MAT 1301 on 260113-4, H7-9: Classes by Sander Kupers on van Kampen

$$\mathrm{Top}_* \longrightarrow \mathrm{Grp}$$

$$(X, x_0) \longmapsto \pi_1(X, x_0) = \{\text{based homotopy classes of } \gamma : (S^1, 1) \to (X, x_0)\}$$

$+$ concatenation

1) Based homotopy category

def a based homotopy between based maps $f_0, f_1 : (X, x_0) \to (Y, y_0)$ is map $H : X \times [0,1] \to Y$ s.t. (i) $H(x, 0) = f_0(x)$, (ii) $H(x,1) = f_1(x)$, (iii) $H(x_0, t) = y_0$

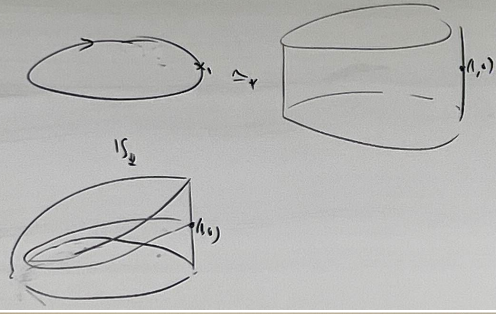
we say f_0, f_1 based homotopic, $f_0 \simeq_* f_1$

def a based map $f : (X, x_0) \to (Y, y_0)$ is a based homotopy equivalence if there is a based map $g : (Y, y_0) \to (X, x_0)$ s.t. $g \circ f \simeq_* \mathrm{id}_X$, $f \circ g \simeq_* \mathrm{id}_Y$

example

$$(S^1, 1) \underset{\mathrm{inc}_0}{\overset{\mathrm{pr}}{\rightleftarrows}} (S^1 \times (-1,1), (1,0))$$

based homotopy equivalences



example $(X, x_0) \simeq_* (X, x_1)$ under mild conditions: (i) x_0, x_1 should be in same path component (ii) HEP wrt points

$$\begin{tikzcd}
\{x\} \times \{0\} \arrow[rr] \arrow[dr] \arrow[dd] & & X \times \{0\} \arrow[dl] \arrow[dd] \\
& X \times [0,1] \arrow[dr, dashed, "\exists"] & \\
\{x\} \times [0,1] \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

$\iff$ $X \times \{0\} \cup \{x\} \times [0,1]$ retract of $X \times [0,1]$

e.g. this holds if X manifold or CW-complex $\underbrace{}_{\text{anything reasonable}}$

Lemma (i) $\simeq_*$ is an equivalence relation on set based maps $(X, x_0) \to (Y, y_0)$

(ii) $f_0 \simeq_* f_1 \Rightarrow g f_0 \simeq_* g f_1$, $f_0 h \simeq_* f_1 h$

def the based homotopy category hoTop_* is category whose objects are based spaces, and whose morphisms are based homotopy classes of based maps

$$\mathrm{Top}_* \longrightarrow \mathrm{hoTop}_*$$

prop f is a based homotopy equivalence $\iff$ its image in hoTop_* is an isomorphism

$$f : (X, x_0) \to (Y, y_0) \rightsquigarrow \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f\gamma]$$

$$\begin{tikzcd}
\mathrm{Top}_* \arrow[r] \arrow[d] & \mathrm{Grp} \\
\mathrm{hoTop}_* \arrow[ur] &
\end{tikzcd}$$

visibly only depends on based homotopy class of f

cor a based homotopy equivalence induces an iso on π_1

2) Seifert-van Kampen theorem

$U \subset X$ open, $V \subset X$ open, $x_0 \in U \cap V$, $x_0 \in U$, $x_0 \in V$, $X \ni x_0$ $\overset{\pi_1}{\rightsquigarrow}$

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[dr] & \\
\pi_1(U \cap V, x_0) \arrow[ur] \arrow[dr] & & \pi_1(X, x_0) \\
& \pi_1(V, x_0) \arrow[ur] &
\end{tikzcd}$$

def $G \xleftarrow{i} K \xrightarrow{j} H$ groups and homomorphisms, the amalgamated free product

$$G *_K H = \langle \bar{g} \in G, \bar{h} \in H \mid \bar{g}_1 \bar{g}_2 = \overline{g_1 g_2}, \bar{h}_1 \bar{h}_2 = \overline{h_1 h_2}, \overline{i(k)} = \overline{j(k)}, \bar{e} = e \rangle$$

ex if $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle$

$H = \langle y_1, \ldots, y_\ell \mid s_1, \ldots, s_n \rangle$

$K =$ generated by $z_1, \ldots, z_p$

$$G *_K H = \langle x_1, \ldots, x_k, y_1, \ldots, y_\ell \mid r_1, \ldots, r_m, s_1, \ldots, s_n, i(z_1) = j(z_1), \ldots, i(z_p) = j(z_p) \rangle$$

ex $K = \{e\}$, $G *_{\{e\}} H =$ free product $G * K$

$\mathbb{Z}^{*k} =$ free group on k generators

$C_2 * C_2 \cong D_\infty = \langle r, s \mid r^2 = e, rsr = s^{-1} \rangle$

$G \xhookrightarrow{inc_G} G * H \xhookleftarrow{inc_H} H$

$$\left\{ \begin{array}{c} \text{homomorphisms} \\ F: G*H \to L \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{pair of homomorphisms} \\ f: G \to L \\ f': H \to L \end{array} \right\}$$

$$F \longmapsto (F \circ inc_G, F \circ inc_H)$$

This is literally saying $G \xhookrightarrow{inc_G} G*H \hookleftarrow inc_H$ exhibit $G*H$ as the coproduct in category Grp of groups

what is a coproduct?

$\mathcal{C}$, $X, Y \in ob\,\mathcal{C}$

their coproduct $X \xrightarrow{inc_X} X \sqcup Y \xleftarrow{inc_Y} Y$

s.t.

$$(*) \quad \left\{ \begin{array}{c} \text{morphisms} \\ X \sqcup Y \to Z \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{pairs of morphisms} \\ X \to Z,\ Y \to Z \end{array} \right\}$$

$$F \longmapsto (f \circ inc_X, f \circ inc_Y)$$

coproducts are unique up to iso

$$X \xrightarrow{inc'_X} (X \sqcup Y)' \xleftarrow{inc'_Y} Y$$

$(*) \rightsquigarrow X \sqcup Y \to (X \sqcup Y)'$

$(X \sqcup Y)' \to X \sqcup Y$

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "j"] & G \arrow[d, "inc_G"] \\ H \arrow[r, "inc_H"] & G *_K H \end{tikzcd}$$

this exhibits $G *_K H$ as the pushout in Grp of

$$\begin{tikzcd} & G \arrow[d] \arrow[ddr, bend left] & & K \arrow[r, "i"] \arrow[d, "j"] & G \arrow[d] \arrow[ddr, bend left] & \\ H \arrow[r] \arrow[drr, bend right] & G * H \arrow[dr, dashed, "\exists!"] & \text{compare} & H \arrow[r] \arrow[drr, bend right] & G *_K H \arrow[dr, dashed, "\exists!"] & \\ & & L & & & L \end{tikzcd}$$

$$\begin{tikzcd} & U \arrow[dr, hook] & \\ U \cap V \arrow[ur] \arrow[dr] & & X \\ x_0 & V \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left] & \\ \pi_1(U \cap V, x_0) \arrow[ur] \arrow[dr] & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \arrow[r, dashed, "\exists!"] & \pi_1(X, x_0) \\ & \pi_1(V, x_0) \arrow[u] \arrow[ur, bend right] & \end{tikzcd}$$

thm (Seifert-van Kampen) if $U, V \subset X$ open, $X = U \cup V$, $U \cap V$ path-connected, $x_0 \in U \cap V$ then

$$\pi_1(U, x_0) \underset{\pi_1(U \cap V, x_0)}{*} \pi_1(V, x_0) \xrightarrow[\cong]{} \pi_1(X, x_0)$$

i.e. $\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$ is a pushout in Grp

examples (i) (wedge sums)

(X, x_0) $\quad (Y, y_0)$ $\quad (X \sqcup Y) / {x_0 \sim y_0} =: X \vee Y$

z_0

want: $\pi_1(X \vee Y, z_0)$

$\cong \pi_1(X, x_0) * \pi_1(Y, y_0)$

true if $U' \subset Y$ open nbhd of y_0 that deformation retracts onto y_0

$V' \subset X$ $\quad x_0$

open $U = X \cup U' \simeq_* X$ $\qquad U \cap V = U' \vee V' \simeq_* *$

open $V = V' \cup Y \simeq_* Y$

$\rightsquigarrow$ SvK $\Rightarrow \pi_1(X, x) * \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(X \vee Y, z)$

$X = Y = S^1$  $\pi_1(\bigvee_k S^1) \cong F_k$ free group on k generators

Seifert-van Kampen II

Examples

Proof

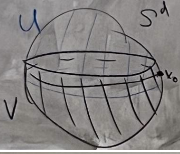
thm (S-vK) $U, V \subseteq X$ open, $U \cup V = X$, $U \cap V$ path-connected, $x_0 \in U \cap V$

$$\pi_1(U, x_0) \underset{\pi_1(U \cap V, x_0)}{*} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

example (spheres)

$\pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(S^d) \cong \{e\}$ $\quad d \geq 2$



$$\{e\} \underset{\mathbb{Z}}{*} \{e\} \xrightarrow{\cong} \pi_1(S^d)$$

$\underbrace{\qquad}_{\cong \{e\}}$

$U \simeq *$

$V \simeq *$

$U \cap V \simeq S^{d-1}$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \{e\} \arrow[d] \\ \{e\} \arrow[r] & \{e\} *_{\mathbb{Z}} \{e\} \end{tikzcd}$$

pattern

U, V simply-connected

$\Longrightarrow X$ simply-connected

$\mathbb{C}P^1 \cong S^2$ $\qquad \mathbb{C}P^n$ has an open cover $U \simeq \mathbb{C}P^{n-1}$, $V \simeq *$, $U \cap V \simeq S^{2n-1}$

$\mathbb{C}P^n$ simply-connected $\Longleftarrow$

example (complements of points)

$X \supset W \ni w_0$

path-cntd, open, $W \cong \mathbb{R}^d$, $d \geq 3$

$w_0 \neq x_0 \in X$

$\pi_1(X \setminus w_0) = ?$

Consider X covered by:

$U = X \setminus w_0$, $V = W \simeq *$, $U \cap V \cong \mathbb{R}^d \setminus pt \simeq S^{d-1}$

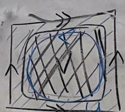
$$\pi_1(X \setminus w_0) * \{e\} \xrightarrow{\cong} \pi_1(X)$$

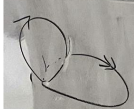
$\{e\}$ simply connected

$$\begin{tikzcd} \pi_1(X\setminus w_0) * \{e\} \arrow[d, phantom, "\cong"] \\ \pi_1(X\setminus w_0) \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r, "\cong"] \arrow[d] & \pi_1(U) \arrow[d] \arrow[ddr] & \\ \pi_1(V) \arrow[r, "\cong"] \arrow[drr, bend right] & \pi_1(X) \arrow[ul, phantom, "po"] & \\ & & G \end{tikzcd}$$

example (surfaces)

 $= T^2$



$U \simeq S^1 \vee S^1$

$V \simeq *$

$U \cap V \simeq S^1$

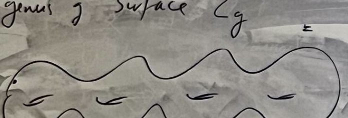
$\langle x, y \mid [x,y] = e \rangle \cong \mathbb{Z}^2$

$$\begin{tikzcd} \langle x, y \mid [x,y] = e \rangle \arrow[d, phantom, "\cong"] \\ F_2 *_{\mathbb{Z}} \{e\} \xrightarrow{\cong} \pi_1(T^2) \end{tikzcd}$$

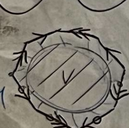
$$\begin{tikzcd} 1 \in \mathbb{Z} \arrow[r] \arrow[d] & \{e\} \arrow[d] \\ xyx^{-1}y^{-1} \in F_2 \arrow[r] \arrow[ur, phantom, "po"] & \pi_1(T^2) \end{tikzcd}$$

$xyx^{-1}y^{-1} = [x,y]$

genus g surface Σ_g (*)



$g = 4$



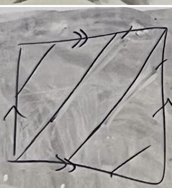
$2g$-gon

$U \simeq \bigvee_{2g} S^1$

$V \simeq *$

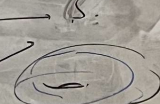
$U \cap V \simeq S^1$

$\langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] = e \rangle \cong \pi_1(\Sigma_g)$



$T^2 = \mathbb{R}^2/\mathbb{Z}^2$ $(px, qx) \hookrightarrow S^3$

$S^1 = \mathbb{R}/\mathbb{Z} \ni x$



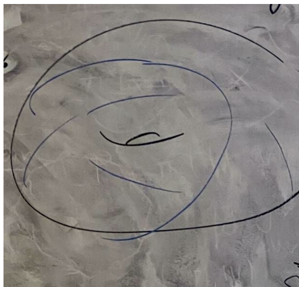
(p,q) torus knot

$p, q \geq 1$ $\gcd(p,q)$

e.g. $p = 2$, $q = 3$

$\pi_1(S^3 \setminus K_{p,q})$

trefoil for $(p,q) = (2,3)$



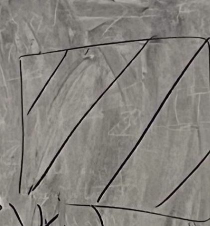
$S^3 \setminus T^2$ = interior solid torus $\cup$ exterior solid torus

$S^3 \setminus K_{p,q}$ = 2 solid tori $\cup$ thickened strip in between strand

U = interior solid torus $\cup$ strip $\simeq S^1$

V = exterior solid torus $\cup$ strip $\simeq S^1$

$U \cap V$ = strip $\simeq S^1$



$$\underbrace{\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}}_{\text{what is this?}} \xrightarrow{\cong} \pi_1(S^3 \setminus K_{p,q})$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1 \mapsto 3"] \arrow[d, "1 \mapsto 2"'] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

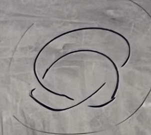
$$\underbrace{\langle x, y \mid x^p = y^q \rangle}$$

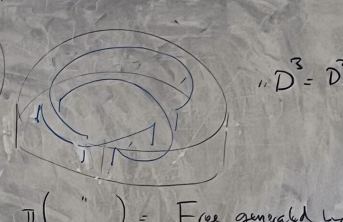
often not $\cong \mathbb{Z} = \pi_1(S^3 \setminus \text{unknot})$

so $K_{p,q}$ is not unknot

example (Wirtinger presentation)

$K \subset \mathbb{R}^3$



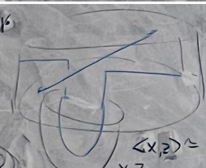


$D^3 = D^2 \times [0,1]$

$\pi_1(\mathbb{R}^3 \setminus K) =$

$\langle$ free group on arcs $\mid$ single relation for every crossing $\rangle$

$\pi_1(D^3 \text{ " }) =$ Free generated by arcs



for each crossing we glue V = cup / arc along $U \cap V$

$z = xy$

$$\begin{tikzcd} \langle x, z \rangle \cong \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \mathbb{Z} \cong \pi_1(V) \arrow[r] \arrow[ur, phantom, "po"] & \pi_1(U \cup V) \end{tikzcd}$$

$x \mapsto 1$, $z \mapsto 0$

$\Rightarrow \pi_1(U \cup V) = \pi_1(U) /$ normal subgroup gen by image of z

Surjective?

$\gamma: [0,1] \to X$ — land in U

(1) Subdivide

(2) Designate

(3) String to base

(4) Read off

(1) By Lebesgue number lemma $\exists N \gg 0$ s.t. $\gamma|[\frac{i}{2^N}, \frac{i+1}{2^N}]$ has image in U or V (or both)

γ_i

pick once and for all η_x path from x to x_0 s.t. if $x \in U \cap V$ (or U or V) image $\eta_x \subset U \cap V$ (or U or V) w/ η_{x_0} constant

(2) assign each to U or V depending on whether γ_i has image in U or V

(3) $x_i = \gamma(i/2^N)$

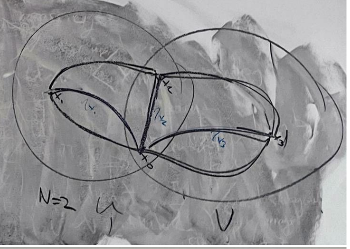
homotope

$\gamma = \gamma_0 * \gamma_1 * \cdots * \gamma_{2^N-1}$

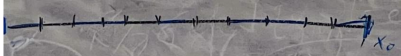
to $\gamma = \gamma_0' * \gamma_1' * \cdots * \gamma'_{2^N-1}$

where $\gamma_i' = \eta_{x_i}^{-1} * \gamma_i * \eta_{x_{i+1}}$

loop from x_0 to x_0 completely in U or V



$\gamma : [0,1] \to X$ — kind in U



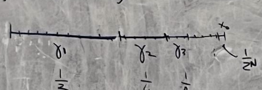
Read off: think of γ_i' as image of element of $\pi_1(U)$, $\pi_1(V)$ according to designation ⇝ element of $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$ mapping to $[\gamma]$

assign each to U or V (or both) depending on whether γ_i has image in U or V

(1) Subdivide
(2) Designate
(3) String to base
(4) Read off

Ψ

for injectivity: for particular choices

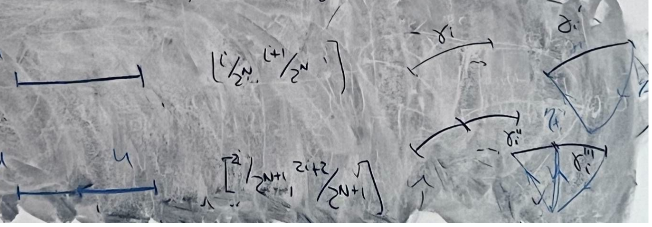


$\Psi(\Psi($ string in amalgamated free product

" string in amalgamated free product

→ choices (a) N
(b) designation
(c) representative γ of $[\gamma]$

(a) N ⇝ $N+1$



get two loops instead of one

(1) Subdivide
(2) Designate
(3) String to base
(4) Read off

Ψ

$[\gamma_i''][\gamma_i'''] \in \pi_1(U)$

$= [\gamma_i'' * \gamma_i'''] = [\gamma_i']$

$\sim \gamma_i'$

designate $[\gamma_i'] \in \pi_1(U)$

(b) only choice in $\mathrm{im}(\gamma_i) \subset U \cap V$ ⇝ γ_i' loop in $U\cap V$

$[\gamma_i']$ and $[\gamma_i']$ $\in \pi_1(U)$ $\in \pi_1(V)$

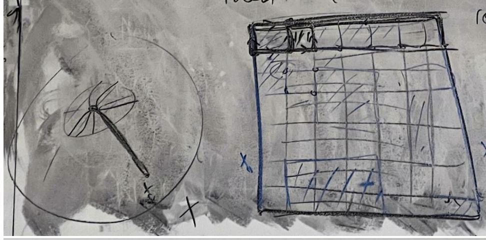
are identified due to relations from $\pi_1(U\cap V)$

(c) $H : [0,1]\times[0,1] \to X$

based homotopy from γ_0 to γ_1

idea: (1)–(4) but now for square

read off: $\Psi(\gamma_1)$



read off $\Psi(\gamma_0)$

Seifert-van Kampen II

Examples

thm (S-vK) $X \ni x_0$

Proof

$\{U_i\}_{i\in I}$ open cover $x_0 \in U_i \ \forall i \in I$

$U_i \cap U_j$, $U_i\cap U_j \cap U_k$ path-connected

$$\pi_1(X) \cong \mathrm{colim}\left(\begin{array}{ccc} \pi_1(U_i) & & \pi_1(U_j) \\ & \pi_1(U_i\cap U_j) & \end{array} \right)$$

$\Pi(X)$ fundamental groupoid

category ob $x \in X$

mor from x to x' are homotopy classes of paths from x to x'

composition: concatenation

(i) for any U, V open cover

$$\begin{array}{ccc} \Pi(U\cap V) & \longrightarrow & \Pi(U) \\ \downarrow & & \downarrow \\ \Pi(V) & \longrightarrow & \Pi(X) \end{array}$$

(ii) if X is path-connected

$\pi_1(X) \hookrightarrow \Pi(X)$

$\simeq$ equiv of cats