Homework Assignment 7

Hocking and Young "Topology"

4–6] **KNOTS AND RELATED IMBEDDING PROBLEMS** **177**

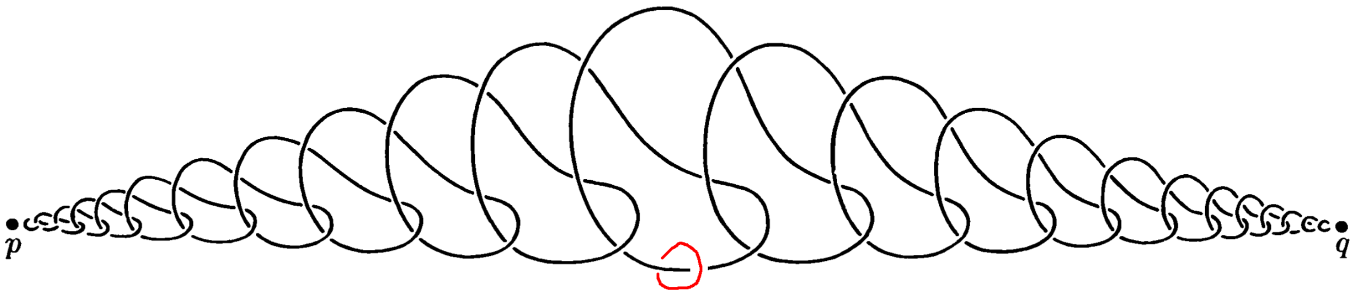

FIG. 4–12. A simple arc in E^3 whose complement is not simply connected.

Problem 1. On page 177 of their topology textbook, Hocking and Young display an embedded interval in $\mathbb{R}^3$ whose complement X is not simply connected. I took the liberty of adding a little red circle to the picture, which represents a class γ in $H_1(X)$. But by a theorem from class, $H_1(X) = 0$, so γ must be the boundary of some 2D object β in X. Draw it!

If you need scratch paper, I've left multiple paper copies of the above picture in an envelope near my office door (Bahen 6178). Feel free to take some (yet leave some for others).

Not for credit, ponder the following: Everything we did in class was in-principle constructive: the prizm construction, barycentric subdivisions, the long exact sequence of a short exact sequence, etc. How exactly did these relatively benign constructions "discover" the relatively sophisticated surface that you must have discovered when you answered this problem?

Problem 2. Search your memories and I'm sure you can go back to these times when you were lying in a crib looking up at a baby mobile, a lovely toy such as in the picture on the right. Little did you expect that twenty-something years later baby mobiles will come back to haunt you in an algebraic topology homework assignment.

https://www.canadiantire.ca/en/pdp/tiny-love-tiny-princess-soothe-n-groove-mobile-2744075p.html

If (X_i, x_i) are connected based topological spaces for $i = 1, \ldots, n$, we let $BM((X_i, x_i))$ be the topological space obtained by connecting each of the X_i's by a string to some central point y_0. In formulas, let Y be a star-shaped tree with centre y_0 and leafs $y_1, \ldots, y_n$, and let

$$BM((X_i, x_i)) \coloneqq (Y \sqcup X_1 \sqcup \cdots \sqcup X_n) \,/ (\forall i \; x_i \sim y_i).$$

Using the Mayer-Vietoris sequence and/or whatever else we studied, compute the homology of $BM((X_i, x_i))$ in terms of the homologies of the individual X_i's.

Problem 3. The suspension ΣX of a topological space X is X multiplied by an interval, with the top and the bottom sides crashed into points S and N (that are not in X):

$$\Sigma X \coloneqq (X \times [-1, 1] \sqcup \{S, N\}) \,/ (\forall x \; (x, 1) \sim S, \; (x, -1) \sim N).$$

1. (0 points) Identify the colonial roots of the discomfort you felt regarding the choice of directions, signs, and poles used in this definition.

2. Using the same tools as in Problem 2, compute the homology of ΣX in terms of the homology of X.

Problem 4.

1. Compute the homology groups of the torus $T^2 = S^1 \times S^1$.

2. (Hatcher's problem 28a on page 157). Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus T^2 by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

Problem 5.

1. Formulate and prove a naturality property for the Mayer-Vietoris sequence. Your property must be at least strong enough to answer part 2 of this question.

2. Use part 1 of this question to prove that if $f \colon S^n \to S^n$ then $\deg(f) = \deg(\Sigma f)$ where Σ is the suspension functor, mentioned previously both in class and in HW7.

Problem 6. Suppose n is even.

1. Show that for any continuous map $f \colon S^n \to S^n$ there is a point x such that $f(x) = \pm x$.

2. Show that any continuous map $f \colon \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point.