Pensieve header: A guest class in Lisa Jeffrey's UTSC reading course (edited).

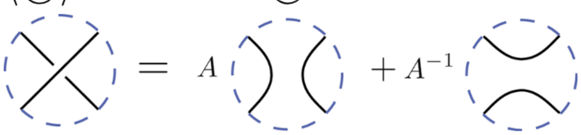
In[∘]:=
```
BarcodeImage["https://drorbn.net/AcademicPensieve/Classes/25-GuestAtUTSC"]
```

Out[∘]=

From https://en.wikipedia.org/wiki/Bracket_polynomial:

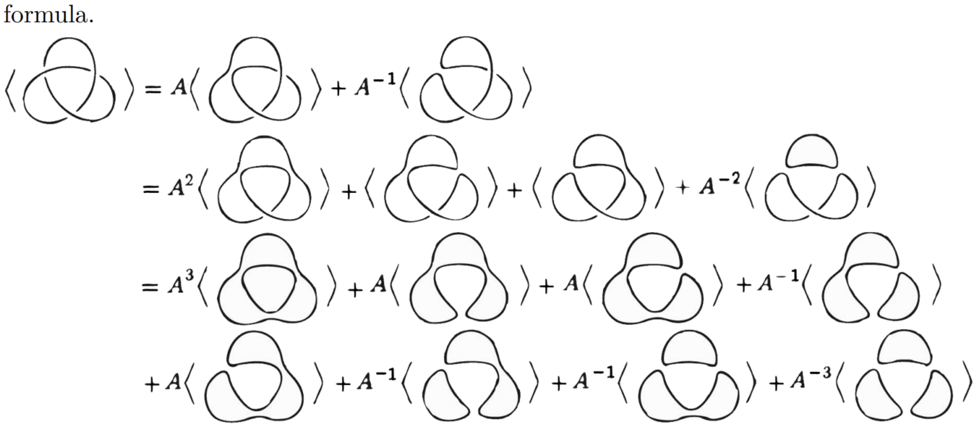
Definition [edit]

The bracket polynomial of any (unoriented) link diagram L, denoted $\langle L \rangle$, is a polynomial in the variable A, characterized by the three rules:

- $\langle \bigcirc \rangle = 1$, where $\bigcirc$ is the standard diagram of the unknot
- 
- $\langle \bigcirc \sqcup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$

From https://www.math.uni-hamburg.de/home/runkel/Material/SS20/L1hand.pdf:

Example 3.2. Let D be the canonical diagram of the Trefoil knot. We calculat the Kaufmann bracket of the Trefoil knot $\langle D \rangle$. At first resolve all crossings with the first formula.



Since a diagram without crossings is the disjoint union of loops we obtail the value of the bracket recursively by (II) and (III).

$$\langle D \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7}).$$

In[∘]:=
```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
PD[Knot[3, 1]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
```
PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]
```

In[∘]:=
```
PD[Mirror@Knot[3, 1]]
```

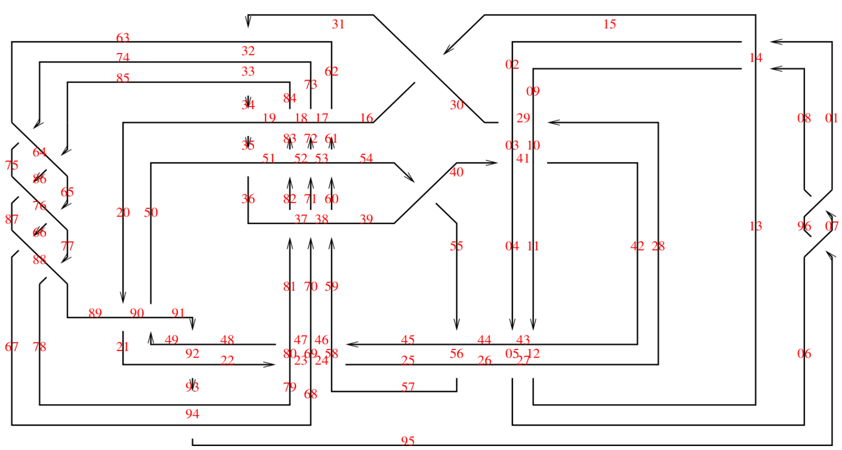
Out[∘]=
```
PD[X[4, 2, 5, 1], X[6, 4, 1, 3], X[2, 6, 3, 5]]
```

In[∘]:=
```
PD[Mirror@Knot[3, 1]] /. i_Integer :> If[i < 6, i + 1, 1]
```

Out[∘]=
```
PD[X[5, 3, 6, 2], X[1, 5, 2, 4], X[3, 1, 4, 6]]
```

The 48-crossing Gompf-Scharlemann-Thompson knot [GST] is significant because it may be a counterexample to the slice-ribbon conjecture:

Gompf Scharlemann Thompson



[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

In[∘]:=
```
PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

In[∘]:=
```
SetAttributes[p, Orderless]
```

In[∘]:=
```
pd = PD[Knot[3, 1]]
```

Out[∘]=

PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]]

In[∘]:=
```
t1 = pd /. X[i_, j_, k_, l_] :→ A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k]
```

Out[∘]=

$$\mathrm{PD}\left[A\,p[1,4]\,p[2,5]+\frac{p[1,5]\,p[4,2]}{A},\ A\,p[3,6]\,p[4,1]+\frac{p[3,1]\,p[6,4]}{A},\ \frac{p[2,6]\,p[5,3]}{A}+A\,p[5,2]\,p[6,3]\right]$$

In[∘]:=
```
t2 = Expand[Times @@ t1]
```

Out[∘]=

$$A\,p[1,4]\,p[2,5]\,p[2,6]\,p[3,6]\,p[4,1]\,p[5,3]+\frac{p[1,5]\,p[2,6]\,p[3,6]\,p[4,1]\,p[4,2]\,p[5,3]}{A}+$$
$$A^3\,p[1,4]\,p[2,5]\,p[3,6]\,p[4,1]\,p[5,2]\,p[6,3]+A\,p[1,5]\,p[3,6]\,p[4,1]\,p[4,2]\,p[5,2]\,p[6,3]+$$
$$\frac{p[1,4]\,p[2,5]\,p[2,6]\,p[3,1]\,p[5,3]\,p[6,4]}{A}+\frac{p[1,5]\,p[2,6]\,p[3,1]\,p[4,2]\,p[5,3]\,p[6,4]}{A^3}+$$
$$A\,p[1,4]\,p[2,5]\,p[3,1]\,p[5,2]\,p[6,3]\,p[6,4]+\frac{p[1,5]\,p[3,1]\,p[4,2]\,p[5,2]\,p[6,3]\,p[6,4]}{A}$$

In[•]:=
```
t3 = t2 //. p[i_, j_] p[j_, k_] :> p[i, k]
```
Out[•]=

$$\frac{p[2,4]^2}{A} + A\, p[1,1]\, p[2,6]^2 + A^3\, p[1,1]\, p[2,2]\, p[3,3] + A\, p[3,3]\, p[4,2]^2 + \frac{p[3,3]\, p[4,4]}{A^3} + \frac{p[4,6]^2}{A} + \frac{p[6,2]^2}{A} + A\, p[2,2]\, p[6,4]^2$$

In[•]:=
```
Expand[t3 /. {p[i_, i_] :> -A^2 - A^-2, p[i_, j_]^2 -> -A^2 - A^-2}]
```
Out[•]=

$$\frac{1}{A^7} + \frac{1}{A^3} + A - A^9$$

In[•]:=
```
pd = PD[Knot[8, 17]]
```
Out[•]=

```
PD[X[6, 2, 7, 1], X[14, 8, 15, 7], X[8, 3, 9, 4], X[2, 13, 3, 14],
 X[12, 5, 13, 6], X[4, 9, 5, 10], X[16, 12, 1, 11], X[10, 16, 11, 15]]
```

In[•]:=
```
t1 = pd /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k]
```
Out[•]=

$$\mathrm{PD}\Big[A\, p[1,7]\, p[2,6] + \frac{p[1,6]\, p[2,7]}{A},\ A\, p[7,15]\, p[8,14] + \frac{p[7,14]\, p[8,15]}{A},\ \frac{p[3,9]\, p[4,8]}{A} + A\, p[3,8]\, p[4,9],\ \frac{p[2,14]\, p[3,13]}{A} + A\, p[2,13]\, p[3,14],\ \frac{p[5,13]\, p[6,12]}{A} + A\, p[5,12]\, p[6,13],\ \frac{p[4,10]\, p[5,9]}{A} + A\, p[4,9]\, p[5,10],\ \frac{p[1,12]\, p[11,16]}{A} + A\, p[1,11]\, p[12,16],\ A\, p[10,16]\, p[11,15] + \frac{p[10,15]\, p[11,16]}{A}\Big]$$

In[•]:=
```
t2 = Expand[Times @@ t1]
```
Out[•]=

$$\frac{p[2,7]\, p[2,13]\, p[3,9]\, p[3,14]\, p[4,9]\, p[5,10]\, p[5,13]\, p[\cdots 1 \cdots]\ \cdots 1 \cdots\ p[8,15]\, p[10,16]\, p[11,15]\, p[12,1]\, p[12,6]\, p[14,7]\, p[16,11]}{A^2} + \cdots 254 \cdots + A^2\, p[1,11]\, p[2,14]\, p[4,10]\, p[6,2]\, p[7,1]\, p[8,3]\ \cdots 3 \cdots\ p[12,5]\, p[13,3]\, p[13,6]\, p[14,8]\, p[15,7]\, p[16,11]\, p[16,12]$$

Full expression not available (original memory size: 419 kB)

In[•]:=
```
t3 = t2 //. p[i_, j_] p[j_, k_] :> p[i, k]
```
Out[•]=

$$\frac{p[6,12]^2\, p[7,14]^2\, p[9,13]^2\, p[10,15]^2\, p[11,16]^2}{A^8} + \frac{p[7,14]^2\, p[10,15]^2\, p[11,16]^2\, p[12,13]^2}{A^6} + \frac{p[6,13]^2\, p[10,15]^2\, p[11,16]^2\, p[12,14]^2}{A^2} + \frac{p[9,13]^2\, p[10,15]^2\, p[11,16]^2\, p[12,14]^2}{A^6} + A^2\, p[6,13]^2\, p[8,14]^2\, p[11,16]^2\, p[12,15]^2 + A^4\, p[4,9]^2\, p[6,13]^2\, p[8,14]^2\, p[11,16]^2\, p[12,15]^2 + A^2\, p[6,13]^2\, p[9,14]^2\, p[11,16]^2\, p[12,15]^2 +$$

$$\frac{p[7, 14]^2 p[9, 13]^2 p[10, 15]^2 p[12, 16]^2}{A^6} + A^6 p[6, 13]^2 p[8, 14]^2 p[11, 15]^2 p[12, 16]^2 +$$
$$A^8 p[4, 9]^2 p[6, 13]^2 p[8, 14]^2 p[11, 15]^2 p[12, 16]^2 +$$
$$A^6 p[6, 13]^2 p[9, 14]^2 p[11, 15]^2 p[12, 16]^2 + \frac{3 p[10, 15]^2 p[11, 16]^2 p[13, 14]^2}{A^4} +$$
$$\frac{p[6, 12]^2 p[10, 15]^2 p[11, 16]^2 p[13, 14]^2}{A^6} + 2 p[11, 16]^2 p[12, 15]^2 p[13, 14]^2 +$$
$$A^2 p[4, 9]^2 p[11, 16]^2 p[12, 15]^2 p[13, 14]^2 + \frac{p[10, 15]^2 p[12, 16]^2 p[13, 14]^2}{A^4} +$$
$$2 A^4 p[11, 15]^2 p[12, 16]^2 p[13, 14]^2 + A^6 p[4, 9]^2 p[11, 15]^2 p[12, 16]^2 p[13, 14]^2 +$$
$$\frac{2 p[7, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^4} + \frac{p[4, 9]^2 p[7, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^2} +$$
$$\frac{2 p[6, 12]^2 p[7, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^6} + \frac{p[4, 9]^2 p[6, 12]^2 p[7, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^4} +$$
$$2 p[8, 14]^2 p[11, 16]^2 p[13, 15]^2 + 2 A^2 p[4, 9]^2 p[8, 14]^2 p[11, 16]^2 p[13, 15]^2 +$$
$$\frac{p[6, 12]^2 p[8, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^2} + p[4, 9]^2 p[6, 12]^2 p[8, 14]^2 p[11, 16]^2 p[13, 15]^2 +$$
$$2 p[9, 14]^2 p[11, 16]^2 p[13, 15]^2 + \frac{p[6, 12]^2 p[9, 14]^2 p[11, 16]^2 p[13, 15]^2}{A^2} +$$
$$\frac{2 p[11, 16]^2 p[12, 14]^2 p[13, 15]^2}{A^4} + \frac{p[4, 9]^2 p[11, 16]^2 p[12, 14]^2 p[13, 15]^2}{A^2} +$$
$$2 p[7, 14]^2 p[12, 16]^2 p[13, 15]^2 + \frac{2 p[7, 14]^2 p[12, 16]^2 p[13, 15]^2}{A^4} +$$
$$\frac{p[4, 9]^2 p[7, 14]^2 p[12, 16]^2 p[13, 15]^2}{A^2} + A^2 p[4, 9]^2 p[7, 14]^2 p[12, 16]^2 p[13, 15]^2 +$$
$$p[8, 14]^2 p[12, 16]^2 p[13, 15]^2 + A^4 p[8, 14]^2 p[12, 16]^2 p[13, 15]^2 +$$
$$A^2 p[4, 9]^2 p[8, 14]^2 p[12, 16]^2 p[13, 15]^2 + A^6 p[4, 9]^2 p[8, 14]^2 p[12, 16]^2 p[13, 15]^2 +$$
$$p[9, 14]^2 p[12, 16]^2 p[13, 15]^2 + A^4 p[9, 14]^2 p[12, 16]^2 p[13, 15]^2 +$$
$$\frac{p[7, 14]^2 p[10, 15]^2 p[13, 16]^2}{A^4} + A^4 p[8, 14]^2 p[11, 15]^2 p[13, 16]^2 +$$
$$A^6 p[4, 9]^2 p[8, 14]^2 p[11, 15]^2 p[13, 16]^2 + A^4 p[9, 14]^2 p[11, 15]^2 p[13, 16]^2 +$$
$$\frac{13 p[11, 16]^2 p[14, 15]^2}{A^2} + 5 p[4, 9]^2 p[11, 16]^2 p[14, 15]^2 + \frac{5 p[6, 12]^2 p[11, 16]^2 p[14, 15]^2}{A^4} +$$
$$\frac{2 p[4, 9]^2 p[6, 12]^2 p[11, 16]^2 p[14, 15]^2}{A^2} + 3 p[6, 13]^2 p[11, 16]^2 p[14, 15]^2 +$$
$$A^2 p[4, 9]^2 p[6, 13]^2 p[11, 16]^2 p[14, 15]^2 + \frac{p[9, 13]^2 p[11, 16]^2 p[14, 15]^2}{A^4} +$$
$$\frac{p[6, 12]^2 p[9, 13]^2 p[11, 16]^2 p[14, 15]^2}{A^6} + \frac{p[11, 16]^2 p[12, 13]^2 p[14, 15]^2}{A^4} +$$
$$\frac{5 p[12, 16]^2 p[14, 15]^2}{A^2} + 6 A^2 p[12, 16]^2 p[14, 15]^2 + 2 p[4, 9]^2 p[12, 16]^2 p[14, 15]^2 +$$

$3 A^4 p[4, 9]^2 p[12, 16]^2 p[14, 15]^2 + 2 A^4 p[6, 13]^2 p[12, 16]^2 p[14, 15]^2 +$
$A^6 p[4, 9]^2 p[6, 13]^2 p[12, 16]^2 p[14, 15]^2 + \frac{p[9, 13]^2 p[12, 16]^2 p[14, 15]^2}{A^4} +$
$\frac{p[13, 16]^2 p[14, 15]^2}{A^2} + 2 A^2 p[13, 16]^2 p[14, 15]^2 + A^4 p[4, 9]^2 p[13, 16]^2 p[14, 15]^2 +$
$\frac{3 p[10, 15]^2 p[14, 16]^2}{A^2} + p[6, 13]^2 p[10, 15]^2 p[14, 16]^2 + \frac{p[9, 13]^2 p[10, 15]^2 p[14, 16]^2}{A^4} +$
$4 A^2 p[11, 15]^2 p[14, 16]^2 + A^4 p[4, 9]^2 p[11, 15]^2 p[14, 16]^2 + A^4 p[6, 13]^2 p[11, 15]^2 p[14, 16]^2 +$
$p[9, 13]^2 p[11, 15]^2 p[14, 16]^2 + \frac{2 p[13, 15]^2 p[14, 16]^2}{A^2} + A^2 p[13, 15]^2 p[14, 16]^2 +$
$p[4, 9]^2 p[13, 15]^2 p[14, 16]^2 + 37 p[15, 16]^2 + 13 A^2 p[4, 9]^2 p[15, 16]^2 +$
$\frac{5 p[6, 12]^2 p[15, 16]^2}{A^2} + 2 p[4, 9]^2 p[6, 12]^2 p[15, 16]^2 + 7 A^2 p[6, 13]^2 p[15, 16]^2 +$
$2 A^4 p[4, 9]^2 p[6, 13]^2 p[15, 16]^2 + \frac{7 p[7, 14]^2 p[15, 16]^2}{A^2} + 3 p[4, 9]^2 p[7, 14]^2 p[15, 16]^2 +$
$\frac{2 p[6, 12]^2 p[7, 14]^2 p[15, 16]^2}{A^4} + \frac{p[4, 9]^2 p[6, 12]^2 p[7, 14]^2 p[15, 16]^2}{A^2} +$
$5 A^2 p[8, 14]^2 p[15, 16]^2 + 5 A^4 p[4, 9]^2 p[8, 14]^2 p[15, 16]^2 +$
$p[6, 12]^2 p[8, 14]^2 p[15, 16]^2 + A^2 p[4, 9]^2 p[6, 12]^2 p[8, 14]^2 p[15, 16]^2 +$
$2 A^4 p[6, 13]^2 p[8, 14]^2 p[15, 16]^2 + 2 A^6 p[4, 9]^2 p[6, 13]^2 p[8, 14]^2 p[15, 16]^2 +$
$\frac{4 p[9, 13]^2 p[15, 16]^2}{A^2} + \frac{p[6, 12]^2 p[9, 13]^2 p[15, 16]^2}{A^4} + \frac{p[7, 14]^2 p[9, 13]^2 p[15, 16]^2}{A^4} +$
$\frac{p[6, 12]^2 p[7, 14]^2 p[9, 13]^2 p[15, 16]^2}{A^6} + 5 A^2 p[9, 14]^2 p[15, 16]^2 +$
$p[6, 12]^2 p[9, 14]^2 p[15, 16]^2 + 2 A^4 p[6, 13]^2 p[9, 14]^2 p[15, 16]^2 + \frac{p[12, 13]^2 p[15, 16]^2}{A^2} +$
$\frac{p[7, 14]^2 p[12, 13]^2 p[15, 16]^2}{A^4} + \frac{2 p[12, 14]^2 p[15, 16]^2}{A^2} + p[4, 9]^2 p[12, 14]^2 p[15, 16]^2 +$
$p[6, 13]^2 p[12, 14]^2 p[15, 16]^2 + \frac{p[9, 13]^2 p[12, 14]^2 p[15, 16]^2}{A^4} + \frac{4 p[13, 14]^2 p[15, 16]^2}{A^2} +$
$4 A^2 p[13, 14]^2 p[15, 16]^2 + 2 A^4 p[4, 9]^2 p[13, 14]^2 p[15, 16]^2 + \frac{p[6, 12]^2 p[13, 14]^2 p[15, 16]^2}{A^4}$

In[∘]:= `Expand[t3 /. {p[i_, i_] :> -A^2 - A^-2, p[i_, j_]^2 -> -A^2 - A^-2}]`

Out[∘]=

$$-\frac{1}{A^{18}} + \frac{2}{A^{14}} - \frac{2}{A^{10}} + \frac{1}{A^6} - \frac{1}{A^2} - A^2 + A^6 - 2 A^{10} + 2 A^{14} - A^{18}$$

In[∘]:=
```
KB1[K_] := Module[{pd, t1, t2, t3},
  pd = PD[K];
  t1 = pd /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k];
  t2 = Expand[Times @@ t1];
  t3 = t2 //. p[i_, j_] p[j_, k_] :> p[i, k];
  Expand[t3 /. {p[i_, i_] :> -A^2 - A^-2, p[i_, j_]^2 -> -A^2 - A^-2}]
 ]
```

In[∘]:=
```
KB1[Knot[10, 165]]
```

Out[∘]=

$$-1-\frac{1}{A^{20}}+\frac{2}{A^{16}}-\frac{1}{A^{12}}+\frac{2}{A^{8}}-\frac{1}{A^{4}}-2A^{8}+2A^{12}-2A^{16}$$

```
(*KB1[GST48]*)
```

Out[∘]=

```
$Aborted
```

In[∘]:=
```
pd = PD[Knot[4, 1]]
```

Out[∘]=

```
PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]]
```

In[∘]:=
```
front = {}; kb = 1; todo = List @@ pd
```

Out[∘]=

```
{X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]}
```

In[∘]:=
```
v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}]
```

In[∘]:=
```
v /@ todo
```

Out[∘]=

```
{0, 0, 0, 0}
```

In[∘]:=
```
?MaximalBy
```

Out[∘]=

Symbol

MaximalBy[*data*, f] returns a list of the elements e_i of *data* for which the value of $f[e_i]$ is maximal.

MaximalBy[*data*, f, n] returns a list of the elements e_i of *data* corresponding to the n largest $f[e_i]$.

MaximalBy[*data*, f, n, p] uses the ordering function p for sorting.

MaximalBy[f] represents an operator form of MaximalBy that can be applied to an expression.

In[∘]:=
```
MaximalBy[todo, v]
```

Out[∘]=

```
{X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]}
```

In[∘]:=
```
c = RandomChoice[MaximalBy[todo, v]]
```

Out[∘]=

```
X[2, 7, 3, 8]
```

```
In[*]:= t1 = Expand[kb * (c /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k])]
```

Out[*]=

$$\frac{p[2,8]\,p[3,7]}{A} + A\,p[2,7]\,p[3,8]$$

```
In[*]:= t2 = t1 //. p[i_, j_] p[j_, k_] :> p[i, k]
```

Out[*]=

$$\frac{p[2,8]\,p[3,7]}{A} + A\,p[2,7]\,p[3,8]$$

```
In[*]:= kb = Expand[t2 /. {p[i_, i_] :> -A^2 - A^-2, p[i_, j_]^2 -> -A^2 - A^-2}]
```

Out[*]=

$$\frac{p[2,8]\,p[3,7]}{A} + A\,p[2,7]\,p[3,8]$$

```
In[*]:= todo = DeleteCases[todo, c]
```

Out[*]=

```
{X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4]}
```

```
In[*]:= front = front ⋃ List @@ c
```

Out[*]=

```
{2, 3, 7, 8, i, j, k, l}
```

```
In[*]:= v /@ todo
```

Out[*]=

```
{1, 1, 2}
```

```
In[*]:= MaximalBy[todo, v]
```

Out[*]=

```
{X[6, 3, 7, 4]}
```

```
In[*]:= c = RandomChoice[MaximalBy[todo, v]]
```

Out[*]=

```
X[6, 3, 7, 4]
```

```
In[*]:= t1 = Expand[kb * (c /. X[i_, j_, k_, l_] :> A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k])]
```

Out[*]=

$$\frac{p[2,8]\,p[3,7]^2\,p[4,6]}{A^2} + p[2,7]\,p[3,7]\,p[3,8]\,p[4,6] + p[2,8]\,p[3,6]\,p[3,7]\,p[4,7] + A^2\,p[2,7]\,p[3,6]\,p[3,8]\,p[4,7]$$

```
In[*]:= t2 = t1 //. p[i_, j_] p[j_, k_] :> p[i, k]
```

Out[*]=

$$2\,p[2,8]\,p[4,6] + \frac{p[2,8]\,p[3,7]^2\,p[4,6]}{A^2} + A^2\,p[2,4]\,p[6,8]$$

```
In[*]:= kb = Expand[t2 /. {p[i_, i_] :> -A^2 - A^-2, p[i_, j_]^2 -> -A^2 - A^-2}]
```

Out[*]=

$$p[2,8]\,p[4,6] - \frac{p[2,8]\,p[4,6]}{A^4} + A^2\,p[2,4]\,p[6,8]$$

```
In[•]:= todo = DeleteCases[todo, c]
Out[•]=
        {X[4, 2, 5, 1], X[8, 6, 1, 5]}

In[•]:= front = front ⋃ List @@ c
Out[•]=
        {2, 3, 4, 6, 7, 8, i, j, k, l}

In[•]:= KB2[K_] := Module[{front, kb, todo, v, c, t1, t2},
          pd = PD[K];
          front = {}; kb = 1; todo = List @@ pd;
          v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}];
          While[todo =!= {},
           c = RandomChoice[MaximalBy[todo, v]];
           t1 = Expand[kb * (c /. X[i_, j_, k_, l_] :→ A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k])];
           t2 = t1 //. p[i_, j_] p[j_, k_] :→ p[i, k];
           kb = Expand[t2 /. {p[i_, i_] :→ -A^2 - A^-2, p[i_, j_]^2 → -A^2 - A^-2}];
           todo = DeleteCases[todo, c];
           front = front ⋃ List @@ c
          ];
          kb
         ]

In[•]:= KB2[Knot[3, 1]]
Out[•]=
```

$$\frac{1}{A^7}+\frac{1}{A^3}+A-A^9$$

```
In[•]:= KB2[Knot[8, 17]]
Out[•]=
```

$$-\frac{1}{A^{18}}+\frac{2}{A^{14}}-\frac{2}{A^{10}}+\frac{1}{A^6}-\frac{1}{A^2}-A^2+A^6-2A^{10}+2A^{14}-A^{18}$$

```
In[•]:= KB2[GST48]
Out[•]=
```

$$\frac{1}{A^{60}}-\frac{2}{A^{56}}+\frac{1}{A^{52}}+\frac{1}{A^{48}}-\frac{2}{A^{44}}-\frac{1}{A^{24}}+\frac{2}{A^{20}}+\frac{1}{A^{16}}-\frac{2}{A^{12}}+\frac{4}{A^8}-\frac{2}{A^4}-2A^8-2A^{16}+A^{20}-A^{28}+A^{36}$$

In[∘]:=
```
KB3[K_] := Module[{front, kb, todo, v, c, t1, t2},
  front = {}; kb = 1; todo = List @@ PD[K];
  v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}];
  While[todo =!= {},
   c = RandomChoice[MaximalBy[todo, v]];
   t1 = Expand[kb * (c /. X[i_, j_, k_, l_] :→ A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k])];
   t2 = t1 //. p[i_, j_] p[j_, k_] :→ p[i, k];
   kb = Expand[t2 /. {p[i_, i_] :→ -A^2 - A^-2, p[i_, j_]^2 → -A^2 - A^-2}];
   todo = DeleteCases[todo, c];
   front = Echo@Complement[front ⋃ List @@ c, front ⋂ List @@ c]
  ];
  kb
 ]
```

In[∘]:=
```
KB3[Knot[3, 1]]
```

» {1, 2, 4, 5}

» {1, 3, 4, 6}

» {}

Out[∘]=

$$\frac{1}{A^7} + \frac{1}{A^3} + A - A^9$$

In[∘]:=
```
KB3[GST48]
```

» {24, 25, 57, 58}

» {24, 26, 56, 58}

» {24, 26, 45, 46, 56, 59}

» {24, 26, 44, 46, 55, 59}

» {24, 26, 44, 47, 55, 59, 69, 70}

» {23, 26, 44, 47, 55, 59, 68, 70}

» {5, 6, 23, 27, 44, 47, 55, 59, 68, 70}

» {4, 6, 23, 27, 43, 47, 55, 59, 68, 70}

» {4, 6, 23, 27, 43, 47, 55, 59, 67, 70, 94, 95}

» {4, 7, 23, 27, 43, 47, 55, 59, 67, 70, 94, 96}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 67, 70, 94}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 66, 70, 87, 88, 94}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 65, 70, 76, 77, 87, 88, 94}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 65, 70, 75, 77, 86, 88, 94}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 65, 70, 75, 78, 86, 89, 94}

» {1, 4, 8, 23, 27, 43, 47, 55, 59, 65, 70, 75, 79, 86, 89, 93}

» {1, 4, 8, 22, 27, 43, 47, 55, 59, 65, 70, 75, 80, 86, 89, 93}

» {1, 4, 8, 22, 27, 43, 48, 55, 59, 65, 70, 75, 81, 86, 89, 93}

» {1, 4, 8, 22, 27, 43, 48, 55, 59, 64, 70, 75, 81, 85, 89, 93}

» {1, 4, 8, 22, 27, 43, 48, 55, 59, 63, 70, 74, 81, 85, 89, 93}

» {1, 4, 8, 21, 27, 43, 48, 55, 59, 63, 70, 74, 81, 85, 89, 92}

» {1, 4, 8, 21, 27, 43, 49, 55, 59, 63, 70, 74, 81, 85, 89, 91}

» {1, 4, 8, 20, 27, 43, 49, 55, 59, 63, 70, 74, 81, 85, 90, 91}

» {1, 4, 8, 20, 27, 43, 50, 55, 59, 63, 70, 74, 81, 85}

» {1, 4, 8, 20, 27, 32, 33, 43, 50, 55, 59, 63, 70, 73, 81, 85}

» {1, 4, 8, 20, 27, 31, 33, 43, 50, 55, 59, 62, 70, 73, 81, 85}

» {1, 4, 8, 20, 27, 31, 34, 43, 50, 55, 59, 62, 70, 73, 81, 84}

» {1, 4, 8, 19, 27, 31, 35, 43, 50, 55, 59, 62, 70, 73, 81, 84}

» {1, 4, 8, 18, 27, 31, 35, 43, 50, 55, 59, 62, 70, 73, 81, 83}

» {1, 4, 8, 18, 27, 31, 36, 43, 51, 55, 59, 62, 70, 73, 81, 83}

» {1, 4, 8, 18, 27, 31, 36, 43, 52, 55, 59, 62, 70, 73, 81, 82}

» {1, 4, 8, 18, 27, 31, 37, 43, 52, 55, 59, 62, 70, 73}

» {1, 4, 8, 17, 27, 31, 37, 43, 52, 55, 59, 62, 70, 72}

» {1, 4, 8, 17, 27, 31, 37, 43, 53, 55, 59, 62, 70, 71}

» {1, 4, 8, 17, 27, 31, 38, 43, 53, 55, 59, 62}

» {1, 4, 8, 16, 27, 31, 38, 43, 53, 55, 59, 61}

» {1, 4, 8, 16, 27, 31, 39, 43, 53, 55, 60, 61}

» {1, 4, 8, 16, 27, 31, 39, 43, 54, 55}

» {1, 4, 8, 16, 27, 31, 40, 43}

» {1, 3, 8, 16, 27, 31, 41, 43}

» {1, 3, 8, 15, 27, 30, 41, 43}

» {1, 2, 8, 15, 27, 29, 41, 43}

» {8, 14, 27, 29, 41, 43}

» {9, 13, 27, 29, 41, 43}

» {10, 13, 27, 28, 41, 43}

» {10, 12, 41, 43}

» {10, 11, 41, 42}

» {}

Out[∘]=

$$\frac{1}{A^{60}}-\frac{2}{A^{56}}+\frac{1}{A^{52}}+\frac{1}{A^{48}}-\frac{2}{A^{44}}-\frac{1}{A^{24}}+\frac{2}{A^{20}}+\frac{1}{A^{16}}-\frac{2}{A^{12}}+\frac{4}{A^{8}}-\frac{2}{A^{4}}-2A^{8}-2A^{16}+A^{20}-A^{28}+A^{36}$$

```
In[•]:= KB4[K_] := Module[{front, kb, todo, v, c, t1, t2},
   front = {}; kb = 1; todo = List @@ PD[K];
   v[X[i_, j_, k_, l_]] := Length[front ⋂ {i, j, k, l}];
   While[todo =!= {},
    c = First[todo];
    t1 = Expand[kb * (c /. X[i_, j_, k_, l_] :→ A p[i, j] p[k, l] + A^(-1) p[i, l] p[j, k])];
    t2 = t1 //. p[i_, j_] p[j_, k_] :→ p[i, k];
    kb = Expand[t2 /. {p[i_, i_] :→ -A^2 - A^-2, p[i_, j_]^2 → -A^2 - A^-2}];
    todo = DeleteCases[todo, c];
    front = Echo@Complement[front ⋃ List @@ c, front ⋂ List @@ c]
   ];
   kb
  ]

In[•]:= KB4[GST48]
» {1, 2, 14, 15}
» {1, 3, 14, 15, 29, 30}
» {1, 4, 14, 15, 29, 30, 40, 41}
» {1, 5, 14, 15, 29, 30, 40, 41, 43, 44}
» {1, 6, 14, 15, 26, 27, 29, 30, 40, 41, 43, 44}
» {1, 7, 14, 15, 26, 27, 29, 30, 40, 41, 43, 44, 95, 96}
» {8, 14, 15, 26, 27, 29, 30, 40, 41, 43, 44, 95}
» {9, 13, 15, 26, 27, 29, 30, 40, 41, 43, 44, 95}
» {10, 13, 15, 26, 27, 28, 30, 40, 41, 43, 44, 95}
» {11, 13, 15, 26, 27, 28, 30, 40, 42, 43, 44, 95}
» {12, 13, 15, 26, 27, 28, 30, 40, 44, 95}
» {15, 26, 30, 40, 44, 95}
» {16, 26, 31, 40, 44, 95}
» {17, 26, 31, 40, 44, 61, 62, 95}
» {18, 26, 31, 40, 44, 61, 62, 72, 73, 95}
» {19, 26, 31, 40, 44, 61, 62, 72, 73, 83, 84, 95}
» {20, 26, 31, 34, 35, 40, 44, 61, 62, 72, 73, 83, 84, 95}
» {21, 26, 31, 34, 35, 40, 44, 61, 62, 72, 73, 83, 84, 89, 90, 95}
» {22, 26, 31, 34, 35, 40, 44, 61, 62, 72, 73, 83, 84, 89, 90, 92, 93, 95}
» {23, 26, 31, 34, 35, 40, 44, 61, 62, 72, 73, 79, 80, 83, 84, 89, 90, 92, 93, 95}
» {24, 26, 31, 34, 35, 40, 44, 61, 62, 68, 69, 72, 73, 79, 80, 83, 84, 89, 90, 92, 93, 95}
» {25, 26, 31, 34, 35, 40, 44, 57, 58, 61, 62, 68, 69, 72, 73, 79, 80, 83, 84, 89, 90, 92, 93, 95}
» {31, 34, 35, 40, 44, 56, 58, 61, 62, 68, 69, 72, 73, 79, 80, 83, 84, 89, 90, 92, 93, 95}
» {32, 34, 35, 40, 44, 56, 58, 61, 63, 68, 69, 72, 73, 79, 80, 83, 84, 89, 90, 92, 93, 95}
```

» {33, 34, 35, 40, 44, 56, 58, 61, 63, 68, 69, 72, 74, 79, 80, 83, 84, 89, 90, 92, 93, 95}

» {35, 40, 44, 56, 58, 61, 63, 68, 69, 72, 74, 79, 80, 83, 85, 89, 90, 92, 93, 95}

» {36, 40, 44, 50, 51, 56, 58, 61, 63, 68, 69, 72, 74, 79, 80, 83, 85, 89, 90, 92, 93, 95}

» {37, 40, 44, 50, 51, 56, 58, 61, 63, 68, 69, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {38, 40, 44, 50, 51, 56, 58, 61, 63, 68, 69, 70, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {39, 40, 44, 50, 51, 56, 58, 59, 60, 61, 63, 68,
69, 70, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {44, 50, 51, 54, 55, 56, 58, 59, 60, 61, 63, 68,
69, 70, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {45, 50, 51, 54, 58, 59, 60, 61, 63, 68, 69, 70, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {46, 50, 51, 54, 60, 61, 63, 68, 69, 70, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {47, 50, 51, 54, 60, 61, 63, 68, 71, 72, 74, 79, 80, 81, 82, 83, 85, 89, 90, 92, 93, 95}

» {48, 50, 51, 54, 60, 61, 63, 68, 71, 72, 74, 79, 82, 83, 85, 89, 90, 92, 93, 95}

» {49, 50, 51, 54, 60, 61, 63, 68, 71, 72, 74, 79, 82, 83, 85, 89, 90, 91, 93, 95}

» {51, 54, 60, 61, 63, 68, 71, 72, 74, 79, 82, 83, 85, 89, 93, 95}

» {52, 54, 60, 61, 63, 68, 71, 72, 74, 79, 85, 89, 93, 95}

» {53, 54, 60, 61, 63, 68, 74, 79, 85, 89, 93, 95}

» {63, 68, 74, 79, 85, 89, 93, 95}

» {64, 68, 75, 79, 85, 89, 93, 95}

» {65, 68, 75, 79, 86, 89, 93, 95}

» {66, 68, 75, 76, 77, 79, 86, 89, 93, 95}

» {67, 68, 75, 76, 77, 79, 86, 87, 88, 89, 93, 95}

» {75, 76, 77, 79, 86, 87, 88, 89, 93, 94}

» {77, 79, 88, 89, 93, 94}

» {78, 79, 93, 94}

» {}

Out[•]=

$$\frac{1}{A^{60}}-\frac{2}{A^{56}}+\frac{1}{A^{52}}+\frac{1}{A^{48}}-\frac{2}{A^{44}}-\frac{1}{A^{24}}+\frac{2}{A^{20}}+\frac{1}{A^{16}}-\frac{2}{A^{12}}+\frac{4}{A^{8}}-\frac{2}{A^{4}}-2A^{8}-2A^{16}+A^{20}-A^{28}+A^{36}$$