

Course Idea: 12 Definitions of the Alexander Polynomial.

1. The "cars" matrix: knots, R-moves, rotation numbers, proof of invariance.
2. π_1 and fox derivatives.
3. H_1 .
4. Seifert.
5. Burau.
6. The Moscow definition.
7. Conway.
8. Meta-monoids.
9. w .
10. Kontsevich.
11. Hopf/Kerler.
12. Surprise!

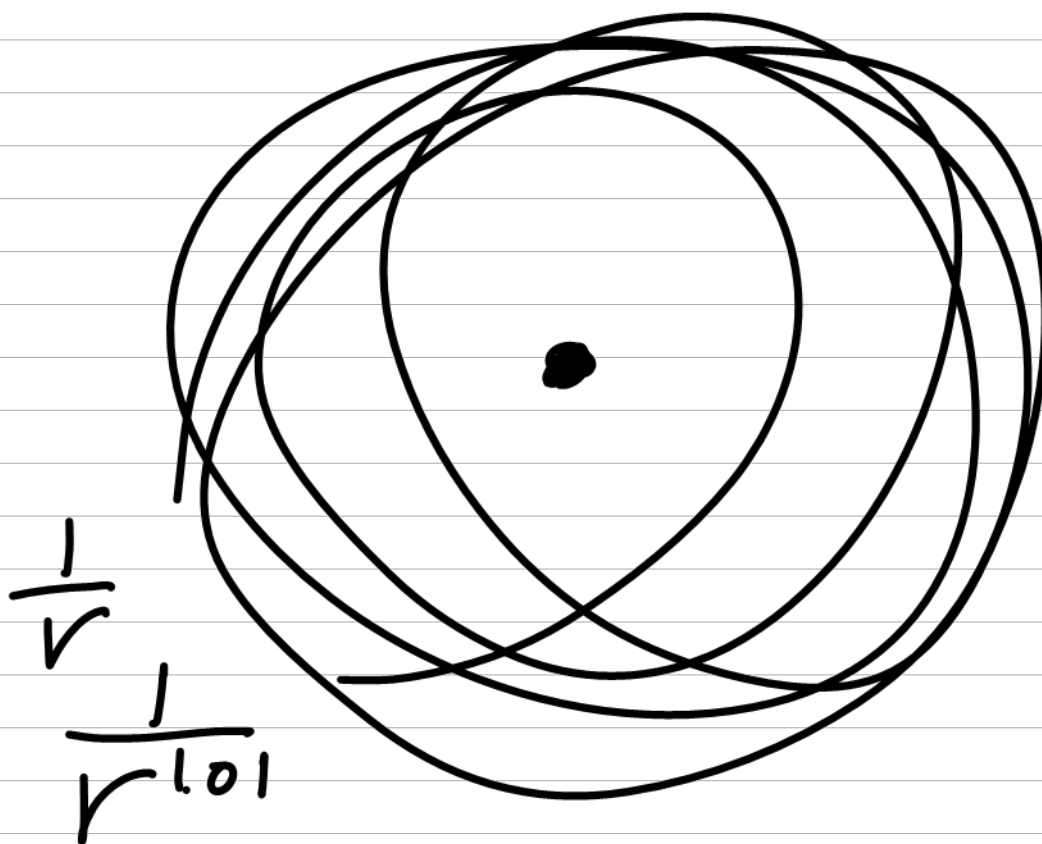
In math, we often care for things for how well-connected they are. In themselves, 57 and 1729 and 196884 are just members of an infinitely long dull and monotone procession of "numbers". Yet the Theory of Numbers talks to nearly everything in mathematics, and everything talks to it. Knots are likewise dull, and Knot Theory is likewise interesting, and within Knot Theory the Alexander Polynomial plays a special role: it is arguably the most successful "Knot Invariant", it talks to everything, and everything talks to it.

In this class we will cover 12 of the 50 or so definitions of the Alexander Polynomial and discuss how they are related to formal probability and determinants, fundamental groups, and Fox derivatives, homology, covering spaces and finitely presented modules, Seifert surfaces and linking forms, braids and their Burau representation, exterior algebras and the Berezin integral, skein relations, tangles and meta-monoids, finite type invariants and the Kontsevich integral, 2-knots in 4D and the w -expansion and Hopf algebras and algebraic knot theory.

We will aim to implement in Mathematica almost everything that we will talk about, so this class is also about turning sophisticated mathematics into concise and effective code.

Prerequisites: Excellent grasp of everything in Core Algebra I (MAT1100) and in Core Topology (MAT1301), and no fear of computers.

Evaluation: Near-weekly problem sets, a possible full day of student lectures at the end.



240122 ■: With Z a domain and Q its field of fractions, given $\partial: R \rightarrow G$ invertible over Q and $(\cdot, \cdot): R \otimes G \rightarrow Z$ with $(r_1, \partial r_2) = (r_2, \partial r_1)$ get a symmetric $\langle \cdot, \cdot \rangle: (\text{coker}_Z \partial)^{\otimes 2} \rightarrow Q/Z$ by $\langle \bar{g}_1, \bar{g}_2 \rangle := (\partial_Q^{-1} g_1, g_2)/Z$. **Q.** When do two presentations yield equivalent $\langle \cdot, \cdot \rangle$'s?

$$\text{Indeed, } \langle g_1 + \partial r, g_2 \rangle = (\partial^{-1} g_1 + \partial^{-1} \partial r, g_2) = (\partial^{-1} g_1, g_2) + (\cancel{\partial^{-1} \partial r}, g_2) \in Z$$

$$\langle g_1, g_2 + \partial r \rangle = (\partial^{-1} g_1, g_2) + (\partial^{-1} g_1, \partial r) = (\partial^{-1} g_1, g_2) + (r, \partial \partial^{-1} g_1) \in Z$$

$$\begin{aligned} \text{Symmetry: } \langle g_2, g_1 \rangle &= (\partial^{-1} g_2, g_1) = \\ &= (\partial^{-1} g_2, \partial \partial^{-1} g_1) = (\partial^{-1} g_1, \partial \partial^{-1} g_2) \\ &= (\partial^{-1} g_1, g_2) = \langle g_1, g_2 \rangle \end{aligned}$$

$\Rightarrow$ well-definedness

In Borodzik - Friedl "The Unknoting Number and Classical Invariants", arXiv:1203.3225, page 20:

Proposition 3.1 ([Ran81, Proposition 1.7.1]). *Let A and B be hermitian matrices over $\mathbb{Z}[t^{\pm 1}]$ with $\det(A(1)) = \det(B(1)) = \pm 1$. Then $\lambda(A) \cong \lambda(B)$ if and only if A and B are related by a sequence of the following three moves:*

- (1) replace C by $PC\bar{P}^t$ where P is a matrix over $\mathbb{Z}[t^{\pm 1}]$ with $\det(P) = \pm 1$,
- (2) replace C by the block sum $C \oplus D$ where D is a hermitian matrix over $\mathbb{Z}[t^{\pm 1}]$ with $\det(D) = \pm 1$,
- (3) the inverse of (2).

We can now prove the following lemma:

Lemma 3.2. *Let $A(t)$ be a hermitian matrix over $\mathbb{Z}[t^{\pm 1}]$ with $\lambda(A(t)) \cong \lambda(K)$, then*

$$\begin{aligned} \text{sign}(A(z)) - \text{sign}(A(1)) &= \sigma_z(K), \quad \text{for any } z \in S^1 \text{ and} \\ \text{null}(A(z)) &= \eta_z(K), \quad \text{for any } z \in \mathbb{C} \setminus \{0, 1\}. \end{aligned}$$

Proof. First let $D(t)$ be any hermitian matrix over $\mathbb{Z}[t^{\pm 1}]$. It is well-known that the function

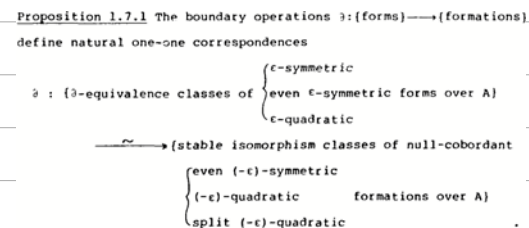
$$\begin{aligned} S^1 &\rightarrow \mathbb{Z} \\ z &\mapsto \text{sign}(D(z)) \end{aligned}$$

is constant outside of the set of zeros of $D(t)$. In particular if $\det(D(t)) = \pm 1$, then the signature function is constant. It now follows easily from Proposition 2.1 that if $A(t)$ and $B(t)$ are hermitian matrices over $\mathbb{Z}[t^{\pm 1}]$ with $\lambda(A(t)) \cong \lambda(B(t))$, then

$$\text{sign}(A(z)) - \text{sign}(A(1)) = \text{sign}(B(z)) - \text{sign}(B(1)) \text{ for any } z \in S^1.$$

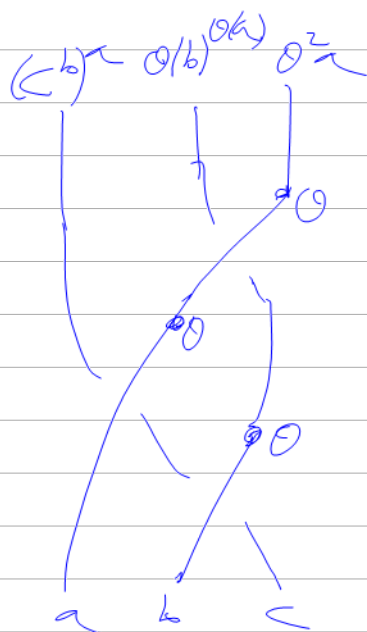
The first claim now follows from (2.4) and Proposition 2.1. The proof of the second statement also follows from a similar argument. $\square$

[Ran81, Prop 1.7.1] is Ranicki's book, page 90. A touch read:



Bloomington questions.

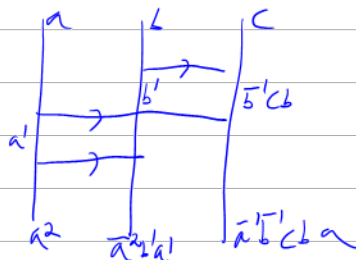
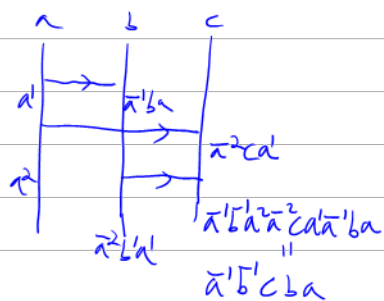
- * A Wirtinger perspective on why is the Alexander polynomial palindromic?
- * A Wirtinger signature formula?
- * Milnor signatures and their relation with Tristram-Lewine.
- * Can you go directly from the Blanchfield pairing to the T2 signatures?
- * How best to compute the Blanchfield pairing?



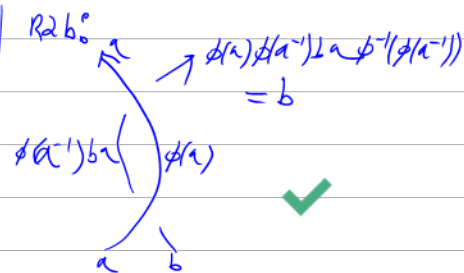
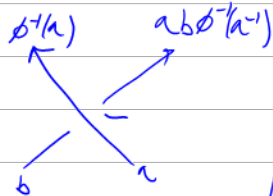
$$\phi \in \text{Aut}(C)$$

$$G \times \mathbb{Z} \quad (a_k)(b, l) = (a^{b^k}, k+l)$$

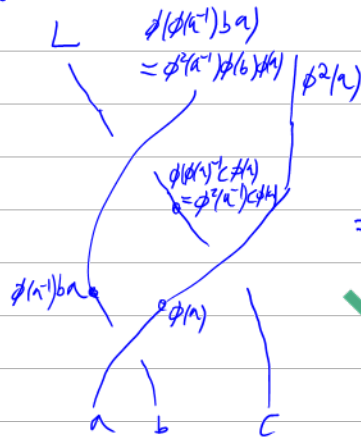
$$\overline{(a_k)} = (\bar{a}^k, -k)$$



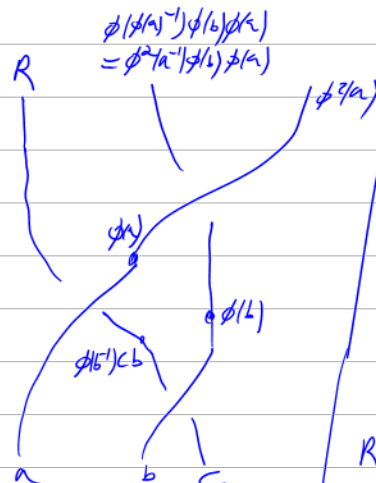
If G is a group and $\phi: G \rightarrow G$ is an automorphism of G , we can define a representation of the braid group B_n on G^n by



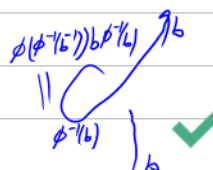
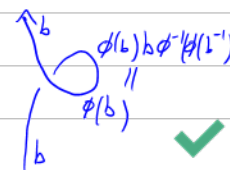
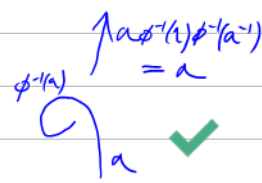
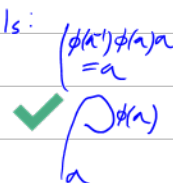
R3:



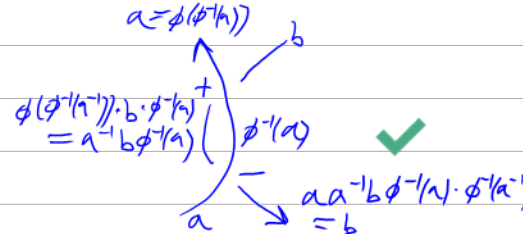
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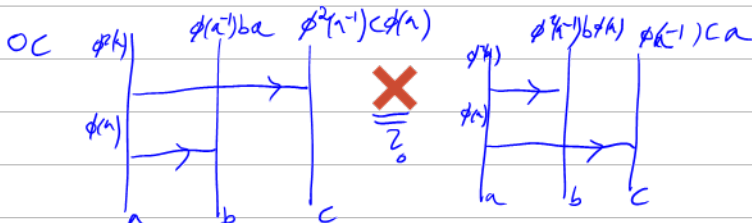
R1s:



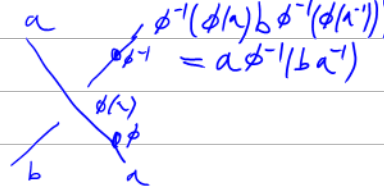
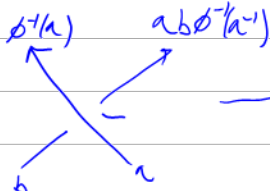
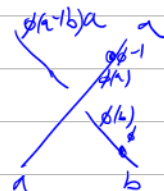
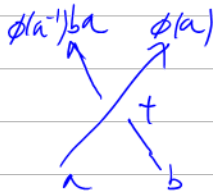
R2c:



$$\begin{aligned} L &= \phi(\phi(a^{-1})ba)^{-1} \cdot \phi^2(a^{-1})c\phi(a) \cdot \phi(a^{-1})ba \\ &= \phi(a^{-1}b^{-1}\phi(a)) \cdot \phi^2(a^{-1})cba = \phi(a^{-1})\phi(b^{-1})\phi(a)\phi^2(a^{-1})cba \\ &= \phi(a^{-1})\phi(b^{-1})cba = R \end{aligned}$$



conjugating by ϕ^v : (v : Alexander number)



Matches on $(G, -1)$

$$GX_{\phi} \mathbb{Z} : (a, k) \cdot (b, l) = (a\phi^k(b), k+l)$$

$$(a, k)^{-1} = (\phi^{-k}(a), -k)$$

$$\begin{aligned} (b, l)^{(a, k)} &= (a, k)^{-1}(b, l)(a, k) \\ &= (\phi^{-k}(a), -k)(b, l)(a, k) \\ &= (\phi^{-k}(a^{-1}b\phi^l(a)), l) \end{aligned}$$

$$\begin{aligned} (b, l)^{(a, k)^{-1}} &= (a, k)(b, l)(\phi^k(a^{-1}), -k) \\ &= (a\phi^k(b)\phi^l(a^{-1}), l) \end{aligned}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad v=y$$

$$u = ax + by$$

$$v = cx + dy$$

$$v = y$$

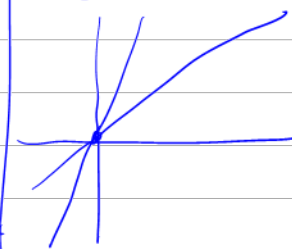
$$u = ax + by$$

$$\Rightarrow 0 = cx + (d-1)y \Rightarrow u = \left(a + \frac{bc}{1-d}\right)x$$

$$y = \frac{cx}{1-d}$$

$$y = ax$$

$$y = x$$

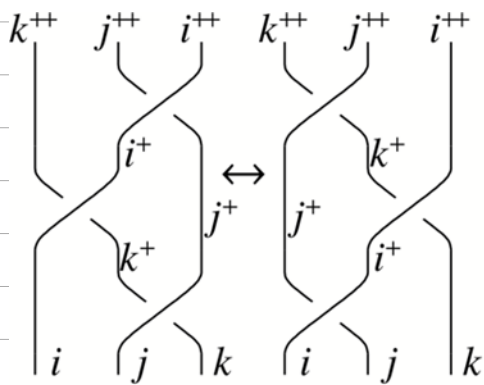


$$y = ax + b$$

$$y = x$$

$$0 = (a-1)x + b$$

$$x = \frac{b}{1-a}$$



$$\text{lhs} = R_\theta[1, j, k] R_\theta[1, i, k^+] R_\theta[1, i^+, j^+]$$

$$\text{EQP} \left[-p_i x_i + (1 - T) p_{k^{++}} x_i + T p_{i^-} x_i - p_j x_j + T p_{j^-} x_j + (1 - T) p_{k^+} x_j - p_k x_k + p_{k^+} x_k + T p_{i^{++}} x_{i^+} + (1 - T) p_{j^{++}} x_{i^+} - p_{i^+} x_{i^+} + p_{j^{++}} x_{j^+} - p_{j^+} x_{j^+} + p_{k^{++}} x_{k^+} - p_{k^+} x_{k^+}, \theta \right]$$

$$\text{rhs} = R_\theta[1, i, j] R_\theta[1, i^+, k] R_\theta[1, j^+, k^+]$$

$$\text{EQP} \left[-p_i x_i + T p_{i^-} x_i + (1 - T) p_{j^+} x_i - p_j x_j + p_{j^+} x_j - p_k x_k + p_{k^+} x_k + T p_{i^{++}} x_{i^+} - p_{i^+} x_{i^+} + (1 - T) p_{k^+} x_{i^+} + T p_{j^{++}} x_{j^+} + (1 - T) p_{k^{++}} x_{j^+} - p_{j^+} x_{j^+} + p_{k^{++}} x_{k^+} - p_{k^+} x_{k^+}, \theta \right]$$

Series[$e^{\beta} e^{n \epsilon} X_1$, { ϵ , 0, 4}]

$$e^{\beta X_1} + e^{\beta X_1} n \beta X_1 \epsilon + \frac{1}{2} e^{\beta X_1} n^2 \beta X_1 (1 + \beta X_1) \epsilon^2 +$$

$$\frac{1}{6} e^{\beta X_1} n^3 \beta X_1 (1 + 3 \beta X_1 + \beta^2 X_1^2) \epsilon^3 +$$

$$\frac{1}{24} e^{\beta X_1} n^4 \beta X_1 (1 + 7 \beta X_1 + 6 \beta^2 X_1^2 + \beta^3 X_1^3) \epsilon^4 + O[\epsilon]^5$$

Series[$\frac{\alpha^n}{n!} e^{\beta} e^{n \epsilon} X_1 X_2^n$, { ϵ , 0, 4}]

$$\frac{e^{\beta X_1} \alpha^n X_2^n}{n!} + \frac{e^{\beta X_1} n \alpha^n \beta X_1 X_2^n \epsilon}{n!} + \frac{e^{\beta X_1} n^2 \alpha^n \beta X_1 (1 + \beta X_1) X_2^n \epsilon^2}{2 n!} +$$

$$\frac{e^{\beta X_1} n^3 \alpha^n \beta X_1 (1 + 3 \beta X_1 + \beta^2 X_1^2) X_2^n \epsilon^3}{6 n!} +$$

$$\frac{e^{\beta X_1} n^4 \alpha^n \beta X_1 (1 + 7 \beta X_1 + 6 \beta^2 X_1^2 + \beta^3 X_1^3) X_2^n \epsilon^4}{24 n!} + O[\epsilon]^5$$

Collect[Sum[Normal@Series[$\frac{\alpha^n}{n!} e^{\beta} e^{n \epsilon} X_1 X_2^n$, { ϵ , 0, 4}], {n, 0, ∞ }],
 ϵ , Simplify]

$$e^{\beta X_1 + \alpha X_2} + e^{\beta X_1 + \alpha X_2} \alpha \beta \epsilon X_1 X_2 +$$

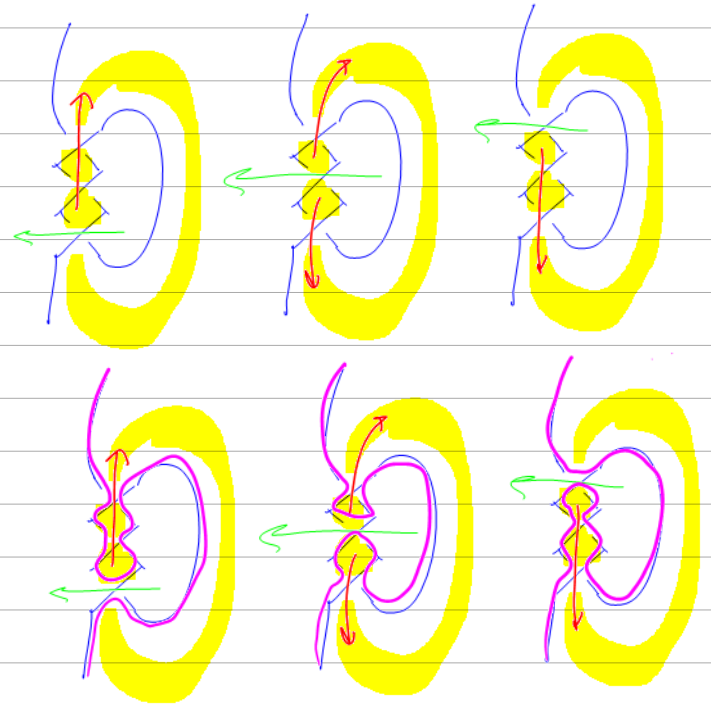
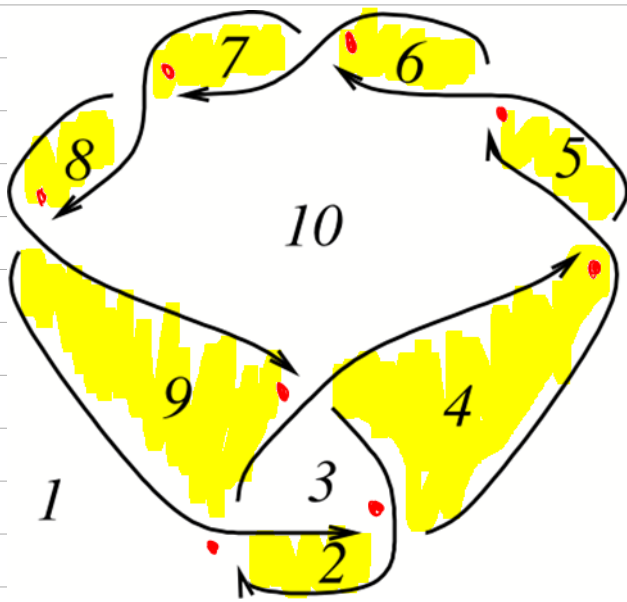
$$\frac{1}{2} e^{\beta X_1 + \alpha X_2} \alpha \beta \epsilon^2 X_1 (1 + \beta X_1) X_2 (1 + \alpha X_2) +$$

$$\frac{1}{6} e^{\beta X_1 + \alpha X_2} \alpha \beta \epsilon^3 X_1 (1 + 3 \beta X_1 + \beta^2 X_1^2) X_2 (1 + 3 \alpha X_2 + \alpha^2 X_2^2) +$$

$$\frac{1}{24} e^{\beta X_1 + \alpha X_2} \alpha \beta \epsilon^4 X_1 (1 + 7 \beta X_1 + 6 \beta^2 X_1^2 + \beta^3 X_1^3) X_2 (1 + 7 \alpha X_2 + 6 \alpha^2 X_2^2 + \alpha^3 X_2^3)$$

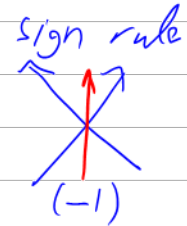
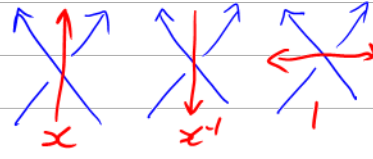
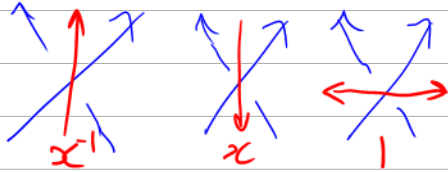
$$\begin{aligned} e^{\alpha X_2} e^{\beta X_1} &= \sum_{n,m} \frac{\alpha^n \beta^m}{n! m!} X_2^n X_1^m \\ &= \sum_{n,m} \frac{\alpha^n \beta^m}{n! m!} q^{nm} X_1^m X_2^n \\ &= \sum_n \frac{\alpha^n}{n!} e^{\beta q^n X_1} X_2^n \\ &= \sum_n \frac{\alpha^n}{n!} e^{\beta (1 + \epsilon n + \frac{\epsilon^2 n^2}{2} + \dots) X_1} X_2^n \end{aligned}$$

See <https://oeis.org/A008277>, "Stirling numbers of the second kind"

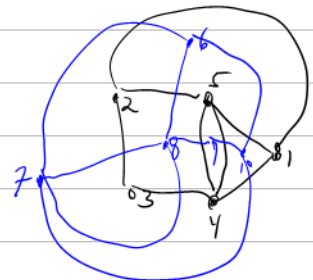
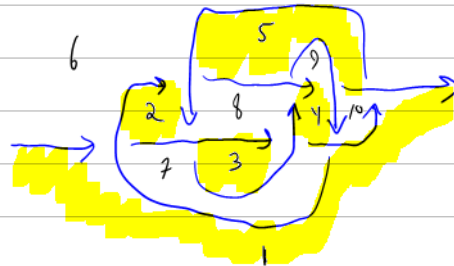
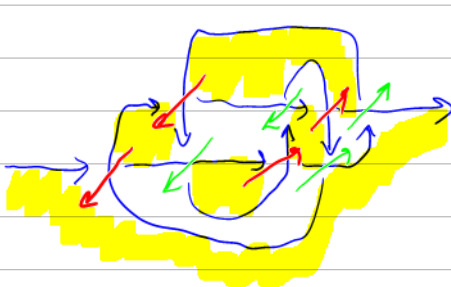
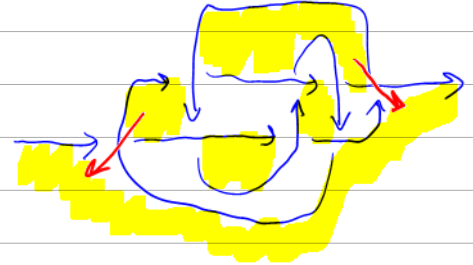
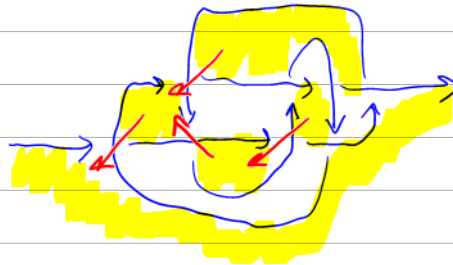
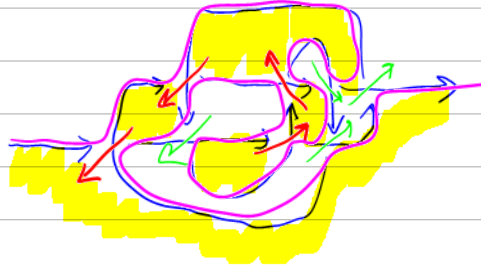
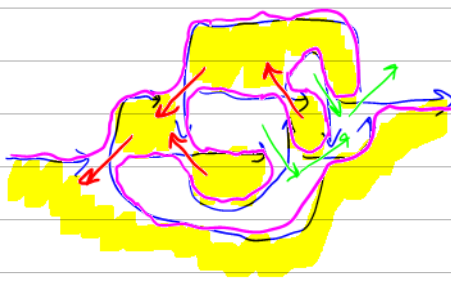


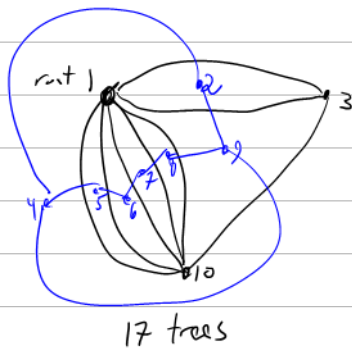
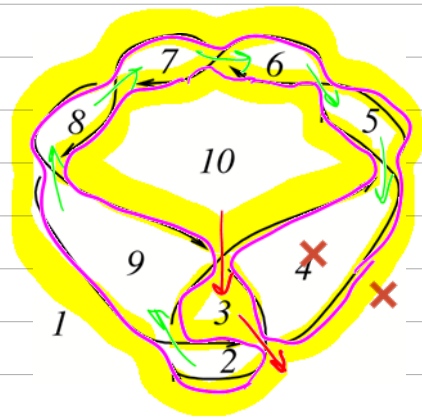
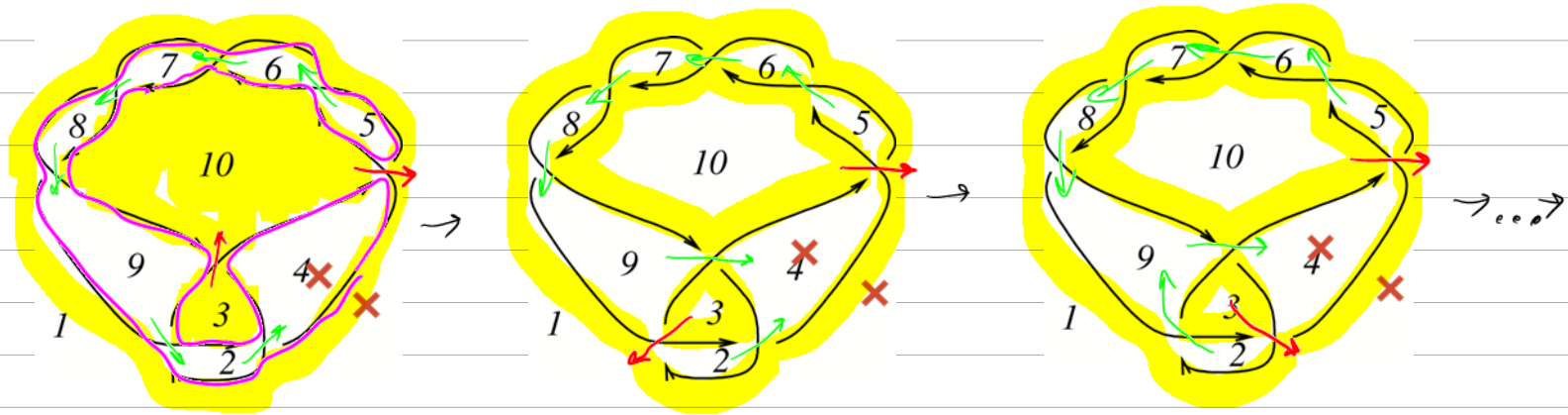
$\{\text{Kauffman styles}\} \longleftrightarrow \{\text{Tree-Dual-Tree pairs in the checkerboard graph}\} \longleftrightarrow \{\text{one long path smoothing}\}$

weight rules:

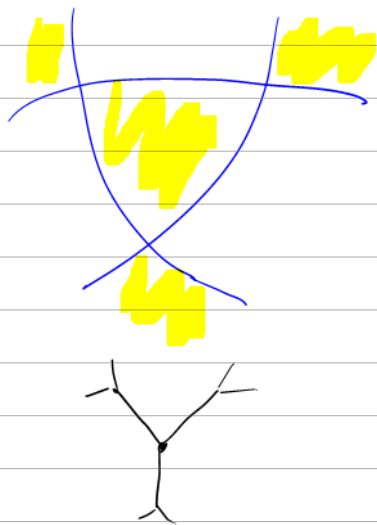


8/7:

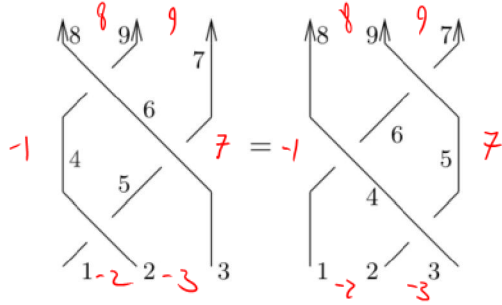




$$7 + 2 \cdot 5 = 17$$
 trees



Reidemeister 3



$$\text{In[*]} := \text{lhs} = \text{KSI}[\text{PD}[\mathbf{X}_{-2,5,4,-1}, \mathbf{X}_{-3,7,6,-5}, \mathbf{X}_{-6,9,8,-4}]] /. m \rightarrow -1$$

Out[*] =

$$\begin{aligned} & \text{GB}[\{-3, 7, 9, 8, -1, -2\}] \left[\right. \\ & - \frac{Y[]_{-1} Y[]_7 Y[-3]_9 Y[-2]_8}{T^3} - \frac{Y[]_{-3} Y[]_8 Y[-2]_7 Y[-1]_9}{T} + T Y[]_{-3} Y[]_8 Y[-1]_9 Y[7]_{-2} \\ & - \frac{Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[7]_8}{T} - \frac{Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[7]_8}{T} - T Y[]_{-1} Y[]_7 Y[-3]_9 Y[8]_{-2} + \\ & T Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[8]_7 + T Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[8]_7 + \frac{Y[]_{-1} Y[]_7 Y[-2]_8 Y[9]_{-3}}{T} + \\ & - \frac{T^3 Y[]_{-1} Y[]_7 Y[8]_{-2} Y[9]_{-3}}{T} - \frac{Y[]_{-3} Y[]_8 Y[-2]_7 Y[9]_{-1}}{T} + T Y[]_{-3} Y[]_8 Y[7]_{-2} Y[9]_{-1} + \\ & - \frac{Y[]_{-2} Y[]_7 Y[]_8 Y[-3, -1]_9}{T^2} - \frac{Y[]_{-2} Y[]_7 Y[]_8 Y[-3, 9]_{-1}}{T} + \frac{Y[]_{-3} Y[]_{-1} Y[]_9 Y[-2, 7]_8}{T^2} - \\ & - \frac{Y[]_{-3} Y[]_{-1} Y[]_9 Y[-2, 8]_7}{T^2} + T^2 Y[]_{-2} Y[]_7 Y[]_8 Y[-1, 9]_{-3} + T^2 Y[]_{-3} Y[]_{-1} Y[]_9 Y[7, 8]_{-2} \left. \right] \end{aligned}$$

$$\text{In[*]} := \text{rhs} = \text{KSI}[\text{PD}[\mathbf{X}_{-3,5,4,-2}, \mathbf{X}_{-4,6,8,-1}, \mathbf{X}_{-5,7,9,-6}]] /. m \rightarrow -1$$

Out[*] =

$$\begin{aligned} & \text{GB}[\{-3, 7, 9, 8, -1, -2\}] \left[\right. \\ & - \frac{Y[]_{-1} Y[]_7 Y[-3]_9 Y[-2]_8}{T^3} - \frac{Y[]_{-3} Y[]_8 Y[-2]_7 Y[-1]_9}{T} - \frac{Y[]_{-3} Y[]_8 Y[-1]_9 Y[7]_{-2}}{T} - \\ & - \frac{Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[7]_8}{T} + T Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[7]_8 + \frac{Y[]_{-1} Y[]_7 Y[-3]_9 Y[8]_{-2}}{T} - \\ & - \frac{Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[8]_7}{T} + T Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[8]_7 - T Y[]_{-1} Y[]_7 Y[-2]_8 Y[9]_{-3} + \\ & - \frac{T^3 Y[]_{-1} Y[]_7 Y[8]_{-2} Y[9]_{-3}}{T} + T Y[]_{-3} Y[]_8 Y[-2]_7 Y[9]_{-1} + T Y[]_{-3} Y[]_8 Y[7]_{-2} Y[9]_{-1} + \\ & - \frac{Y[]_{-2} Y[]_7 Y[]_8 Y[-3, -1]_9}{T^2} - \frac{Y[]_{-2} Y[]_7 Y[]_8 Y[-3, 9]_{-1}}{T} + \frac{Y[]_{-3} Y[]_{-1} Y[]_9 Y[-2, 7]_8}{T^2} - \\ & - \frac{Y[]_{-3} Y[]_{-1} Y[]_9 Y[-2, 8]_7}{T^2} + T^2 Y[]_{-2} Y[]_7 Y[]_8 Y[-1, 9]_{-3} + T^2 Y[]_{-3} Y[]_{-1} Y[]_9 Y[7, 8]_{-2} \left. \right] \end{aligned}$$

In[]:=* **lhs[[1]] - rhs[[1]]**

Out[]=*

$$\begin{aligned}
 & \frac{Y[]_{-3} Y[]_8 Y[-1]_9 Y[7]_{-2}}{T} + T Y[]_{-3} Y[]_8 Y[-1]_9 Y[7]_{-2} - \frac{Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[7]_8}{T} - \\
 & T Y[]_{-2} Y[]_9 Y[-1]_{-3} Y[7]_8 - \frac{Y[]_{-1} Y[]_7 Y[-3]_9 Y[8]_{-2}}{T} - T Y[]_{-1} Y[]_7 Y[-3]_9 Y[8]_{-2} + \\
 & \frac{Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[8]_7}{T} + T Y[]_{-2} Y[]_9 Y[-3]_{-1} Y[8]_7 + \frac{Y[]_{-1} Y[]_7 Y[-2]_8 Y[9]_{-3}}{T} + \\
 & T Y[]_{-1} Y[]_7 Y[-2]_8 Y[9]_{-3} - \frac{Y[]_{-3} Y[]_8 Y[-2]_7 Y[9]_{-1}}{T} - T Y[]_{-3} Y[]_8 Y[-2]_7 Y[9]_{-1}
 \end{aligned}$$

chim talk?
COMX talk?

- * Multi-variable $\mathcal{P}_1$?
- * understand Roland's "n" program?
- * writeup of DOPEADO?
- * Kauffman-state homology?
- * Writeup of palindromicity?
- * Writeup of pushforwards?

China talk: "I-Type knot Invariants"

Abstract: An I-Type knot invariant is a knot inv. obtained by integrating the exponential of a Lagrangian, defined locally from a knot diagram D , over a product of spaces locally associated to the features of D . Q&A: Are there any? (Yes)

Are they any good? (They are the strongest w/ CPU cycle, and there are reasons to hope for more).
Integration? (squicky chain algebra)

chapters.

- * Elucidation of the definition.

- * Example: Alexander

- * What's integration? * 6

- * Implementation.

- * Example: $\mathcal{P}_1(T)$ * Example: $\mathcal{P}_2(T)$

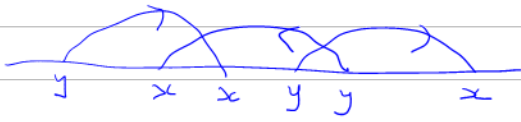
- * What's integration? * PG

- * Implementation.

- * Example: $\mathcal{P}_1(U, V)$

- * Implementation

- * Origins & prospects.



$$am_k^{is} = \exp[(\alpha_i + \alpha_j')x_k + (e^{-\alpha_i \xi_i' + \xi_j'}x_k)]$$

G a group, $G \stackrel{\rho_1}{\curvearrowright} V_1$, $G \stackrel{\rho_2}{\curvearrowright} V_2$ representations,

$\gamma: G \rightarrow V_1^* \otimes V_2$ is a $V_1 \rightarrow V_2$ cross-representation "glow" if

$$\gamma(g_1 g_2) = \gamma(g_1) \rho_2(g_2) + \rho_1(g_1) \gamma(g_2) \quad \begin{array}{|c|c|} \hline g_1 & g_2 \\ \hline \end{array}$$

Let $\mu: G \rightarrow V_1^* \otimes V_2$ be given by $\mu(g) = \rho_1(g)^{-1} \gamma(g)$.

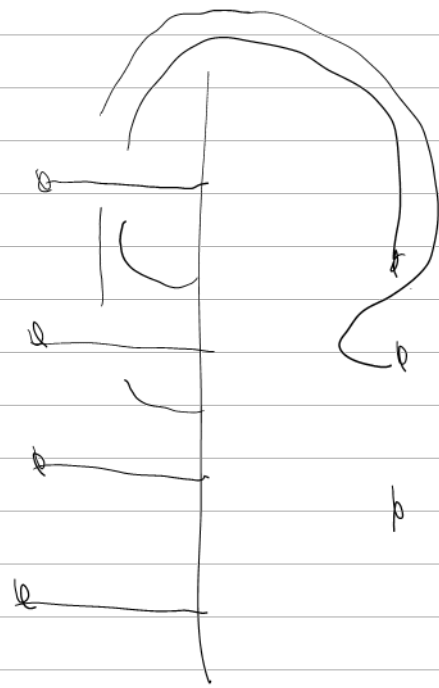
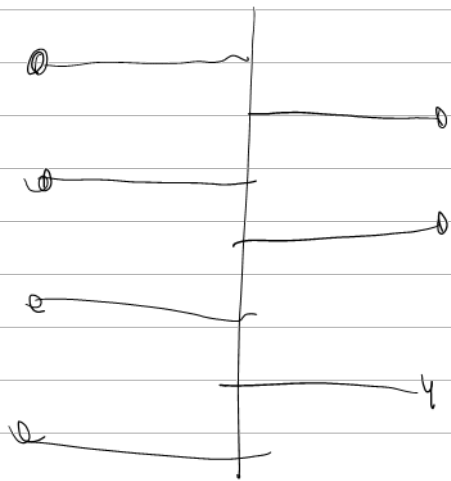
Claim γ is a $V_1 \rightarrow V_2$ cross-rep glow $\Leftrightarrow \mu$ is a 1-cocycle for $V_1^* \otimes V_2$
 $(\Leftrightarrow \mu$ is a $1 \rightarrow V_1^* \otimes V_2$ cross-rep glow)

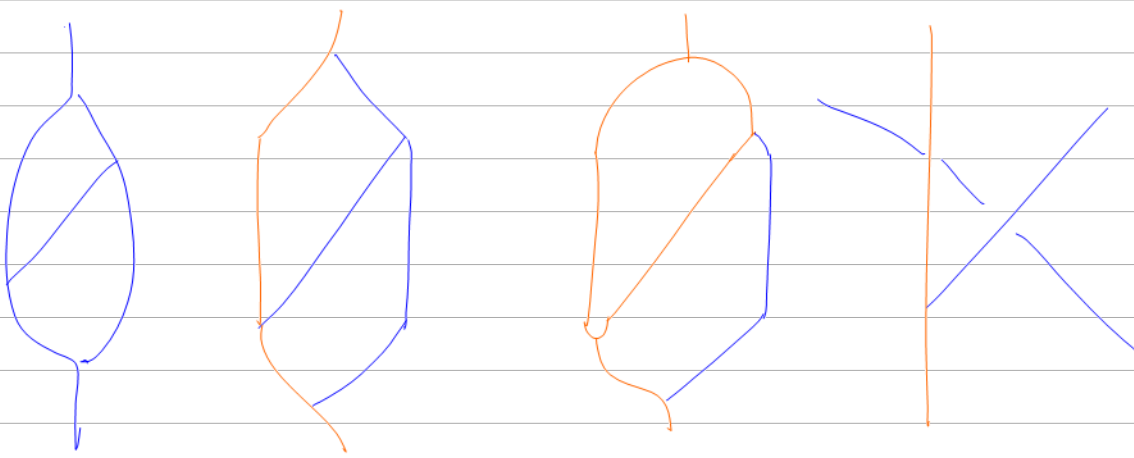
$$\begin{aligned} \text{Indeed, } \mu(g_1 g_2) &= \rho_1(g_2^{-1} g_1^{-1}) \gamma(g_1 g_2) = \rho_1(g_2^{-1} g_1^{-1}) (\gamma(g_1) \rho_2(g_2) + \rho_1(g_1) \gamma(g_2)) \\ &= \rho_1(g_2^{-1}) \mu(g_1) \rho_2(g_2) + \rho_1(g_2^{-1}) \gamma(g_2) \end{aligned}$$

Let $\mu: G \rightarrow V_1^* \otimes V_2$ be given by $\mu(g) = \rho_1(g)^{-1} \gamma(g) \rho_2(g)^{-1}$

Claim γ is a $V_1 \rightarrow V_2$ cross-rep glow $\Leftrightarrow \mu$ is a 1-cocycle for $V_1^* \otimes V_2$
 $(\Leftrightarrow \mu$ is a $1 \rightarrow V_1^* \otimes V_2$ cross-rep glow)

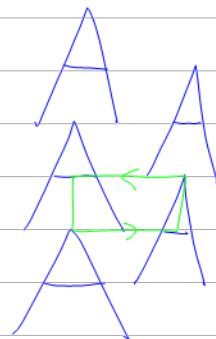
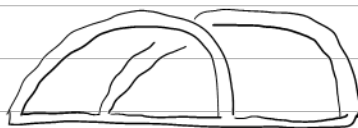
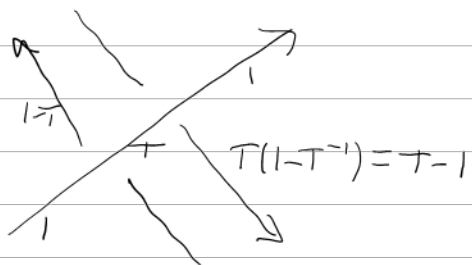
$$\begin{aligned} \text{Indeed, } \mu(g_1 g_2) &= \rho_1(g_2^{-1} g_1^{-1}) \gamma(g_1 g_2) \rho_2(g_2^{-1} g_1^{-1}) \\ &= \rho_1(g_2^{-1} g_1^{-1}) (\gamma(g_1) \rho_2(g_2) + \rho_1(g_1) \gamma(g_2)) \rho_2(g_2^{-1} g_1^{-1}) \\ &= \rho_1(g_2^{-1}) \mu(g_1) + \mu(g_2) \rho_2(g_1^{-1}) \end{aligned}$$



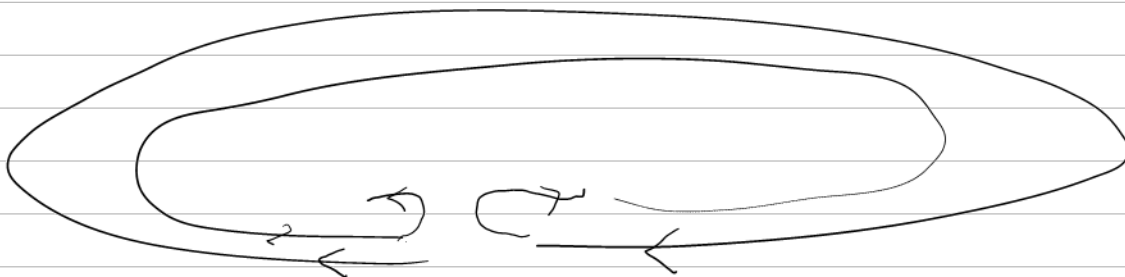
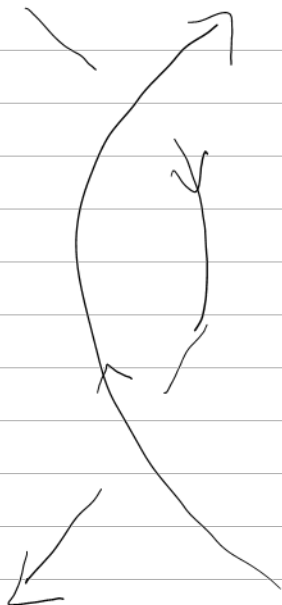


$$\underbrace{A^w / \begin{array}{|c|} \hline \text{---} \end{array}}_{\text{must be it?}} \rightarrow A^w / \begin{array}{|c|} \hline \rightarrow \end{array} \rightarrow A^w / \begin{array}{|c|} \hline \rightarrow \end{array}$$

in the IPIS case, these grow polynomially.



g g g
R R R

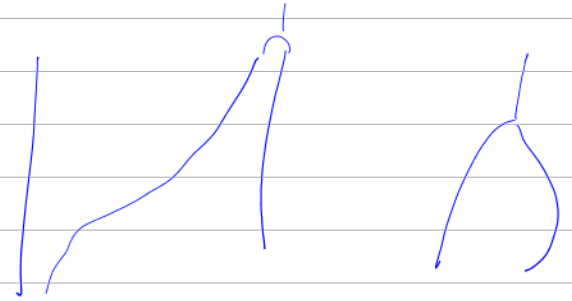


$grt: \text{Additive mor}(Pap) \rightarrow DK$ that are
compatible w/ the face maps.

AT: $\psi: gr_1 \rightarrow kr_2$ via

$$\psi \mapsto (\psi(y, x), \psi(x, y))$$

a Lie algebra morphism.



$$PaB \longrightarrow PaT \longrightarrow wKTG$$



$$PaDK \longrightarrow PaCD \longrightarrow Aw$$

