M.Sc. Math Workshop — Assignment #1

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Dror Bar-Natan

(1) (a) Prove that the fundamental group of the complement of a circle in $\mathbf{R}^3$ is $\mathbf{Z}$.
(b) Prove that the fundamental group of the complement of a trefoil knot is

$$\left\langle \alpha, \beta, \gamma : \alpha = \beta^\gamma, \beta = \gamma^\alpha, \gamma = \alpha^\beta \right\rangle,$$

where $a^b \stackrel{\text{def}}{=} b^{-1}ab$.

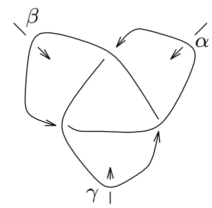

Figure 1. A hint.

(c) Is this group Abelian? (Hint: think about maps into S_3)
(d) Can you figure out a presentation for the fundamental group of an arbitrary knot given in term of a planar projection?

(2) Prove: If $\lambda > 0$ is irrational, $\epsilon > 0$, and $f : [0,1] \times [0,1] \to \mathbf{R}$ is continuous, then there exists a continuous function $\phi : [0,1] \to [0,1]$ and a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on at least 98% of the area of $[0,1] \times [0,1]$. (I.e., given addition, functions of two variables "almost factor" through functions of one variable).

(3) Can you find $2^{\aleph_0}$ different sets of natural numbers, so that for any two of them A and B, either $A \subset B$ or $B \subset A$?

(4) Let G be a trivalent planar map (that is, a trivalent graph embedded in the plane). Prove that there are as many edge 3-colorings of G (colorings of the edges of G by 3 colors, so that the 3 edges meeting at any given vertex are of different colors) as face 4-colorings of G (that is, map 4-colorings) in which "the state at infinity" is blue.

(5) Prove: The unit sphere in an infinite-dimensional Hilbert space is contractible.

(6) What is the configuration space of the machine M_2 in the picture below?

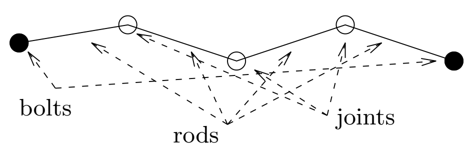

Figure 2. M_2 is a planar machine (a machine constrained to move in the plane) made of 4 rods, three joints, and two bolts arranged as above. The rods are straight and inflexible. The joints allow perfect bending, and the bolts are fixed to some fixed points in the plane. The relative sizes and distances are as shown.

(7) Prove that the group of symmetries of the dodecahedron is A_5.