Math 273 Course Description

Knot Theory as an Excuse

- *Time and place:* MWF 11AM, Science Center 507. *Instructor:* Dror Bar-Natan, Science Center 426G, 5-8797, dror@math.
- *Goal:* Use knot theory as an excuse to learning amusing mathematics and to having fun.
- *Intended for:* Math graduate students and everybody else. *Prerequisites:* Not being too far behind everybody else on the enclosed prerequisites quiz.
- *Texts:* My own papers (will be distributed), additional photocopied handouts, Kauffman's "Knots and Physics", and maybe more.
- *Course plan, main strand* (about 75% of classes): Explain the following paragraph:

> Vassiliev invariants are extremely simple to define, seem to be very powerful, and known to be at least as powerful as the standard knot polynomials. There are two natural and contradicting conjectures about Vassiliev invariants —
>
> 1. that they all come (in an appropriate sense) from Lie algebras.
> 2. that they separate knots.
>
> Besides, being so easy to define and so closely related to other knot invariants, they lead to new insight about these other invariants.

First, we will study the general theory — definitions, the relation with the Jones-like polynomials, the relation with weight systems and chord diagrams, the relation with Lie algebras and the Hopf algebra structure. Then, we will say more on the "Lie algebras" conjecture: the diagrammatic PBW theorem, the map into surfaces and the relation with the classical groups. Then we will say some about the "separation" conjecture — proving it for braids and for string links up to homotopy. Finally, we will see how Vassiliev invariants are useful in proving theorems about knot invariants — the Melvin-Morton-Rozansky conjecture and the power of HOMFLY over braids.

Along the way we will need to use many results and ideas from knot theory and from several other disciplines (and part of my motivation in giving the course is to finally understand these results and ideas myself). We will cover some of these results, many of them in lectures given by the students. These include: consistency of the classical knot polynomials, Hopf algebras, some representation theory, connections and holonomies, braids and free groups, the topological theory of the Alexander polynomial, etc.

We will postpone the discussion of two related, necessary, deep, and beautiful subjects, quantum groups and quasi-Hopf algebras, to the second semester. In the second semester we will study quasi-Hopf algebras following Drinfel'd's original papers on the subject, discussing their relation to the existence of a canonical universal Vassiliev invariant. We will also say some things about quantum groups, but I haven't decided yet on the source. It may be the relatively low-level but readable exposition in Kauffman, or some deeper and less comprehensible other source.

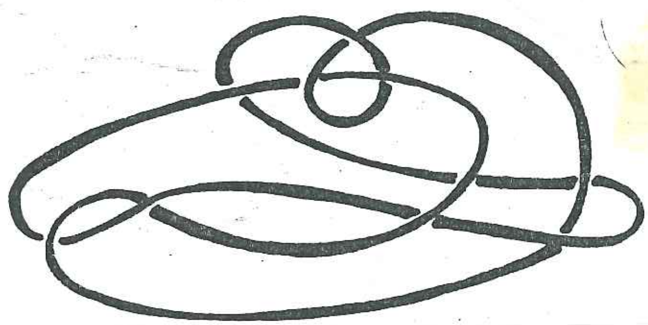
We may or may not say something about the relation with perturbative Chern-Simons theory.

We may or may not say something about related three-manifold invariants.

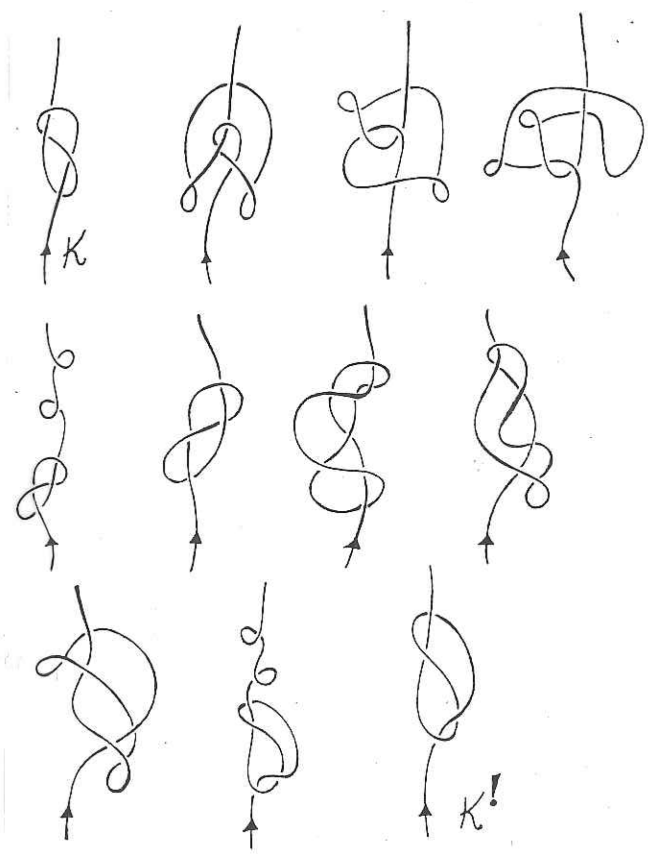
- *Course plan, secondary strand* (about 25% of classes): Play with elementary, isolated, and fun topics in and around knot theory, such as the 15 or so such topics in the second part ("miscellany") of Kauffman's book, the 20 or so such topics in the older Kauffman book ("On Knots"), alternating links, resistor networks, rational tangles, Kuperberg's polynomial, non-invertible knots, and more.

  I hope that many of these topics will be covered by visitors and/or students.

- *Grading:* Hopefully, almost everybody taking this course does not need a grade. About the rest we will worry later. *Homework:* Sporadic exercises and projects. If you do need a grade for the course, do the homework and impress me with the projects.



Millett's example.



Regular Isotopy of Figure Eight and its Mirror Image

Name: Everybody (optional).

Probability of taking the course: 80%: 20-80 <20

Will you need a grade? Yes No Maybe

Please rate the your level of understanding of the following topics, using the scale:

1 - Huh?
2 - Heard about it.
3 - Can state it.
4 - Feel confident; can prove including most details.
5 - Feel very confident; know it in and out.

The tensor product of two vector spaces.

1 2 3 4 5

The tensor algebra, symmetric algebra, and exterior algebra of a vector space.

1 2 3 4 5

The definition of a Lie algebra, structure constants.

1 2 3 4 5

The Poincare-Birkhoff-Witt theorem.

1 2 3 4 5

Finite dimensional representations of sl(2,C).

1 2 3 4 5

The classification of simple Lie algebras and their irreducible representations.

1 2 3 4 5

The definition of a Hopf algebra.

1 2 3 4 5

The structure theorem of co-commutative Hopf algebras.

1 2 3 4 5

Connections, curvatures, and holonomies.

1 2 3 4 5

The Van-Kampen theorem about the fundamental group of a union.

1 2 3 4 5

The relation between the fundamental group the the first homology of a space.

1 2 3 4 5

Galua theory for covering spaces.

1 2 3 4 5

The Poincare duality theorem.

1 2 3 4 5

The Alexander duality theorem.

1 2 3 4 5

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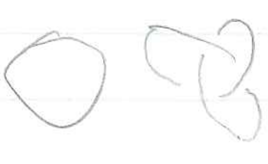
Read out course description

A knot: imbedding $S^1 \to \mathbb{R}^3$ considered up to "ambient isotopy"

examples , Millett, Fig-8, , ...

Is trefoil really knotted? Classify knots.

The Kauffman bracket (a.k.a the Jones polynomial)

$$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$$

$$\langle \underbrace{\circ\circ\cdots\circ}_{k} \rangle = d^{k-1}$$ (e.g. $\langle$  $\rangle = A^3 d + 3A^2 B + 3AB^2 d + B^3 d^2$)

Reid moves:

1. $\gamma \leftrightarrow)($ $\quad \to B = A^{-1}$; $d = -A^2 - \frac{1}{A^2} = \frac{A^4+1}{-A^2}$
2. $\times \to \times$ $\quad \to \checkmark$
3. $\quad \to \checkmark$

$$\langle \text{curl} \rangle = -A^3 \langle | \rangle$$

writhe: $\nearrow$ over $= +1$, $\nearrow$ under $= -1$

$\Rightarrow$ $J(K) = (-A^3)^{-w(K)} \langle K \rangle$

is a knot invariant!! Evaluated at $q^{-1/4}$, we get Jones.

Distribute Handouts!

Matteo Mainetti
(may do R-moves)
623-2699

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$$q^{-1} J(\times) - q J(\times) =$$

$$= -q^{-1} q^{3/4} \langle \times \rangle + q q^{-3/4} \langle \times \rangle =$$

$$= q^{1/4}\left(q^{-1/4} \langle \asymp \rangle + q^{1/4} \langle)(\rangle\right) - q^{-1/4}\left(q^{-1/4} \langle)(\rangle + q^{1/4} \langle \asymp \rangle\right)$$

$$= (q^{1/2} - q^{-1/2}) \langle)(\rangle$$

Review: 2. $\times \leftrightarrow \times$

1. $\times \leftrightarrow)($

3. $\rho \leftrightarrow |$

$$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$$

$$\langle \underbrace{O \cdots O}_{k} \rangle = d^{k-1}$$

$$R1,2 \Rightarrow B = A^{-1},\ d = -A^2 - \frac{1}{A^2}$$

$$\langle \rho \rangle = (-A^3) \langle | \rangle$$

$$w(\times) = +1 \qquad w(\times) = -1$$

$$J(K) = (-A^3)^{-w(K)} \langle K \rangle \Big|_{A = q^{-\frac{1}{4}}}$$

Do the trefoil example!
Do the Jones relation
Discuss Conway, HOMFLY, SU(N)-HOMFLY.

If time - Define Vassiliev.

The Taxonomy of Knot (Link) Polynomials

The Alexander polynomial $\left(\frac{d}{dN} GL(N)\Big|_{N=0}\right)$

$$A(\nearrow\!\!\!\!\nwarrow) - A(\nwarrow\!\!\!\!\nearrow) = (t^{1/2} - t^{-1/2}) A(\uparrow\uparrow) \qquad A(O^{(k)}) = \begin{cases} 1 & k=1 \\ 0 & k>0 \end{cases}$$

The Conway polynomial

$$C(\nearrow\!\!\!\!\nwarrow) - C(\nwarrow\!\!\!\!\nearrow) = z\, C(\uparrow\uparrow) \qquad C(O^{(k)}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

The Jones polynomial $(SU(2))$

$$q\, J(\nearrow\!\!\!\!\nwarrow) - q^{-1} J(\nwarrow\!\!\!\!\nearrow) = (q^{1/2} - q^{-1/2}) J(\uparrow\uparrow) \qquad J(O^{(k)}) = (q^{1/2} + q^{-1/2})^K$$

Framed version: $J^f(\nearrow\!\!\!\!\nwarrow) - J^f(\nwarrow\!\!\!\!\nearrow) = (q^{1/2} - q^{-1/2}) J^f(\uparrow\uparrow) \quad J^f(O^{(k)}) =$ same

The HOMFLY (LYMPH-TOFU) Polynomial $(SU(N))$

$$q^{N/2} H(\nearrow\!\!\!\!\nwarrow) - q^{-\frac{N}{2}} H(\nwarrow\!\!\!\!\nearrow) = (q^{1/2} - q^{-1/2}) H(\uparrow\uparrow) \qquad H(O^{(k)}) = \left(\frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}\right)^K$$

Framed version: $H^f(\nearrow\!\!\!\!\nwarrow) - H^f(\nwarrow\!\!\!\!\nearrow) = (q^{1/2} - q^{-1/2}) H^f(\uparrow\uparrow) \quad H^f(O^{(k)}) =$ same

The Kauffman polynomial $(SO(N))$

$$q^{\frac{N-1}{2}} F(\diagup\!\!\!\!\diagdown) - q^{\frac{1-N}{2}} F(\diagdown\!\!\!\!\diagup) = (q^{1/2} - q^{-1/2}) \left(F()() - F(\asymp) \right)$$

$$F(O^{(k)}) = \left(1 + \frac{q^{\frac{N-1}{2}} - q^{\frac{1-N}{2}}}{q^{1/2} - q^{-1/2}}\right)^K \qquad F^f(O^{(k)}) = \text{same.}$$

Framed version: $F^f(\diagup\!\!\!\!\diagdown) - F^f(\diagdown\!\!\!\!\diagup) = (q^{1/2} - q^{-1/2}) \left(F^f()() - F^f(\asymp) \right)$

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Grades: At least give a talk or two

ftp: finger dror@math.harvard.edu (dror/273)
within Harvard: ~dror/ftp/273

Taxonomical remarks:

1. Links !
2. Paramatrization (usually $q^{N/2} \to \alpha$, $q^{1/2}-q^{-1/2} \to z$)
3. up.f: 0. regular isotopy: 1. Framed knots (links)
   2. blackboard framing, R3.
   3. linking numbers ($\frac{1}{2}\Sigma$ crossings) (Symm. fun.
   4. Writhe = l.n w/ frame !

Vassiliev invariants:

1. def
2. big hope
3. Conway is, HOMFLY is

4. W.S.

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Distribute paper.
Remind Vassiliev invariants.

~~Define v.i.s.~~ show how to get a w.i.s.
Compute for Conway & HOMFLY & Kauffman

(Kauffman: $\downarrow\text{- -}\uparrow = -(N-1)\downarrow\,\uparrow +$ $-$)

Fram ind & 4T
Define w.i.s.
<u>Thm 1</u>

(Thm 1' for framed links)

Lie algebras & tensors.

Math 273, Sep 28 1994

Reminder for gradables!

Pavel's proof of ~~singular knots~~ / Flips = CD's.

Remind W_{Homfly}, Postpone a better rep.

Fram ind & 4T

Define wcs.

Thm 1

(Thm 1' for framed links)

Lie algebras and tensors

STU, IHX, AS

gl(N), SU(N)

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Reminder for gradables.

Remind Thm 1,

"actuality table"

Thm 1'.

Tensors

Lie algebras & Tensors.

STU IHX, AS.

Math 273, Oct 3 1994 Ask Matteo.

Remind Lie algebra as a tensor
Reps as tensors.

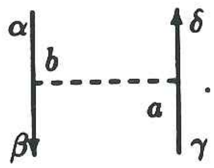
AS, STU, IHX.

gl(N) all the way.

8.5 Complete evaluation for the classical algebras

By the remark at the end of the previous section, to calculate $C_{\mathcal{G}}$ for the classical algebras (in their defining representations) it is enough to consider the four complex classical algebras.

The first step is to use relation STU repeatedly, with each usage reducing the number of $\mathcal{G}^3$ vertices by one, until we are left with a diagram D that has no $\mathcal{G}^3$ vertices. The basic building block of such diagrams is the tensor

$$T^{\alpha\gamma}_{\beta\delta} =$$ 

This tensor will be evaluated explicitly for each of the complex classical algebras, and the results will turn out to have representations in terms of diagrams that have no propagators in them. Using this repeatedly, we are left with disjoint unions of circles which again are easy to evaluate explicitly.

I will show in detail the computations for $so(N,\mathbf{C})$, and just state the results for $gl(N,\mathbf{C})$, $sl(N,\mathbf{C})$, and $sp(N,\mathbf{C})$.

8.5.1 The algebra $so(N,\mathbf{C})$.

A convenient choice of generators for $so(N,\mathbf{C})$ are the $N\times N$ matrices M_{ij} $(i<j)$, given by

$$(M_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}.$$

That is, the ij entry of M_{ij} is $+1$, the ji entry of M_{ij} is -1, and all other entries of M_{ij} are zero. The invariant bilinear form that we pick on $so(N,\mathbf{C})$ is the matrix trace in the defining representation, and so

$$t_{(ij)(kl)} \stackrel{\text{def}}{=} tr(M_{ij}M_{kl}) = -2\delta_{ik}\delta_{jl}.$$

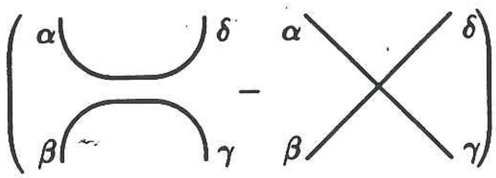
Inverting the $\frac{N(N-1)}{2}\times\frac{N(N-1)}{2}$ matrix $t_{(ij)(kl)}$ we get

$$t^{(ij)(kl)} = -\frac{1}{2}\delta^{ik}\delta^{jl}, \tag{8.14}$$

and so

$$T^{\alpha\gamma}_{\beta\delta} = \sum_{i<j;k<l} t^{(ij)(kl)}(M_{ij})_{\alpha\beta}(M_{ij})_{\gamma\delta}. \tag{8.15}$$

Using (8.14) and some algebraic manipulations we can simplify (8.15), and then represent it by a diagram:

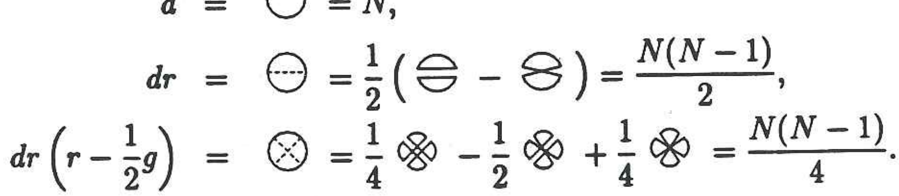
$$(8.15) = \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) = \frac{1}{2}$$ 

(8.16)

The last thing to note is that

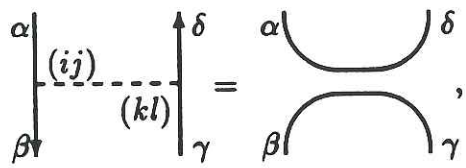
$$C_{so(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $so(N,\mathbf{C})$ in its defining representation we can calculate d, r, and g using: (suppressing the '$C_{so(N,\mathbf{C})}$' symbols)



8.5.2 The algebra $gl(N,\mathbf{C})$.

Similar considerations lead to the even simpler rule



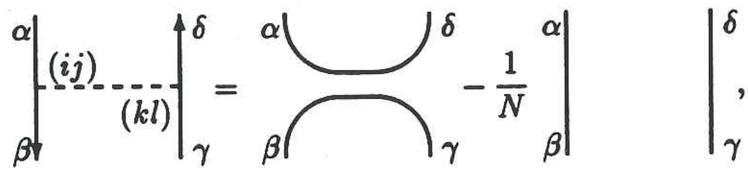
while retaining

$$C_{gl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $gl(N,\mathbf{C})$ in its defining representation,

8.5.3 The algebra $sl(N,\mathbf{C})$.

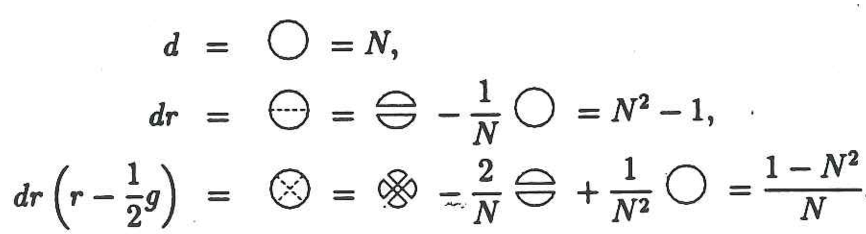
The rule here is the so-called "Fierz identity",



with the usual

$$C_{sl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $sl(N,\mathbf{C})$ in its defining representation we can calculate d, r, and g using:

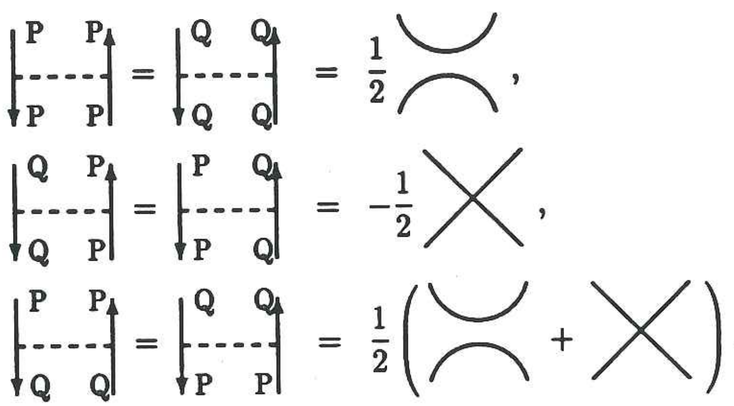


8.5.4 The algebra $sp(N, \mathbf{C})$.

This is the most complicated case. Let D be a diagram with no $\mathcal{G}^3$ vertices. The computation of $C_{sp(N,\mathbf{C})}(D)$ now proceeds in two steps:

1. Mark each Wilson loop segment in D with either the symbol P or the symbol Q, in such a way that the number of P's entering each subdiagram of D of the form  is equal to the number of P's leaving it. (Remember that the Wilson loops are *directed*).

2. Simplify D using the following rules:

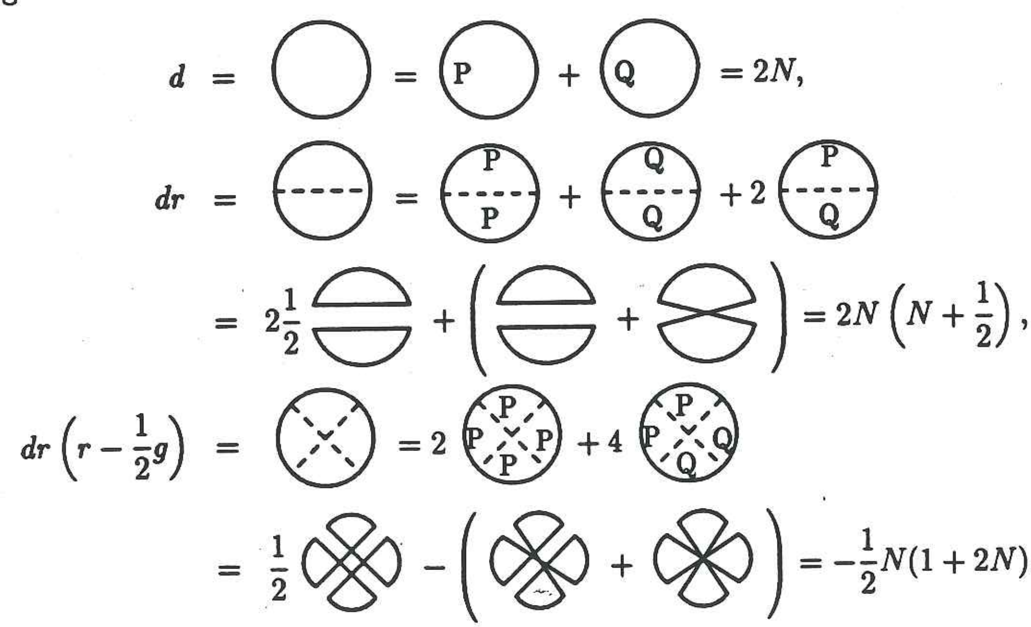


3. Similarly to the usual,

$$C_{sp(N,\mathbf{C})}(k \text{ disjoint } \textit{marked} \text{ circles}) = N^k.$$

(Notice that this time $\dim R = 2N \neq N$).

Example For $sp(N, \mathbf{C})$ in its defining representation we can calculate d, r, and g using:



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① Distribute handout

$\phi^c : \mathcal{A}^c \longrightarrow \mathcal{A}^t$

build

$\phi^t : \cancel{\mathcal{A}^t} \longrightarrow \mathcal{A}^c$ (k = # of internal, induct on k)

IHX, AS.

$\phi^l : \mathcal{A}^l \longrightarrow \mathcal{A}^t$ is iso!

The product on $\mathcal{A}^l$, $\mathcal{A}^t$; Connected sum of knots.

The product of Vass. is Vass; Hopf algebras.

Igor Pak (October 7, 1994)

The Chromatic polynomial

Definition for a graph $P(G,N)$

$$P(\text{empty}_k) = N^k$$

$$P(\text{complete}_k) = N(N-1)\cdot\ldots(N-k+1)$$

$$P(\underbrace{\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ}_{k}) = N(N-1)^{k-1}$$

Def $G-e$ deletion, G/e restriction

Thm $P(G,X) = P(G-e) - P(G/e)$

Example $P(\circ\!-\!\circ) = P(\cdot\ \cdot) - P(\cdot) = x^2 - x$

$$P(\triangle) = P(<) - P(\circ\!-\!\circ) = x(x-1)^2 - x(x-1)$$
$$= x(x-1)(x-2)$$

Thm (Whitny) $P(G,x) = \sum (-1)^{n-i} a_i x^i$ $n = \#$ of verts.

$a_i = \#\{S \subset E : |S| = i,$ S doesn't contain a broken cycle$\}$

<u>Def</u> assume the edges of G are ordered $e_1, \dots e$.

$$E = \{e_1, e_2 \dots\}$$

$$C \subseteq E$$

is called "a broken cycle" if it is a cycle with the maximal edge removed.

<u>Idea of proof of Whitney</u> use chromatic relations for the maximal edge.

<u>Tutte polynomial</u>

$$r(G) = \#E - \underset{\substack{\uparrow \\ \text{connected comps}}}{\#C} - \underset{\substack{\downarrow \\ \text{number of}}}{\#L}$$

$$\underset{\substack{\downarrow \\ \text{"The Tutte polynomial"}}}{T(G,x,y)} = \sum_{S \subseteq E} (x-1)^{r(G)-r(S)} \cdot (y-1)^{\#E - r(S)}$$

<u>Thm</u> (Tutte)

$$T(G,x,y) = \begin{cases} T(G-e,x,y) + T(G/e,x,y) & \\ y\,T(G-e,x,y) & e \text{ is a loop} \\ x\,T(G/e,x,y) & e \text{ is a bridge} \\ 1 & G = \text{[illegible]} \quad |E| = 0 \end{cases}$$

Cor

$$T(G, 1-z, 0) = z \cdot P(G, z)(-1)^{n-1}$$

Examples

$T($  $) = x$ $T($  $) = y$

$T($  $) = x + y$

$T($  $) = x^2 + x + y$

Thm If G is planar, G^* its dual.

$$T(G, x, y) = T(G^*, y, x)$$

Obs $T(G, 2, 2) = 2^{|E|}$

$T(G, 1, 2) = \#$ of connected subgraphs

Th (Stanley)

$P(G, -1) = (-1)^n \#$ acyclic orientations

Math 273, October 12 1994

Hopf algebra ≅ an algebra whose dual is also an algebra, in a compatible way.

Examples: 1. G finite group, $\mathbb{Z}G = \{\sum c_i g_i\}$
2. Generalizations
3. knots; knot invariants.
4. Vassiliev invariants.

claim $V_1 (\in \mathcal{F}_{m_1} \mathcal{V})$ $V_2 (\in \mathcal{F}_{m_2} \mathcal{V}) \Rightarrow (V_1 \cdot V_2) \in \mathcal{F}_{m_1+m_2} \mathcal{V}$

pf

$$(V_1 \cdot V_2)(dbl) = (V_1 \cdot V_2)(+) - (V_1 \cdot V_2)(-) = (V_1(+) - V_1(-)) V_2(+) + V_1(-)(V_2(+) - V_2(-))$$
$$= V_1(dbl) V_2(+) + V_1(-) V_2(dbl)$$

More Formally: A is an algebra if $A \otimes A \to A$ s.t.

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r] \arrow[d] & A\otimes A \arrow[d] \\ A\otimes A \arrow[r] & A \end{tikzcd}$$

unit: $\eta : Q \to A$ s.t.

$$A = Q \otimes A \xrightarrow{\eta} A \otimes A \xrightarrow{m} A$$

Commutative

Co-algebra: $M : A^* \otimes A^* \to A^*$, or simply $\Delta : A \to A \otimes A$ s.t.

counit

Hopf alg:

$$\begin{tikzcd} A\otimes A \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & A \arrow[d, "\Delta"] \\ A\otimes A\otimes A\otimes A \arrow[r, "m_{13}\otimes m_{24}"] & A\otimes A \end{tikzcd}$$

Δ is an algebra morphism
m is a co-algebra morphism.

Thm $\mathcal{A}$ is a commutative & co-commutative Hopf algebra

Factoring θ out

9/11/94

Operators involved:

$\theta: \mathcal{G}_m\mathcal{A} \to \mathcal{G}_{m+1}\mathcal{A}$ – multiplication by θ

$F: \mathcal{G}_m\mathcal{A} \to \mathcal{G}_{m-1}\mathcal{A}$ – Forget one (any) chord

$[F,\theta] = F\theta - \theta F = I$

$S: \mathcal{G}_m\mathcal{A} \to \mathcal{G}_m\mathcal{A}$ – $SD = D - \theta F D + \frac{1}{2}\theta^2 F^2 D - \frac{1}{3!}\theta^3 F^3 D + \dots$

$R: \mathcal{G}_m\mathcal{A} \to \mathcal{G}_{m+1}\mathcal{A}$ – $R = \theta - \frac{1}{2}\theta^2 F + \frac{1}{3!}\theta^3 F^2 - \dots$ $\boxed{RF = I - S}$

__claim__ If $W = W_1 \cdot W'$, then $W(D) = W'(FD)$

__pf__ trivial

__claim__ If $W = W_1 \cdot W'$, then $W'(D) = W(RD)$

__pf__ $W(RD) = W'(FRD) = W'(D)$ because $FRD = D$!
(see below)

__claim__ $FR = I$ $FS = 0$ (trivial if F is interpreted as $d/d\theta$)
$RF = I - S$

__pf__ easy

__claim__ If $W \circ S = 0$, then $W = W_1 \cdot W'$ for some W'.

__pf__ define $W' = W \circ R$ $D = RFD + SD$

$(W_1 \cdot W')(D) = (W_1 \cdot W')(RFD + SD) = W'(FRFD + FSD) =$ $W(D)$
$= W'(FD) = W(RFD) = W(D - SD)$

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$\mathcal{A}$ is generated by $\mathcal{P}(\mathcal{A}) = \{D \in \mathcal{A} : \Delta D = 1 \otimes D + D \otimes 1\}$

1. Δ in terms of knots.

2. Δ in terms of Lie algebras

Primitives & generators

$$A = S\,\mathcal{P}(A) \qquad \mathcal{P}(A) = \{x \in A : \Delta D = 1 \otimes D + D \otimes 1\}$$

Example $\mathcal{P}(A) =$ connected not a great desc.

---

① $\mathcal{W}$ is a sub-Hopf alg of $\mathcal{A}^*$, $\mathcal{P}(\mathcal{W}) = \mathcal{P}(\mathcal{A}^*) \cap \mathcal{W}$

$= P(\mathcal{A}^*)$

$\Rightarrow \exists$ proj. $W \longrightarrow \vec{W}$.

F, ⊕ desc

Math 273, October 19 1994

$$\mathbb{Q}[\theta, D_2, \dots] = \mathcal{A} \underset{S\,?}{\rightleftarrows} \mathcal{A}' = \mathcal{A}/\langle\theta\rangle \cong \mathbb{Q}[D_2, \dots]$$

$$\mathbb{Q}[W_1, W_2, \dots] = \mathcal{A}^* \overset{S^*\,?_0}{\rightleftarrows} \mathcal{W} \overset{?_0}{\cong} \mathbb{Q}[W_2, \dots]$$

1. Indeed $\mathcal{W} = \mathbb{Q}[W_2, \dots]$

   Pf 1. $\mathcal{W}$ is sub-Hopf $\left(\begin{smallmatrix}\text{prod. in}\\ \text{coprod in}\end{smallmatrix}\right)$

   2. $\mathcal{P}(\mathcal{W}) = \mathcal{P}(\mathcal{A}^*)$ $\quad \Delta W = W \otimes \epsilon + \epsilon \otimes W$

   $\Rightarrow W$ is prim iff vanishes on reducible diagrams.

2. Definition of S:

$\hat{\theta} : \mathcal{G}_* \mathcal{A} \to \mathcal{G}_{*+1} \mathcal{A}$ $\quad W_1 : \mathcal{G}_* \mathcal{A}^* \to \mathcal{G}_{*+1} \mathcal{A}^*$

$F : \mathcal{G}_* \mathcal{A} \to \mathcal{G}_{*-1} \mathcal{A}$ $\quad F(D) = \sum$ (forget one chord)

$[F, \hat{\theta}] = 1$ $\;!\;$ $\quad \hat{\theta} = $ mult by θ $\quad F = \frac{d}{d\theta}$

$$S = \sum_{n \geq 0} \frac{(-1)^n}{n!} \hat{\theta}^n F^n = :e^{-\hat{\theta} F}:$$

$\mathcal{A} \leftrightarrow \mathbb{Q}[\theta, D_2, \dots]$, $\hat{\theta} \downarrow F$, $\mathcal{A} \leftrightarrow \mathbb{Q}[\theta, \dots]$

$\hat{W}_1(\theta) = 1$ $\quad \hat{W}_1(\text{other } m) = 0$

$\hat{W} \longmapsto \hat{W}_1 \cdot \hat{W}$

$\hat{W}_1 \cdot \hat{W}(m) = \hat{W}\left(\frac{d}{d\theta} m\right)$

$$R = \sum_{n \geq 0} \frac{1}{(n+1)!} \hat{\theta}^{n+1} S F^n = \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \hat{\theta}^{n+1} F^n$$

Props 1. S is a Hopf morphism:

a. $S(D_1 D_2) = S(D_1) S(D_2)$ $\quad$ (Follows of Leibnitz's rule $F(D_1 D_2) = (FD_1)D_2 + D_1 F D_2$)

b. $\Delta(SD) = (S \otimes S)(\Delta D)$ $\leftarrow$ $\left(\Delta \hat{\theta} = (\hat{\theta} \otimes 1 + 1 \otimes \hat{\theta}) \circ \Delta\right)$

2. $FS = 0$ $\qquad FR = I$

   $S\hat{\theta} = 0$ $\qquad RF = I - S$

Example: $\mathfrak{gl}(N)$

Math 273, October 26 1994

Alexander Postnikov, Hecke Alg & Jones.

Following Jones' paper Ann Math 126 (1987) 335–388.

 no good.

$$S_n = \left\langle \sigma_1 \dots \sigma_{n-1} ; \begin{array}{l} \sigma_i^2 = 1 \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2 \end{array} \right\rangle$$

Br_n = same, dropping $\sigma_i^2 = 1$.

Markov's theorem.

reps of Br_n: 1. Permutations.

$$I_q : \quad \sigma_i \begin{pmatrix} b_i \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} b_{i+1} \\ q b_i + (1-q) b_{i+1} \end{pmatrix}$$

(equiv to adding the relation $(\sigma_i - q)(\sigma_i + 1) = 0$ or $\sigma_i^2 = (q-1)\sigma_i + q$)

$$\Longrightarrow \text{Hecke}_n = \left\langle \sigma_i \dots ; \begin{array}{l} Br_n \text{ relations} \\ \& (\sigma_i - q)(\sigma_i + 1) \end{array} \right\rangle = H_n = H(n,q)$$

Has the same reps as S_n for generic q!

Ocneanu's trace:

$H_0 \subset H_1 \subset \ldots \qquad tr: \bigcup_n H_n \to \mathbb{C}$

s.t. 1. $tr\, 1 = 1$

2. $tr(ab) = tr(ba)$

3. $tr(a\sigma_n) = z \cdot tr\, a$ for some fixed z.

Thm (0) $\exists !$ ~~such~~ such tr.

Sketch of pf: assume for H_{n-1}.

basis of H_n is

$\sigma_{i_1}\sigma_{i_1-1}\cdots\sigma_{i_1-k_1}\sigma_{i_2}\sigma_{i_2-1}\cdots\sigma_{i_2-k_2}\cdots$

$i_1 < i_2 < \cdots < n$

Property $a \in H_{n-1}$ or $b\sigma_n c$ w/ $b, c \in H_{n-1}$ generate basis.

$\Longrightarrow$ forces def of tr!

Have to check to relations! $\square$

We have an "almost knot invariant".

Indeed, define

$$\cancel{X(q,\lambda)} \; X_{q,\lambda}(b) = (z-\sqrt{\lambda})^{-n+1}(\sqrt{\lambda})^{r(b)}\, tr\, \pi(b)$$

w/ λ defined by $z = -\frac{1-q}{1-\lambda q}$

$\pi : Br_n \to H_n$ proj

and

$r(b) = \text{writhe}(\text{link}(b))$

<u>Thm</u> $X_{q,\lambda}$ is a link invariant.

<u>Pf</u> .... $\square$

<u>Thm</u> $X_{q,\lambda}(b) = \text{HOMFLY}(\text{link}(b))$

where HOMFLY is defined by

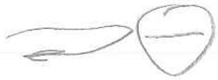
$t = \sqrt{\lambda}\sqrt{q} \qquad x = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)$

<u>Pf</u> ... $\square$.

Math 273, October 28 1994

$$\mathcal{A}^c \longleftrightarrow \mathcal{A}^t \longleftrightarrow \mathcal{D} \xrightarrow{1\text{-}1?} \mathcal{M}$$

 / STU, IHX, AS

$\mathcal{A}^{\text{links}}$ 

$$\mathcal{A}^{sl} \longleftrightarrow \mathcal{D}^{sl} \xrightarrow[1\text{-}1?]{} \mathcal{M}^{sl}$$

$$\underset{\text{(boring chords.}}{\mathcal{A}^{hsl}} \longleftrightarrow \underset{\text{(boring comps, non-forests)}}{\mathcal{D}^{hsl}} \xrightarrow[1\text{-}1!]{\Phi^{hsl}} \mathcal{M}^{hsl}$$

Need to prove:

$$\mathcal{D}^{res} \longrightarrow \mathcal{M}^{res}$$

1. $\tilde{\chi}$ descends. (start w/ boring)
2. σ descends ("D^{cc} is a relation in $\mathcal{D}^{hsl}$" is preserved)
3. Φ^{hsl} is 1-1 !

Math 273 October 30 1994

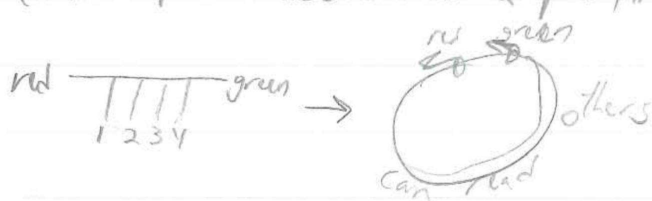
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left] \arrow[r, bend right, "x"'] & \mathcal{D} \arrow[r, "\bar\Phi"] & \mathcal{M} \end{tikzcd}$$

$$\mathcal{A}^{sl} \qquad \mathcal{D}^{sl} \qquad \mathcal{M}^{sl}$$

alternatively:

$$\begin{tikzcd} \mathcal{A}^{hsl} \arrow[r, leftrightarrow, "\text{done}"'] & \mathcal{D}^{hsl} \arrow[r, "1-1?"'] \arrow[d, leftrightarrow] & \mathcal{M}^{hsl} \arrow[d, leftrightarrow] \\ & FL(n-1) \arrow[r, "1-1?"'] & FA(n-1) \end{tikzcd}$$

(interrupt & discuss Δ & primitives)



graded by some simple function of the genus.

Claim $(\Phi\circ\sigma)^*\mathcal{M}^* \supset$ all classical Lie algs

Pf $\mathfrak{g} = SO(N)$

$$\begin{tikzcd} \mathcal{A} \arrow[r, leftrightarrow] \arrow[dr, "W_{\mathfrak{g},R,m}"'] & \mathcal{D} \arrow[r] & \mathcal{M} \arrow[dl, "M_{\mathfrak{g},R,m}"] \\ & \mathbb{C} & \end{tikzcd}$$

Products:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\Delta"] & \mathcal{M} \arrow[d, dashed, "\exists ? !"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r] \arrow[dr] & \mathcal{M}\otimes\mathcal{M} \arrow[dl] \\ & \mathbb{C}\otimes\mathbb{C} \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

$\cancel{\text{(genus)}}$ δ(genus $= ..$)

$\mathfrak{g} = \mathfrak{gl}(N)$

$$\begin{tikzcd} & & \mathcal{M} \arrow[d, shift left, "\pi\ \text{obv. proj.}"] \\ \mathcal{A} \arrow[r, leftrightarrow] \arrow[dr, "W_{\mathfrak{gl},R,m}"'] & \mathcal{D} \arrow[ur, "\Phi"] \arrow[r, "\Phi^{gl}"'] & \mathcal{M}^{gl} \arrow[u, shift left] \arrow[dl, "M_{\mathfrak{gl},R,m}"] \\ & \mathbb{C} & \end{tikzcd}$$

Φ is to $C_1(D, \mathbb{Z}/2\mathbb{Z})$

as Φ^{gl} is to $C_1(D, \mathbb{Z}/2\mathbb{Z})$

$$\pi\Phi(D) = \sum_{\substack{c\in C_1(D,\mathbb{Z}/2\mathbb{Z})\\ dc=0}} \pi\Phi_c(D)$$

$$= 2^{|H_0|} \sum_{c\in C_1(D,\mathbb{Z}/2\mathbb{Z})} \Phi_{dc} = 2^{|H_0|} \sum_{c\in C^0} \Phi^{gl}_c$$

Math 273, 11/4/94

David Goldberg – SO(3) & four colors.

1. The franks trick for so(3)
2. ϵ-calculus for trivalent graphs.
3.  . . .
4. The definition of ϵ.
5. The franks trick satisfies the relations of before.
6. The relation between ϵ & so(3).
7. The relation between $(\mathbb{Z}/2\mathbb{Z})^2$, map 4-cols & edge 3-cols.
8. Signs using red & blue filters. (The # of intersections of the views is even)

Math 273 November 7 1994

$$\begin{tikzcd} A \arrow[r] \arrow[d, "W_{g,N}"'] & M \arrow[dl, "M_{g,R}"] \\ \mathbb{C} & \end{tikzcd}$$

To show that the $M_{g,R,m}$ span M^* we need to know a little more about reps & characters.

Rep $\longrightarrow$ character (class function)

<u>Thm</u> Every class function is a linear comb of characters

$(\psi^q \chi)(g) = \chi(g^q)$

Thus $\psi^q R = \sum a_i R_i$

> <u>Example</u> $g = U(N)$, $R = N$ dim $\psi^3 R = ?$
>
> $M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$ $\quad \chi_R(M) = \sum \lambda_i \quad \chi_{\Lambda^2 R}(M) = \sum_{i<j} \lambda_i \lambda_j \quad \chi_{\Lambda^3 R}(M) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k$
>
> $(\psi^3 \chi_R)(M) = \sum \lambda_i^3 = \left(\sum \lambda_i\right)^3 - 3\left(\sum_{i<j} \lambda_i \lambda_j\right)\left(\sum \lambda_i\right) + 3 \sum_{i<j<k} \lambda_i \lambda_j \lambda_k$

$\chi_R : U(g) \longrightarrow \mathbb{C}$ by $X_1 X_2 X_3 \longmapsto \operatorname{tr}(R(X_1) R(X_2) R(X_3))$

$$= \partial_{t_1} \partial_{t_2} \partial_{t_3} \chi_R(e^{t_1 X_1} e^{t_2 X_2} e^{t_3 X_3}) \Big|_{t_j = 0}$$

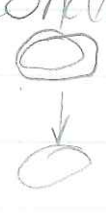
(ordered?)

$$\chi_{\psi^q R}\left(\prod X_j\right) = \sum a_i \chi_{R_i}\left(\prod X_j\right) = \sum a_i \left(\prod_j \partial_{t_j}\right) \chi_{R_i}\left(\prod_j e^{t_j X_j}\right) \Big|_{t_j = 0}$$

$$= \prod_j (\partial_{t_j}) \left(\sum a_i \chi_{R_i}\left(\prod e^{t_j X_j}\right)\right) \Big|_{t_j = 0} = \left(\prod_j \partial_{t_j}\right) (\psi^q \chi_R)\left(\prod e^{t_j X_j}\right) \Big|_{t_j = 0}$$

$$= \left(\prod \partial_{t_j}\right) \chi_R\left(\left(\prod e^{t_j X_j}\right)^q\right) \Big|_{t_j = 0} = \left(\prod \partial_{t_j}\right) \operatorname{tr}\Big[R\big(\underbrace{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_n X_n} e^{t_1 X_1} \cdots e^{t_n X_n} \cdots}_{q}\big)\Big] \Big|_{t_j = 0}$$

Conclude covering formula for ψ^q:



$$W_{g, \psi^q R}(D) = W_{g,R}(\psi^q(D))$$

Covering formulas for tensor products

Math 273, November 14 1994

$tr(M_1 M_2 M_3 M_4)$ $\quad tr(M_4 M_3 M_2 M_1) = tr((M_4 M_3 M_2 M_1)^T) =$

$= tr(\bar{M}_1 \bar{M}_2 \bar{M}_3 \bar{M}_4) = \overline{(tr\, M_1 M_2 M_3 M_4)}$

$\Psi^{-1} W_{g,R} = W_{g,\bar{R}} \overset{?}{=} W_{g,R}$

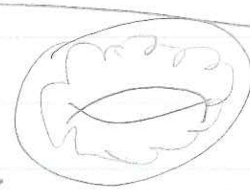
---

1. interpretation of coverings in terms of knots

2. Ψ^{-1} on $\mathcal{U}, \mathcal{A}, \mathcal{D}$; Ψ^{-1} on Knots.

3. $\Phi|_{\mathcal{D}^{odd}} = 0$. $\Rightarrow$ classical Lie algs do not detect orientation.

4. No Lie algs detect orientations.

Kuperburg: $S_T(K) = Tr\ T$ around K 

S_T^* Preserves $\mathcal{V}$: $V \circ S_T$ is vass.

if $K = K^{-1}$ $(V \circ S_T)(K) = (V \circ S_T)(K^{-1})$ for all V

but it may be that

$(V \circ S_T)(K)$ & $(V \circ S_T)(K^{-1})$

are completel different!

Lewis Wolfgang, Math 273, Nov 18 1994

1. Refs Milnor Moore
   Appendix B of Quillen.

2. Def of A a Hopf algebra over a char-0 k.

3. $P(A)$, is a lie algebra (Proof)
   L

4. $U(L)$ by the universal prop.

5. PBW

6. $S(V)$ (a universal def), co-product on $S(V)$.

7. $U(L)$ is a Hopf algebra.

8. $T(V)$ is a Hopf algebra by th cut coproduct, $S(V) \longrightarrow T(V)$ is a coalg map.

9. $U(P(A)) \xrightarrow{\alpha} A$

   α is an algebra map & a co-alg map.

10.
$$\begin{tikzcd} U(P(A)) \arrow[r, "\alpha"] & A \arrow[dl, "\theta"] \\ S(P(A)) \arrow[u, "i", "\wr"'] & \end{tikzcd}$$

11 connected, connected'

12 connected' $\longrightarrow$ connected

Math 273, November 28 1994

KKZ Formula:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_1<\cdots<t_m} \sum_{\substack{\text{a.p. } P \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \in \mathcal{A}^r$$

1. Can move @ time where there's no crit pt.
2. Can (by a needle argument) move critical pts
3. Can cancel critical points? $\nu = 1$?

$$\rightarrow \int_0^1 dt_1 \frac{d(1-t_1)}{1-t_1} \cdot \int_{t_1}^1 dt_2 \frac{d(t_2)}{t_2} = \int_0^1 \frac{dt}{1-t} \log t = \frac{\pi^2}{6}$$

$Z(\infty) = Z(\infty) = \bigcirc + \frac{\pi}{6} \otimes + \cdots$

$\tilde{Z}(K) = Z(K) / (Z(\infty))^{c/2}$ $\quad c - \#$ of crit pts.

$\Rightarrow \tilde{Z}(K)$ is a knot invariant!

<u>Thm</u> $\tilde{Z}(K_D) = D +$ higher order terms.
(i.e. - $\tilde{Z}$ is a universal Vassiliev invariant)

<u>Cor</u> Thm 1

<u>Thm</u> $Z(\infty) \cdot \tilde{Z}$ is a Hopf map (in as much as this makes sense)

Math 273, December 13 1994

next class: Leonid Koregodsky

Pent $\quad \Phi^{123}\cdot(\Delta 1)\Phi\cdot\Phi^{234}=(\Delta 11)\Phi\,(11\Delta)\Phi$

hex$_\pm$ $\quad (\Delta 1)R^{\pm 1}=\Phi^{123}\cdot(R^{\pm 1})^{23}\cdot(\Phi^{-1})^{132}\cdot(R^{\pm})^{13}\cdot\Phi^{312}$

$\Phi^{321}\Phi=I$

Thm (Drinfeld, Le-Murakami, ..) If R is fixed, all Φ's are conjugate.

Explain "conjugate" by an an-isotropic KZ.

Pf of of thm Fix R, suppose Φ & Φ' sat.

Pent$_\pm$ & hex; assume $\Phi'=\Phi+\varphi$, ...

get $\quad d\varphi=0$ & $\varphi^{321}=-\varphi$

& $\varphi-\varphi^{132}+\varphi^{312}=0$

$\varphi-\varphi^{213}+\varphi^{231}=0$

Try $F=1+f$

$\Phi'=(\Delta\otimes 1)F\cdot F^{12}\Phi\,(F^{23})^{-1}\cdot(1\otimes\Delta)F^{-1}$

$\varphi=df$

$f-f^{21}=0$

Prob: is $H^3_{Harr}(A^{pb})=0$?

Math 273, Possible topics for 2^{nd} semester:

1. More on Vassiliev invariants:

   a. Vassiliev invariants separate braids.

   b. The Milnor invariants are Vassiliev.

   c. $gl(N)$ invariants separate braids.

   d. Something on Graph cohomology.

   e. More on Chern-Simons theory.

   f. The Maclane coherence theorem & other ommissions.

   g. The Melvin-Morton-Rozansky conjecture.

2. The Alexander polynomial.

3. A joint attempt to read Drinfel'd's papers on Quasi-Hopf algebras.

Please prioritize! (or make other suggestions...)

Sergiu Moroianu, Braids & Free groups, 12/16/94.

1. Def of B_n, P_n

2. $0 \to P_n \to B_n \to \Sigma_n \to 0$.

3. σ_i, $\quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1$

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

4. Pf of above, by reduction to P_{n-1} in P_n:

a. $P_n = \langle A_{ij} : 1 \le i < j \le n \mid rels \rangle$

rels $= \{ A_{rs}A_{ij}A_{rs}^{-1} = A_{ij}$ if $r<s<i<j$ ...

[illegible]

Leonid Koregodsky, Ribbon categories & invariants of Links, 12/19/94

Ribbon Categories = $\underbrace{\text{balanced braided}}_{\text{quasi triangular}}$, rigid monoidal (quasi-) tensor categories

Example: Ribbon tangles. (global description, unclear if a normal is specified)

def by generators.

Colors, $*: \sigma \to \sigma$ (σ = set of colors)

Def Monoidal: associative tensor prod. w/ identity

Rigid: dual objects exist, st.

$k \to X \otimes X^*$, $X^* \otimes X \to k$

$k \to {}^*X \otimes X$, $X \otimes {}^*X \to k$, ...

s.t. all usual axioms.

braided: ...

balanced $b_X : {}^*X \to X^*$

Quasi-triangular Hopf algebras, reps, ...