l}(Y;R)$.

Theorem 5.8 (Poincaré Duality: Theorem 3.30 [Hat02]). *If M is a closed $\mathbb{Z}$-orientable n-manifold with fundamental class $[M] \in H_n(M;\mathbb{Z})$, then the map $D_M : H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$ defined by $D_M(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.*

Theorem 5.9 (Lefschetz Poincaré Duality: Theorem 3.34 [Hat02]). *Suppose M is a compact $\mathbb{Z}$-orientable n-manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$-dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Then the map $D_M : H^k(M,A;\mathbb{Z}) \to H_{n-k}(M,B;\mathbb{Z})$ defined by $D_M(\alpha) = [M] \cap \alpha$, where $[M] \in H_n(M,\partial M;\mathbb{Z})$, is an isomorphism for all k.*

Setting A and B to be the empty set reduces Theorem 5.9 to Theorem 5.8 The duality theorems also hold when we consider local coefficients. Refer to [Hat02] and [KLW01] for more details.

Applying the Poincaré Duality map to both factors in $\smile : H^k(M;R) \times H^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}$, the cup product is the intersection $\langle \ , \rangle : H_{n-k}(M;\mathbb{Z}) \times H_k(M;\mathbb{Z}) \to \mathbb{Z}$. More specifically, suppose U and V are closed, oriented submanifolds of M, of dimensions k and $n-k$ respectively. Suppose U and V intersect transversely, and $u = D_M[U], v = D_M[V]$, then $\langle u, v\rangle$ is the intersection number of u and v described above. If we replace M with $\widetilde{P}$ and tensor with the field $\mathcal{P}$, then we get the intersection product define in Equation 5.2.

5.4 The homology Gassner invariant is unitary

In this section, we detail the proof of the unitarity of the homology Gassner invariant as presented in [KLW01]. Following this, we will provide a coordinate-based example to illustrate the theorem. In what follows, the ring R is $\mathbb{Z}$.

Theorem 5.10 (Theorem 3.2 of [KLW01]). *For $a \in H_1(X_0;\mathcal{F})$ and $b \in H_1(X_1;\mathcal{F})$, $\langle \mathcal{G}_h(a), b\rangle = \langle a, \mathcal{G}_h^{-1}(b)\rangle$.*

Before proving the theorem, let us look at the following lemmas. First of all, let us understand the boundary $X = (D^2 \times [0,1] - L)$. The boundary of X is made up of the two punctured disks X_0 and X_1, and a disjoint union of cylinders denoted T. Figure 5.7 shows an example of the boundary of the complement of a 3-string link. Note that up to homotopy, all the complement of all n-string links have the same boundary.

Lemma 5.11. *$H_i(S^1;\mathcal{F}) = 0$ and $H^i(S^1;\mathcal{F}) = 0$ for $i = 0,1$, where $\mathcal{F} = \mathbb{Q}(t)$*

Proof. The cell structure of S^1 consist od a 0-cell q and a 1-cell γ. The cellular chain complex for the universal covering of S^1 is $0 \xrightarrow{\partial_2} C_1 = \langle\widetilde{\gamma}\rangle \xrightarrow{\partial_1} C_0 = \langle\widetilde{q}\rangle \xrightarrow{\partial_0} 0$. Note that $\partial_1(\widetilde{\gamma}) = t-1$. The chain complex is therefore $0 \to \mathcal{F} \xrightarrow{t-1} \mathcal{F} \to 0$ with all homology groups equal to 0. Thus $H_i(S^1;\mathcal{F}) = 0$ for $i = 0,1$.

A similar argument shows that the cellular cochain complex of S^1 is $0 \to \mathcal{F} \xrightarrow{t-1} \mathcal{F} \to 0$, and it has cohomology groups $H^i(S^1)$ for $i = 0,1$ is 0. $\square$

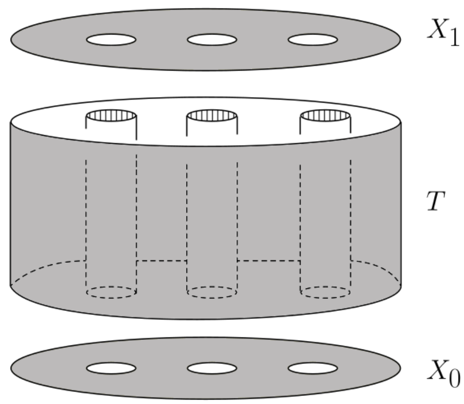


Figure 5.7: The boundary of X; $\partial X = X_1 \sqcup T \sqcup X_0$.

According to Lemma 5.11, $H_1(X_0 \cup T \cup X_1; \mathcal{F}) \cong H_1(X_0 \cup X_1; \mathcal{F})$ since $H_1(T; \mathcal{F}) \cong \bigoplus H_1(S^1; \mathcal{F}) = 0$. Consider the diagram of inclusions and their corresponding induced homomorphisms. We may sometimes abuse notation by using the same symbol for both inclusion and induced maps.

$$\begin{tikzcd}
 & X_0 \cup T \cup X_1 \arrow[dd, "\iota"] & & & & H_1(X_0 \cup X_1; \mathcal{F}) \arrow[dd, "\iota_*"] & \\
X_0 \arrow[ur, "j_0"] \arrow[dr, "i_0"'] & & X_1 \arrow[ul, "j_1"'] \arrow[dl, "i_1"] \arrow[r, "H_1(-;\mathcal{F})"] & H_1(X_0; \mathcal{F}) & & & H_1(X_1; \mathcal{F}) \arrow[ul, "j_{1*}"'] \arrow[dl, "i_{1*}"] \\
 & X & & & & H_1(X; \mathcal{F}) &
\end{tikzcd}$$

$$\begin{tikzcd}
H_1(X_0; \mathcal{F}) \arrow[r, "j_{0*}"] \arrow[dr, "i_{0*}"'] & H_1(X_0 \cup X_1; \mathcal{F}) \arrow[d, "\iota_*"] \\
 & H_1(X; \mathcal{F})
\end{tikzcd}$$

Figure 5.8: Inclusions and their corresponding induced maps

The pair $(X, X_0 \cup T \cup X_1)$ induces a long exact sequence, part of which has been shown in the first row of the diagram below.

$$\begin{tikzcd}
H_2(X, X_0 \cup X_1; \mathcal{F}) \arrow[r, "\partial"] & H_1(X_0 \cup X_1; \mathcal{F}) \arrow[r, "\iota_*"] & H_1(X; \mathcal{F}) \arrow[r] & H_1(X, X_0 \cup X_1; \mathcal{F}) \\
 & H_1(X_0; \mathcal{F}) \oplus H_1(X_1; \mathcal{F}) \arrow[u, "j_* = j_{0*} + j_{1*}"] \arrow[r, equal] & H_1(X_0; \mathcal{F}) \oplus H_1(X_1; \mathcal{F}) \arrow[u, "i_* = i_{0*} + i_{1*}"'] &
\end{tikzcd}$$

Figure 5.9: Long exact sequence

From the diagrams in Figure 5.8 and Figure 5.9, we have the following lemma

Lemma 5.12. *Let $a \in H_1(X_0; \mathcal{F})$. Then there exists $A \in H_2(X, X_0 \cup X_1; \mathcal{F})$ such that $\partial A = a - \mathcal{G}_h(a)$ if and only if $\iota_*(a - \mathcal{G}_h(a)) = 0$.*

Proof of Lemma 5.12. Let $a \in H_1(X_0; \mathcal{F})$. From Figure 5.8, we have the following mappings which are all isomorphisms.

$$\begin{tikzcd} H_1(X_0;\mathcal{F}) \arrow[r, "i_{0*}"] \arrow[rr, bend right, "\mathcal{G}_h"'] & H_1(X;\mathcal{F}) & H_1(X_1;\mathcal{F}) \arrow[l, "i_{1*}"'] \end{tikzcd}$$

$$a \longmapsto i_{0*}(a) \longmapsto \mathcal{G}_h(a)$$

Note that $(a, -\mathcal{G}_h(a)) \in H_1(X_0;\mathcal{F}) \oplus H_1(X_1;\mathcal{F})$. From the first upward arrow in Figure 5.9, we have

$$j(a, -\mathcal{G}_h(a)) = j_{0*}(a) - j_{1*}(\mathcal{G}_h(a)) = a - \mathcal{G}_h(a).$$

Note that the square in Figure 5.9 is commutative, that is $\iota_* \circ j_* = i_*$. Thus,

$$\iota_*(j_*(a, -\mathcal{G}_h(a))) = i_*(a - \mathcal{G}_h(a)\big) = i_{0*}(a) - i_{1*}(\mathcal{G}_h(a)) = i_{0*}(a) - i_{0*}(a) = 0$$

since $i_{1*}(\mathcal{G}_h(a)) = i_{0*}(a)$. This implies that there exists $A \in H_2(X, X_0 \cup X_1;\mathcal{F})$ such that $\partial A = a - \upsilon_h(a)$. □

Now, we state the excision theorem to aid in the proof of the next lemma.

Theorem 5.13 (Excision Theorem: Theorem 2.20 [Hat02])**.** *For subspaces $A, B \subset Y$ whose interior cover Y, the inclusion $(B, A \cap B) \hookrightarrow (Y, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for n.*

The following lemma establishes an isomorphic relation between the homologies of the boundary ∂X and its subspace $X_0 \cup X_1$ by excising the subpace T in ∂X.

Lemma 5.14. *The maps $i_{1*} : H_1(X_0 \cup X_1;\mathcal{F}) \to H_1(\partial X;\mathcal{F})$ and $i_{2*} : H_2(X, X_0 \cup X_1;\mathcal{F}) \to H_2(X, \partial X;\mathcal{F})$ are injective and isomorphism respectively.*

Proof. The short exact sequence $0 \to C_*(X_0 \cup X_1) \xrightarrow{i_1} C_*(\partial X) \twoheadrightarrow C_*(\partial X, X_0 \cup X_1) \to 0$ (of the pair $(\partial X, X_0 \cup X_1)$, where $X_0 \cup X_1 \subset \partial X$) induces a long exact sequence,

$$\begin{tikzcd} H_2(\partial X, X_0 \cup X_1;\mathcal{F}) \arrow[r] & H_1(X_0 \cup X_1;\mathcal{F}) \arrow[r, "i_{1*}"] & H_1(\partial X;\mathcal{F}) \arrow[r] & H_1(\partial X, X_0 \cup X_1;\mathcal{F}) \\ H_2(T, \partial T;\mathcal{F}) \arrow[u, "\cong", "\text{Excision}"'] & & & H_1(T, \partial T;\mathcal{F}) \arrow[u, "\cong", "\text{Excision}"'] \\ H^0(T;\mathcal{F}) \arrow[u, "\cong", "\text{Poincare Duality}"'] \arrow[d, equal] & & & H^1(T;\mathcal{F}) \arrow[u, "\cong", "\text{Poincare Duality}"'] \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

of the pair $(\partial X, X_0 \cup X_1)$. T deformation retract to the disjoint union of S^1. By Lemma 5.11, both $H^0(T;\mathcal{F})$ and $H^1(T;\mathcal{F})$ are 0. The long exact sequence then reduces to

$$0 \to H_1(X_0 \cup X_1;\mathcal{F}) \xrightarrow{i_{1*}} H_1(\partial X;\mathcal{F}) \to 0.$$

This proves that i_1 is isomorphic and hence injective. Similarly, the short exact sequence $0 \to C_*(\partial X, X_0 \cup X_1) \xrightarrow{i_2} C_*(X, X_0 \cup X_1) \twoheadrightarrow C_*(X, \partial X) \to 0$ of the triple $(X; \partial X, X_0 \cup X_1)$, where $X_0 \cup X_1 \subset \partial X \subset X$, induces the following long exact sequence.

$$
\begin{tikzcd}
H_2(\partial X, X_0\cup X_1;\mathcal{F}) \arrow[r] & H_2(X, X_0\cup X_1;\mathcal{F}) \arrow[r, "i_{2*}"] & H_2(X,\partial X;\mathcal{F}) \arrow[r] & H_1(\partial X, X_0\cup X_1;\mathcal{F}) \\
H_2(T,\partial T;\mathcal{F}) \arrow[u, "\cong"', "\text{Excision}"] & & & H_1(T,\partial T;\mathcal{F}) \arrow[u, "\cong"', "\text{Excision}"] \\
H^0(T;\mathcal{F}) \arrow[u, "\cong"', "\text{Poincare Duality}"] \arrow[d, equal] & & & H^1(T;\mathcal{F}) \arrow[u, "\cong"', "\text{Poincare Duality}"] \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}
$$

The same argument above shows that i_2 is an isomorphism. ☐

Remark 5.15. 1. Up to homotopy, the complement of X is a 3-manifold with a 2-dimensional manifold boundary ∂X. Setting $M = X, A = \emptyset$ and $B = \partial X$ in Theorem 5.9, we have the isomorphism

$$D_X : H^1(X;R) \to H_2(X,\partial X;R),$$

where $D_X := \cap[X,\partial X]$

2. ∂X is a 2-dimensional manifold without boundary. Replacing X with its boundary, ∂X, we have

$$D_{\partial X} : H^1(\partial X;R) \to H_1(\partial X;R),$$

where $D_{\partial X} := \cap[\partial X]$

Now, Referring to Lemma 5.14, Theorem 5.8, Theorem 5.9 and Remark 5.15, the commutative diagram in Figure 5.10 can be constructed. This leads us to the next lemma.

$$
\begin{tikzcd}
H_2(X, X_0\cup X_1;\mathcal{F}) \arrow[r, "i_{2*}"] \arrow[d, "\partial"'] & H_2(X,\partial X;\mathcal{F}) \arrow[d, "\partial"'] & H^1(X;\mathcal{F}) \arrow[l, "\cap[X,\partial X]"'] \arrow[d, "i_{3*}"'] \\
H_1(X_0\cup X_1;\mathcal{F}) \arrow[r, "i_{1*}"'] & H_1(\partial X;\mathcal{F}) & H^1(\partial X;\mathcal{F}) \arrow[l, "\cap[\partial X]"]
\end{tikzcd}
.
$$

Figure 5.10: A commutative diagram.

Lemma 5.16. *If $M \in H_2(X,\partial X;\mathcal{F})$ and $N \in H_2(X,\partial X;\overline{\mathcal{F}})$, then $\langle \partial M, \partial N\rangle = 0$.*

Proof. Let $\phi, \psi \in H^1(X;\mathcal{F})$ such that $M = \phi \cap [X,\partial X]$ and $N = \psi \cap [X,\partial X]$. $M \in H_2(X,\partial X;\mathcal{F})$ and $N \in H_2(X,\partial X;\overline{\mathcal{F}})$

Now, from the commutative diagram in Figure 5.10, the intersection product of M and N is computed as follows. Note that since the fundamental class $[\partial X]$ of ∂X is bounded by that of X, then $i_{3*}[\partial X] = 0$. So,

$$
\begin{aligned}
\langle \partial M, \partial N\rangle &:= (i_3^*(\phi) \smile i_3^*(\psi)) \cap [\partial X] \\
&= (i_3^*(\phi \smile \psi) \cap [\partial X] \\
&= (\phi \smile \psi) \cap i_{3*}([\partial X]) \\
&= 0.
\end{aligned}
$$

□

Corollary 5.17. *Let $A \in H_2(X, X_0 \cup X_1; \mathcal{F})$ and $B \in H_2(X, X_0 \cup X_1; \mathcal{F})$. Then $\langle \partial A, \partial B\rangle = 0$.*

Proof. Referring to the commutative diagram in Figure 5.10, it is clear from Lemma 5.16 that $\langle \partial A, \partial B\rangle = 0$. □

Now, we are in a good position to prove Theorem 5.10.

Proof of Theorem 5.10. Let $a \in H_1(X_0; \mathcal{F})$ and $b \in H_1(X_1; \mathcal{F})$, then we want show that $\langle v_h(a), b\rangle = \langle a, \mathcal{G}_h^{-1}(b)\rangle$. By Lemma 5.12 there exist $A \in H_2(X, X_0 \cup X_1; \mathcal{F})$ and $B \in H_2(X, X_0 \cup X_1; \mathcal{F})$ such that $\partial A = a - \mathcal{G}_h(a)$ and $\partial B = \mathcal{G}_h^{-1}(b) - b$.

By Corollary 5.17, $\langle \partial A, \partial B\rangle = 0$. On the other hand,

$$\begin{aligned}
\Big\langle \partial A, \partial B \Big\rangle &= \Big\langle\, a - \mathcal{G}_h(a), -b + \mathcal{G}_h^{-1}(b)\Big\rangle_{X_0 \cup X_1} \\
&= \cancelto{0}{\langle\, a, -b\rangle} + \Big\langle\, a, \mathcal{G}_h^{-1}(b)\Big\rangle_{X_0} + \langle\, -\mathcal{G}_h(a), -b\rangle_{X_1} + \cancelto{0}{\Big\langle\, -\mathcal{G}_h(a), \mathcal{G}_h^{-1}(b)\Big\rangle} \\
&= \Big\langle\, a, \mathcal{G}_h^{-1}(b)\Big\rangle_{X_0} - \langle\, \mathcal{G}_h(a), b\rangle_{X_1}.
\end{aligned}$$

It follows that $\Big\langle\, a, \mathcal{G}_h^{-1}(b)\Big\rangle_{X_0} = \langle\, \mathcal{G}_h(a), b\rangle_{X_1}$ □

The following theorem shows that the reduced Gassner invariant is unitary with respect to the intersection pairing on in Lemma 5.5.

Theorem 5.18 (Unitary Condition)**.** *Let L be a string link. Then the Gassner invariant*

$$\mathcal{G}_h^r(L) : H_1(X_0; \mathcal{F}) \longrightarrow H_1(X_1; \mathcal{F})$$

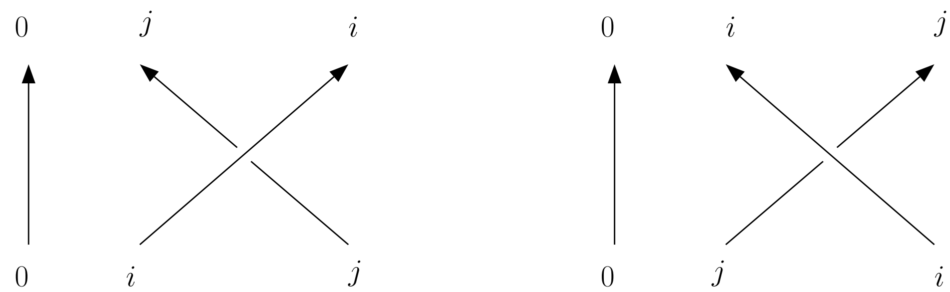
satisfies

$$\langle\, \mathcal{G}_h^r(L)x, \mathcal{G}_h^r(L)y\,\rangle = \langle\, x, y\,\rangle \tag{5.7}$$

Proof. Equation 5.7 follows immediately from the proof of Theorem 5.10 (see Corollary 5.4) which states that if $a \in H_1(X_0; \mathcal{F})$ and $b \in H_1(X_1; \mathcal{F})$, then $\langle \mathcal{G}_h(a), b\rangle = \langle a, \mathcal{G}_h^{-1}(b)\rangle$. This shows that the Gassner invariant is unitary. □

Example 5.19 (The homology Gassner invariant of the generator σ_i is unitary.)**.** In this example we demonstrate the unitary condition of the homology Gassner invariant for the braid generators using the formulas obtained above: $\mathcal{G}_h(R_{i,j}) = \begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}$ and $\mathcal{G}(R_{j,i}) = \begin{pmatrix} \frac{t_j-1}{t_i} & 1 \\ \frac{1}{t_i} & 0 \end{pmatrix}$ These formulas correspond to the diagrams in Figure 5.11. The local coefficient system here is $\mathcal{F} = \mathbb{Q}(t_0, t_i, t_j)$. Let $\eta : t_k \mapsto \frac{1}{t_k}$ be the map that inverts t_k. For $R_{i,j}$, the matrices corresponding to the intersection products on X_0 and X_1 are

$$\begin{pmatrix} \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} & -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j} \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_i} & -\frac{(t_0-1)(t_j-1)(t_0t_j-1)}{t_0t_j} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{(t_0-1)(t_j-1)(1-t_0t_j)}{t_0t_j} & \frac{(1-t_0)(t_i-1)(t_j-1)}{t_i} \\ \frac{(1-t_0)(t_i-1)(t_j-1)}{t_0t_j} & \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} \end{pmatrix}$$



(a) $R_{i,j}$. **(b)** $R_{j,i}$

Figure 5.11: Over and under crossing

respectively. So, we have

$$\begin{pmatrix} 0 & 1 \\ t_i & 1-t_j \end{pmatrix}\begin{pmatrix} \frac{(t_0-1)(t_j-1)(1-t_0t_j)}{t_0t_j} & \frac{(1-t_0)(t_i-1)(t_j-1)}{t_i} \\ \frac{(1-t_0)(t_i-1)(t_j-1)}{t_0t_j} & \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} \end{pmatrix}\left(\begin{pmatrix} 0 & t_i \\ 1 & 1-t_j \end{pmatrix}//\eta\right) = \begin{pmatrix} \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} & -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j} \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_i} & -\frac{(t_0-1)(t_j-1)(t_0t_j-1)}{t_0t_j} \end{pmatrix}.$$

Also, for $R_{j,i}$, the matrices corresponding to the intersection products on X_1 and X_0 are

$$\begin{pmatrix} \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} & -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j} \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_i} & -\frac{(t_0-1)(t_j-1)(t_0t_j-1)}{t_0t_j} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{(t_0-1)(t_j-1)(1-t_0t_j)}{t_0t_j} & \frac{(1-t_0)(t_i-1)(t_j-1)}{t_i} \\ \frac{(1-t_0)(t_i-1)(t_j-1)}{t_0t_j} & \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} \end{pmatrix}$$

respectively. So, we have

$$\begin{pmatrix} \frac{t_j-1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} & -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_j} \\ -\frac{(t_0-1)(t_i-1)(t_j-1)}{t_0t_i} & -\frac{(t_0-1)(t_j-1)(t_0t_j-1)}{t_0t_j} \end{pmatrix}\left(\begin{pmatrix} \frac{t_j-1}{t_i} & 1 \\ \frac{1}{t_i} & 0 \end{pmatrix}//\eta\right) = \begin{pmatrix} \frac{(t_0-1)(t_j-1)(1-t_0t_j)}{t_0t_j} & \frac{(1-t_0)(t_i-1)(t_j-1)}{t_i} \\ \frac{(1-t_0)(t_i-1)(t_j-1)}{t_0t_j} & \frac{(t_0-1)(t_i-1)(1-t_0t_i)}{t_0t_i} \end{pmatrix}.$$

This completes the example.

5.5 Alternative proof of Theorem 5.18

In this section, we present an alternative proof of the unitary condition of the homology Gassner invariant for a string link, which relies on Theorem 4.7, utilizing the relationship between braids and string links as presented in Lemma 4.4. We then conclude with an example demonstrating the unitary property of the homology Gassner invariant for a string link.

Let L be a string link obtained from the partial closure of a braid β using repeated stitching operation. Let $\mathcal{G}_\beta = \mathcal{G}_h^r(\beta)$ be the homology Gassner invariant of β,

$$\begin{aligned} \mathcal{G}_h^r(\beta) &= \left(D_n \cdot \mathcal{C}(\beta) \cdot D_n^{-1}\right)//\rho_{col}//m^t \\ &= \left(\begin{array}{c|ccc} & \rho(K) & \rho(n-1) & \rho(n) \\ \hline K & \Xi & \psi & \phi \\ n-1 & \theta & \beta & \alpha \\ n & \epsilon & \delta & \gamma \end{array}\right), \end{aligned}$$

where Ξ is an $(n-2)\times(n-2)$ matrix, ψ and ϕ are $(n-2)\times 1$ matrices and, θ and ϵ are $1\times(n-2)$ matrices and $Z=\{1,2,\cdots,n-2\}\cup\{n-1,n\}$ is the set of colours assigned to the strands of β.

Suppose $\mathcal{G}_\beta$ is unitary with respect to Ω_0 and Ω_1, where $\Omega_0 = \left(\begin{array}{c|ccc} & Z & n-1 & n \\ \hline Z & \beta_{Z,Z} & \beta_{Z,n-1} & \beta_{Z,n} \\ n-1 & \beta_{n-1,Z} & \beta_{n-1,n-1} & \beta_{n-1,n} \\ n & \beta_{n,Z} & \beta_{n,n-1} & \beta_{n,n} \end{array}\right)$ and $\Omega_1 = \left(\begin{array}{c|ccc} & \rho(Z) & \rho(n-1) & \rho(n) \\ \hline \rho(Z) & \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} & \beta_{\rho(Z,n)} \\ \rho(n-1) & \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} & \beta_{\rho(n-1,n)} \\ \rho(n) & \beta_{\rho(n,Z)} & \beta_{\rho(n,n-1)} & \beta_{\rho(n,n)} \end{array}\right)$ are the matrices corresponding to the intersection products on X_0 and X_1 respectively, where $\beta_{i,j} := \langle\widetilde{\beta_i},\widetilde{\beta_j}\rangle$ (see Equation 5.5), $\rho(Z)$ represents the permutation of the element of Z, $\beta_{\rho(i,j)} := \langle\widetilde{\beta}_{\rho(i)},\widetilde{\beta}_{\rho(j)}\rangle)$ and $\beta_{Z,Z}=\{\beta_{i,j} : i,j\in Z\}$. That is,

$$(\mathcal{G}_\beta//m^t)\,\Omega_1\,(\mathcal{G}_\beta//\eta)=\Omega_0, \tag{5.8}$$

where $\eta : t_k \mapsto \frac{1}{t_k}$. If the strand labeled $\rho(n)$ is stitched to the strand labeled n, where $\rho(n)\neq n$, then we will demonstrated that the homology Gassner invariant of the resulting string link is unitary. It suffices to do this for one stitching operation.

Suppose L is the string link obtained after stitching strand $\rho(n)$ to strand n. The entry of $\mathcal{G}_\beta$ corresponding to the stitching operation is $P_{n,\rho(n)}=\gamma$ and the resulting matrix is

$$S(\gamma)=\begin{pmatrix}\Xi+\frac{1}{1-\gamma}\phi\cdot\epsilon & \psi+\frac{1}{1-\gamma}\delta\cdot\phi\\ \theta+\frac{1}{1-\gamma}\alpha\cdot\epsilon & \beta+\frac{1}{1-\gamma}\alpha\cdot\delta\end{pmatrix}//\,t_{\rho(n)},\ t_n\ \mapsto\ t_{\rho(n)}.$$

The matrix $S(\gamma)$ can be reproduced as a block matrix using elementary matrices as follows: a bold-faced letter here will denote a matrix, or a vector, whose dimension depends on the context. Let

$$H_{\gamma-1}\ =\ \mathcal{G}_\beta-\begin{pmatrix}\mathbf{0}&\mathbf{0}&0\\ \mathbf{0}&0&0\\ 0&0&1\end{pmatrix}=\begin{pmatrix}\Xi&\psi&\phi\\ \theta&\beta&\alpha\\ \epsilon&\delta&\gamma-1\end{pmatrix}$$

and let

$$E_\phi=\begin{pmatrix}\mathbf{1}&\mathbf{0}&-\phi\\ \mathbf{0}&1&0\\ 0&0&1\end{pmatrix},\ E_\alpha=\begin{pmatrix}\mathbf{1}&\mathbf{0}&0\\ \mathbf{0}&1&-\alpha\\ 0&0&1\end{pmatrix},\ E_{\frac{1}{\gamma-1}}=\begin{pmatrix}\mathbf{1}&\mathbf{0}&0\\ \mathbf{0}&1&0\\ 0&0&\frac{1}{\gamma-1}\end{pmatrix}$$

be elementary matrices, where the block matrices $\begin{pmatrix}\mathbf{0}&\mathbf{0}\\ \mathbf{0}&0\end{pmatrix}$ and $\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{0}&1\end{pmatrix}$ corresponds to the block matrix $\begin{pmatrix}\Xi&\psi\\ \theta&\beta\end{pmatrix}$ of $\mathcal{G}_\beta$. Then, notice that

$$E_\phi\cdot E_\alpha\cdot E_{\frac{1}{\gamma-1}}\cdot H_{\gamma-1}=\begin{pmatrix}\mathbf{1}&\mathbf{0}&\frac{1}{1-\gamma}\phi\\ \mathbf{0}&1&\frac{1}{1-\gamma}\alpha\\ 0&0&\frac{1}{\gamma-1}\end{pmatrix}\begin{pmatrix}\Xi&\psi&\phi\\ \theta&\beta&\alpha\\ \epsilon&\delta&\gamma-1\end{pmatrix}=\begin{pmatrix}\Xi+\frac{1}{1-\gamma}\phi\cdot\epsilon&\psi+\frac{1}{1-\gamma}\delta\cdot\phi&0\\ \theta+\frac{1}{1-\gamma}\alpha\cdot\epsilon&\beta+\frac{1}{1-\gamma}\alpha\cdot\delta&0\\ \frac{1}{\gamma-1}\epsilon&\frac{1}{\gamma-1}\delta&1\end{pmatrix}=\begin{pmatrix}\multicolumn{2}{c}{S(\gamma)}&0\\ &&0\\ \frac{1}{\gamma-1}\epsilon&\frac{1}{\gamma-1}\delta&1\end{pmatrix}.$$

So,

$$H_{\gamma-1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & -\phi \\ \mathbf{0} & 1 & -\alpha \\ 0 & 0 & 1-\gamma \end{pmatrix} \begin{pmatrix} \multicolumn{2}{c}{S(\gamma)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \frac{1}{\gamma-1}\epsilon & \frac{1}{\gamma-1}\delta & 1 \end{pmatrix} \tag{5.9}$$

Now, let $Q = \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\mathcal{G}_\beta = H_{\gamma-1} + Q$. So, Equation 5.8 becomes

$$\begin{aligned} (\mathcal{G}_\beta//m^t)\,\Omega_1\,(\mathcal{G}_\beta//\eta) &= ((H_{\gamma-1}+Q)\,//m^t)\,\Omega_1\,((H_{\gamma-1}+Q)\,//\eta) \\ &= (H_{\gamma-1}//m^t)\,\Omega_1\,(H_{\gamma-1}//\eta) + (H_{\gamma-1}//m^t)\Omega_1 Q + Q\Omega_1(H_{\gamma-1}//\eta) + Q\Omega_1 Q \\ &= \Omega_0. \end{aligned} \tag{5.10}$$

The last three summands on the left hand side of Equation 5.10 evaluate to the following:

$$\begin{aligned} (H_{\gamma-1}//m^t)\Omega_1 Q &= \begin{pmatrix} \Xi//m^t & \theta//m^t & \epsilon//m^t \\ \psi//m^t & \beta & \delta \\ \phi//m^t & \alpha & \gamma-1 \end{pmatrix} \begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} & \beta_{\rho(Z,n)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} & \beta_{\rho(n-1,n)} \\ \beta_{\rho(n,Z)} & \beta_{\rho(n,n-1)} & \beta_{\rho(n,n)} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{m_1} \\ \mathbf{0} & 0 & m_2 \\ 0 & 0 & m_3 \end{pmatrix}, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} \mathbf{m_1} &= (\Xi//tr)\beta_{Z,n} + (\theta//m^t)\beta_{\rho(n-1,n)} + (\epsilon//m^t)\beta_{\rho(n,n)} \\ m_2 &= (\psi//tr)\beta_{Z,n} + \beta\beta_{\rho(n-1,n)} + \delta\beta_{\rho(n,n)} \\ m_3 &= (\phi//tr)\beta_{Z,n} + \alpha\beta_{\rho(n-1,n)} + (\gamma-1)\beta_{\rho(n,n)}, \end{aligned}$$

$$\begin{aligned} Q\Omega_1(H_{\gamma-1}//\eta) &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} & \beta_{\rho(Z,n)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} & \beta_{\rho(n-1,n)} \\ \beta_{\rho(n,Z)} & \beta_{\rho(n,n-1)} & \beta_{\rho(n,n)} \end{pmatrix} \begin{pmatrix} \Xi//\eta & \psi//\eta & \phi//\eta \\ \theta//\eta & \beta//\eta & \alpha//\eta \\ \epsilon//\eta & \delta//\eta & (\gamma//\eta)-1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{w_1} & \mathbf{w_2} & w_3 \end{pmatrix}, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} \mathbf{w_1} &= (\Xi//\eta)\beta_{\rho(n,Z)} + (\theta//\eta)\beta_{\rho(n,n-1)} + (\epsilon//\eta)\beta_{\rho(n,n)} \\ w_2 &= (\psi//\eta)\beta_{\rho(n,Z)} + (\beta//\eta)\beta_{\rho(n,n-1)} + (\delta//\eta)\beta_{\rho(n,n)} \\ w_3 &= (\phi//\eta)\beta_{n,Z} + (\alpha//\eta)\beta_{\rho(n,n-1)} + ((\gamma//\eta)-1)\beta_{\rho(n,n)}, \end{aligned}$$

and

$$Q\Omega_1 Q = \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & \beta_{\rho(n,n)} \end{pmatrix}. \tag{5.13}$$

So, we have

$$\begin{aligned}
(H_{\gamma-1}//tr)\,\Omega_1\,(H_{\gamma-1}//\eta) &= \Omega_0 - (H_{\gamma-1}//m^t)\Omega_1 Q - Q\Omega_1(H_{\gamma-1}//\eta) - Q\Omega_1 Q \\
&= \begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} & \beta_{Z,n} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} & \beta_{n-1,n} \\ \beta_{n,Z} & \beta_{n,n-1} & \beta_{n,n} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{m_1} \\ \mathbf{0} & 0 & m_2 \\ 0 & 0 & m_3 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{w_1} & w_2 & w_3 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & \beta_{\rho(n,n)} \end{pmatrix} \\
&= \begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} & \beta_{Z,n} - \mathbf{m_1} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} & \beta_{n-1,n} - m_2 \\ \beta_{n,Z} - \mathbf{w_1} & \beta_{n,n-1} - w_2 & \beta_{n,n} - m_3 - w_3 - \beta_{\rho(n,n)} \end{pmatrix}.
\end{aligned} \tag{5.14}$$

But from Equation 5.9,

$$H^t_{\gamma-1} = \begin{pmatrix} S(\gamma)//m^t & \begin{matrix} \frac{1}{\gamma-1}\epsilon^t \\ \frac{1}{\gamma-1}\delta \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & 1 & 0 \\ -\phi & -\alpha & 1-\gamma \end{pmatrix} \tag{5.15}$$

and

$$H_{\gamma-1}//\eta = \begin{pmatrix} \mathbf{1} & \mathbf{0} & -\phi//\eta \\ \mathbf{0} & 1 & -\alpha//\eta \\ 0 & 0 & 1-\gamma//\eta \end{pmatrix} \begin{pmatrix} S(\gamma)//\eta & \begin{matrix} 0 \\ 0 \end{matrix} \\ \frac{1}{\gamma-1}\epsilon//\eta \quad \frac{1}{\gamma-1}\delta//\eta & 1 \end{pmatrix}. \tag{5.16}$$

Notice that

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & 1 & 0 \\ -\phi & -\alpha & 1-\gamma \end{pmatrix} \Omega_1 \begin{pmatrix} \mathbf{1} & \mathbf{0} & -\phi//\eta \\ \mathbf{0} & 1 & -\alpha//\eta \\ 0 & 0 & 1-\gamma//\eta \end{pmatrix} = \begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} & f_1 \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} & f_2 \\ f_3 & f_4 & f_5 \end{pmatrix},$$

where f_1, f_2, f_3 and f_4 are entries resulting from the matrix multiplication which can be ignored. It follows from Equation 5.14, Equation 5.15 and Equation 5.16 that

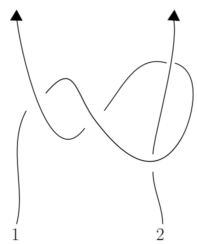
$$(S(\gamma)//m^t)\begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} \end{pmatrix}(S(\gamma)//\eta) = \begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} \end{pmatrix},$$

after the renaming $t_{\rho(n)}, t_n \mapsto t_{\rho(n)}$. That is, $S(\gamma)\, //\, t_{\rho(n)}, t_n \mapsto t_{\rho(n)}$ is unitary with respect to the matrices $\begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} \end{pmatrix} //\, t_{\rho(n)}, t_n \mapsto t_{\rho(n)}$ and $\begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} \end{pmatrix}$, which are the corresponding matrices for the intersection product on the spaces X_1 and X_0 respectively after stitching. This shows that the unitary condition is preserved after the stitching operation. We have proved the following theorem, which is an alternative prove of Theorem 5.18:

Theorem 5.20. *Let L be an $(n+1)$-string link whose strands are labeled by $T = 0, 1, \cdots, n$. Suppose L is the partial closure of an $(n+2)$ braid β. If the homology Gassner invariant $\mathcal{G}_h(\beta)$ of β is unitary with respect to the intersection products Ω_0 and Ω_1, then $\mathcal{G}_h(L)$ is also unitary with respect to the intersection products*

$$\begin{pmatrix} \beta_{\rho(Z,Z)} & \beta_{\rho(Z,n-1)} \\ \beta_{\rho(n-1,Z)} & \beta_{\rho(n-1,n-1)} \end{pmatrix} /\!/ \ t_{\rho(n)},\ t_n \mapsto t_{\rho(n)} \ and \begin{pmatrix} \beta_{Z,Z} & \beta_{Z,n-1} \\ \beta_{n-1,Z} & \beta_{n-1,n-1} \end{pmatrix}. \qquad \square$$

Example 5.21 (The homology Gassner invariant of L_4 is unitary)**.** In Example 4.6, we computed te homology Gassner invariant of the string link L_4, which is



$$\mathcal{G}_h^r(L_4) = \begin{pmatrix} \frac{t_2t_1-t_1-2t_2+1}{t_2t_1-t_1-t_2} & \frac{(t_2-1)t_2}{t_2t_1-t_1-t_2} \\ \frac{t_1-1}{t_2t_1-t_1-t_2} & -\frac{t_1}{t_2t_1-t_1-t_2} \end{pmatrix}.$$

The local efficient system here is $\mathcal{F} = \mathbb{Q}(t_0, t_1, t_2)$. Since the string link L_4 induces the identity permutation, then the matrices corresponding to the intersection products on X_0 and X_1 are the same:

$$\begin{pmatrix} \frac{(t_0-1)(t_1-1)(1-t_0t_1)}{t_0t_1} & -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_2} \\ -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_0t_1} & \frac{(t_0-1)(t_2-1)(1-t_0t_2)}{t_0t_2} \end{pmatrix}.$$

We have

$$(\mathcal{G}_h^r(L_4) /\!/ \ m^t) \begin{pmatrix} \frac{(t_0-1)(t_1-1)(1-t_0t_1)}{t_0t_1} & -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_2} \\ -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_0t_1} & \frac{(t_0-1)(t_2-1)(1-t_0t_2)}{t_0t_2} \end{pmatrix} (\mathcal{G}_h^r(L_4) /\!/ \ \eta) = \begin{pmatrix} \frac{(t_0-1)(t_1-1)(1-t_0t_1)}{t_0t_1} & -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_2} \\ -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_0t_1} & \frac{(t_0-1)(t_2-1)(1-t_0t_2)}{t_0t_2} \end{pmatrix}.$$

This shows that the homology Gassner invariant for the string link L_4 is unitary with respect to the matrix

$$\begin{pmatrix} \frac{(t_0-1)(t_1-1)(1-t_0t_1)}{t_0t_1} & -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_2} \\ -\frac{(t_0-1)(t_1-1)(t_2-1)}{t_0t_1} & \frac{(t_0-1)(t_2-1)(1-t_0t_2)}{t_0t_2} \end{pmatrix}.$$

5.6 A Mathematica implementation of the unitary property

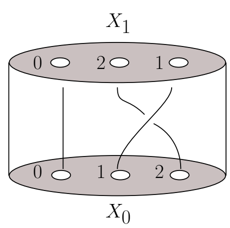
In this section, we implement the unitary condition using Mathematica. Referring to the notations from Section 3.5, we define a Mathematica function $\mu := \langle -, - \rangle$ for the intersection product on X_0 and X_1. This function takes two parameters, $h[T, L1]$ and $h[T, L2]$. We then test the unitary condition for the homology Gassner invariant of the braid β and the string link L_4 presented below. Note that when computing the intersection product, we do not ignore the strand labeled 0. A reader with Mathematica can get the notebook by clicking the following link: *GassnerInvariantMathematicaNotebook.nb*

The intersection product

In[62]:= $\mu[\mathrm{h}[T\_, L1\_], \mathrm{h}[T\_, L2\_]] := \mathrm{Factor}\Big[\mathrm{Expand}\big[L1\,(L2\ /.\ \{t_{i\_} \mapsto t_i^{-1}, \beta_{i\_} \mapsto \overline{\beta}_i\})\big]\ /.$

$$\left\{\beta_{i\_}\,\overline{\beta}_{j\_} \mapsto \begin{cases} \dfrac{(t_0-1)\left(t_{T[[i]]}-1\right)\left(1-t_0\,t_{T[[i]]}\right)}{t_0\,t_{T[[i]]}} & i == j \\[2ex] \dfrac{-(t_0-1)\left(t_{T[[i]]}-1\right)\left(t_{T[[j]]}-1\right)}{t_{T[[j]]}} & i < j \\[2ex] \dfrac{-(t_0-1)\left(t_{T[[i]]}-1\right)\left(t_{T[[j]]}-1\right)}{t_0\,t_{T[[j]]}} & i > j \end{cases}\right\}\Big]$$

Example of intersection product on X_0 and X_1



Here, we compute the matrices of intersection products on X_0 and X_1 corresponding to the braid generator σ_1. The matrices are not equal since the induced permutation is not the identity permutation.

In[234]:= $\sigma\mu_{X_0}$ = Table[μ[h[{1, 2}, β_i], h[{1, 2}, β_j]], {i, 1, 2}, {j, 1, 2}];
$\sigma\mu_{X_1}$ = Table[μ[h[{2, 1}, β_i], h[{2, 1}, β_j]], {i, 1, 2}, {j, 1, 2}];

In[236]:= {$\sigma\mu_{X_0}$ // MatrixForm, $\sigma\mu_{X_1}$ // MatrixForm}

Out[236]= $$\left\{\begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0\,t_1)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_2} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0\,t_2)}{t_0\,t_2} \end{pmatrix}, \begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0\,t_2)}{t_0\,t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_1} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0\,t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0\,t_1)}{t_0\,t_1} \end{pmatrix}\right\}$$

Here, we test the unitary condition for the homology Gassner invariant of the braid generators σ_1 and σ_1^{-1} using matrices.

In[245]:= $\sigma\mu_{X_0}$ == Transpose[M_1[{1, 2}]].$\sigma\mu_{X_1}$.(M_1[{1, 2}] /. $t_{i\_} \mapsto t_i^{-1}$) // Simplify

Out[245]= True

In[246]:= $\sigma\mu_{X_0}$ == Transpose[$\overline{M}_1$[{1, 2}]].$\sigma\mu_{X_1}$.($\overline{M}_1$[{1, 2}] /. $t_{i\_} \mapsto t_i^{-1}$) // Simplify

Out[246]= True

Unitary property for $\mathcal{G}_h^r(\sigma_k), k = 1, 2, 3$

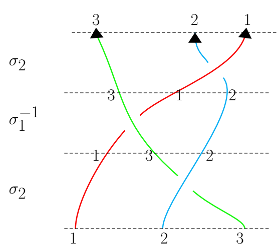
Here, we test the unitary condition for the homology Gassner invariant of the braid generators $\sigma_k, k = 1, 2, 3$. Note that they all evaluate to true.

```
In[124]:= Table[
            Table[
              μ[h[{1, 2, 3, 4}, βi] // σk, h[{1, 2, 3, 4}, βj] // σk] == μ[h[{1, 2, 3}, βi], h[{1, 2, 3}, βj]],
              {i, 1, 3}, {j, 1, 3}
            ] // Simplify // MatrixForm,
            {k, 3}
          ]
```

Out[124]= $\left\{\begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}, \begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}, \begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}\right\}$

Unitary property for $\mathcal{G}_h^r(\beta)$

Here, we compute the matrices of intersection products on X_0 and X_1



$\beta = \sigma_2\sigma_1^{-1}\sigma_2$

corresponding to the braid β. Note that the matrices are not equal since the induced permutation is not the identity permutation. We then test the unitary condition for the homology Gassner invariant of β.

```
In[71]:= MX0 = Table[μ[h[{1, 2, 3}, βi], h[{1, 2, 3}, βj]], {i, 1, 3}, {j, 1, 3}];
         MX1 = Table[μ[h[{3, 2, 1}, βi], h[{3, 2, 1}, βj]], {i, 1, 3}, {j, 1, 3}];

In[90]:= MX0 // MatrixForm
```

Out[90]//MatrixForm=

$$\begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0 t_1)}{t_0 t_1} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_3)}{t_3} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0 t_1} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0 t_2)}{t_0 t_2} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_3)}{t_3} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_3)}{t_0 t_1} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_3)}{t_0 t_2} & -\frac{(-1+t_0)\times(-1+t_3)\times(-1+t_0 t_3)}{t_0 t_3} \end{pmatrix}$$

```
In[91]:= MX1 // MatrixForm
```

Out[91]//MatrixForm=

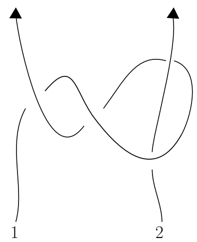
$$\begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_3)\times(-1+t_0 t_3)}{t_0 t_3} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_3)}{t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_3)}{t_1} \\ -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_3)}{t_0 t_3} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0 t_2)}{t_0 t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_1} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_3)}{t_0 t_3} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0 t_2} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0 t_1)}{t_0 t_1} \end{pmatrix}$$

```
In[74]:= Mβ = M2[{3, 1, 2}].M̄1[{1, 3, 2}].M2[{1, 2, 3}] // Simplify;

In[75]:= MX0 == Transpose[Mβ].MX1.(Mβ /. ti_ :> ti^-1) // Simplify

Out[75]= True
```

Unitary property for $\mathcal{G}_h^r(L_4)$



Here, we compute the matrices of intersection products on X_0 and X_1 corresponding to the braid L_4. Note that the matrices are not equal since the induced permutation is not the identity permutation. We then test the unitary condition for the homology Gassner invariant of L_4.

In[254]:= $L\mu_{X_0}$ = Table[μ[h[{1, 2}, β_i], h[{1, 2}, β_j]], {i, 1, 2}, {j, 1, 2}];
$L\mu_{X_1}$ = Table[μ[h[{1, 2}, β_i], h[{1, 2}, β_j]], {i, 1, 2}, {j, 1, 2}];

In[256]:= {$L\mu_{X_0}$ // MatrixForm, $L\mu_{X_1}$ // MatrixForm}

$$\text{Out[256]=}\ \left\{\begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0\,t_1)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_2} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0\,t_2)}{t_0\,t_2} \end{pmatrix},\ \begin{pmatrix} -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_0\,t_1)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_2} \\ -\frac{(-1+t_0)\times(-1+t_1)\times(-1+t_2)}{t_0\,t_1} & -\frac{(-1+t_0)\times(-1+t_2)\times(-1+t_0\,t_2)}{t_0\,t_2} \end{pmatrix}\right\}$$

$$\text{In[257]:=}\ M_{L_4} = \begin{pmatrix} \frac{t_2\,t_1-t_1-2\,t_2+1}{t_1\,t_2-t_1-t_2} & \frac{t_2\,(t_2-1)}{t_1\,t_2-t_1-t_2} \\ \frac{t_1-1}{t_1\,t_2-t_1-t_2} & -\frac{t_1}{t_1\,t_2-t_1-t_2} \end{pmatrix};$$

In[258]:= $L\mu_{X_0}$ == Transpose[M_{L_4}].$L\mu_{X_1}$.(M_{L_4} /. $t_{i\_} \mapsto t_i^{-1}$) // Simplify

Out[258]= True

Chapter 6

Concluding Remarks

The research presented in this thesis has made substantial contribution to knot theory, specifically to the understanding of the Gassner invariant for string links and braids, and the verification of the unitary condition. The key results can be summarized as follows:

1. Utilizing the (co)homological approach presented in [KLW01], we have derived matrices (formulas)

$$\mathcal{G}_h(\sigma_i) = \begin{pmatrix} 0 & t_{T[i]} \\ 1 & 1 - t_{T[i+1]} \end{pmatrix} \text{ and } \mathcal{G}_c(\sigma_i^{-1}) = \begin{pmatrix} \frac{t_{T[i+1]}-1}{t_{T[i]}} & 1 \\ \frac{1}{t_{T[i]}} & 0 \end{pmatrix},$$

   for both the homology and cohomology Gassner invariants base on the topological properties of string links and braids. These matrices can be derived from each other by taking the inverse transpose in appropriate basis, offering a wider perspective on the Gassner invariant and broadening its potential applications.

2. We have introduced the concept of "flying cars", which assigns an invariant $\mathcal{C}(L)$ to a labeled $(n+1)$-component string link L. We have also established a connection between the homology Gassner invariant and of flying cars:

$$\begin{tikzcd} & SL_n \arrow[dl, "\mathcal{C}"'] \arrow[dr, "\mathcal{G}_h^r"] & \\ Mat_n(\mathcal{F}') \arrow[rr, "D_n(-)D_n^{-1}//\rho_{col}//m^t"'] & & Mat_n(\mathcal{F}') \end{tikzcd}$$

   In [BNb] and [BNS13], the author introduces a tangle invariant known as Γ-Calculus, which is further discussed in [Hal16] and [Vo18]. This invariant applies to both string links (tangles) and w-tangles (a more generalized form of string links) and is represented as an $n \times n$ matrix. Interestingly, the matrices derived from flying cars are transposes of those obtained from Γ-Calculus. This establishes a connection between the homology Gassner invariant and Γ-Calculus through their respective relationships with flying cars.

3. Additionally, this thesis provides formulas for the intersection product

$$\mu := \langle -,- \rangle : H_1(X_j;\mathcal{F}) \times H_1(X_j;\mathcal{F}) \to \mathcal{F},$$

defined on the cycles of the homology group $H_1(X_j;\mathcal{F})$ for $j = 0, 1$. The formulas are:

$$\langle \widetilde{\beta_i}, \widetilde{\beta_j} \rangle = \begin{cases} \frac{(t_0-1)(t_{T[i]}-1)(1-t_0 t_{T[i]})}{t_0 t_{T[i]}}, & i = j \qquad (self-intersection) \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_{T[j]}}, & i < j \\ -\frac{(t_0-1)(t_{T[i]}-1)(t_{T[j]}-1)}{t_0 t_{T[j]}}, & i > j \end{cases},$$

Recall that in [Abd97] and [BN14], the authors explicitly define different Hermitian matrices to prove the unitary condition, but they do not provide details on how these matrices were derived. In contrast, we have not only provided formulas that define Hermitian matrices, but we have also demonstrated their natural origins. The cup product offers an even more natural method to obtain a Hermitian matrix, presenting an area for further exploration.

4. Furthermore, we have confirmed that the homology Gassner invariant is unitary with respect to the Hermitian matrices derived from the intersection product. This verification opens new avenues for exploring the analytic properties of the Gassner invariant.

In conclusion, the findings of this research not only enhance our understanding of the Gassner invariant and its computation, but also offer the flexibility to work with either the homology or cohomology Gassner invariant. The provided formulas facilitate the use of computer programs to simplify computations. The insights gained from this thesis hold the potential to inspire and guide future advancements in the field.

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