Finitely Generated Abelian Groups

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November 2025

Definition: A group G is *finitely generated* if there exist finitely many $g_1, g_2, \dots, g_n$ such that $G = \langle g_1, g_2, \dots, g_n \rangle$.

Theorem 1. *If M is a finitely generated Abelian group, then there exists nonnegative integers r, s_i and primes p_i, $i = 1, 2, \dots, k$ such that*

$$M \cong \mathbb{Z}^r \times \prod_{i=1}^{k} \mathbb{Z}/p_i^{s_i}\mathbb{Z}$$

Furthermore, r is unique, and the pairs (p_i, s_i) are unique up to permutation.

Definition: For a possibly infinite set X, $\mathbb{Z}^X := \{f : X \to \mathbb{Z} : \text{support } f \text{ is finite}\}$, that is, the set of all functions $f : X \to \mathbb{Z}$ such that f maps only finitely many elements of X to nonzero elements of $\mathbb{Z}$. It is easy to check that $\mathbb{Z}^X$ is an Abelian group, where the operation is function addition. In practice, we denote elements of $\mathbb{Z}^X$ by v, and its x-th coordinate $v(x)$ by v_x as in the ordinary case.

Definition: Let X, Y be sets, a $Y \times X$-matrix in $\mathbb{Z}$ is a function $A : Y \times X \to \mathbb{Z}$, $(y, x) \to A_{y,x}$. Fixing $x \in X$, $A_{\cdot,x}$ is a function that takes y, that is, in $\mathbb{Z}^Y$. We can think of it as the column of the matrix. We denote the set of all column-finite (the support of $A_{\cdot,x}$ is finite for each $x \in X$) $Y \times X$-matrices in $\mathbb{Z}$ by $M_{Y \times X}(\mathbb{Z})$.

Just like in the ordinary case, A induces the multiplication by A map, $A : \mathbb{Z}^X \to \mathbb{Z}^Y$,

$$Av = \sum_{x \in X} A_{\cdot,x} v_x$$

The multiplication by A map is a homomorphism because

$$\begin{aligned} A(v_1 + v_2) &= \sum_{x \in X} A_{\cdot,x}(v_1 + v_2)_x \\ &= \sum_{x \in X} A_{\cdot,x}(v_1)_x + \sum_{x \in X} A_{\cdot,x}(v_2)_x \\ &= Av_1 + Av_2 \end{aligned}$$

Definition: Given $A \in M_{Y \times X}(\mathbb{Z})$, we define

$$M_A = \mathbb{Z}^Y / \text{im } A$$

(we can do this since $\mathbb{Z}^Y$ is Abelian)

Lemma 1. *If M is a finitely generated Abelian group, then $M \cong M_A$ for some $A \in M_{G \times X}(\mathbb{Z})$, where G is finite.*

Proof. Pick G to be a finite set of generators of M, define $\pi_G : \mathbb{Z}^G \to M$ by $\pi_G(v) = \sum_{g \in G} v_g g$. It is clear that π_G is a homomorphism (we used the fact that M is Abelian here), and since G is a set of generators, π_G is surjective, so

$$M \cong \mathbb{Z}^G / \ker \pi_G$$

by the first isomorphism theorem.

Now let X be a set of generators [1] of $\ker \pi_G$, and define $\pi_X : \mathbb{Z}^X \to \mathbb{Z}^G$ by

$$\pi_X(u) = \sum_{x \in X} u_x x$$

But then π_X is multiplication by the matrix $A \in M_{G \times X}(\mathbb{Z})$, given by

$$A_{\cdot,x} = x$$

Indeed,

$$\begin{aligned} \pi_X(u) &= \sum_{x \in X} u_x x \\ &= \sum_{x \in X} x u_x \\ &= \sum_{x \in X} A_{\cdot,x} u_x \\ &= Au \end{aligned}$$

It is clear that $\ker \pi_G = \text{im } \pi_X = \text{im } A$, so

$$M \cong \mathbb{Z}^G / \text{im } A = M_A$$

$\square$

Proposition 1. *Let $A \in M_{G \times X}(\mathbb{Z})$. If $G = G_1 \cup G_2, X = X_1 \cup X_2$ are partitions of X, G respectively, and that A is 0 outside of $G_1 \times X_1$ and $G_2 \times X_2$, then*

$$M_A \cong M_{A_1} \times M_{A_2}$$

where A_1, A_2 are the restrictions of A to $G_1 \times X_1, G_2 \times X_2$ respectively.

Remark 1.1. *In this case we write $A = A_1 \oplus A_2$. We can interpret this as A being block-diagonal, with block matrices A_1, A_2.*

Proof. Define $\phi : M_{A_1} \times M_{A_2} \to M_A$ by

$$\phi(u\text{im } A_1, v\text{im } A_2) = (u, v)\text{im } A$$

where $(u, v) \in \mathbb{Z}^{G_1 \cup G_2} = \mathbb{Z}^G, (u, v) : G \to \mathbb{Z}$ is

$$(u, v)_g = \begin{cases} u_g, & g \in G_1 \\ \\ v_g, & g \in G_2 \end{cases}$$

In fact, we can write (v, u) instead of (u, v), since the set G is not ordered. We make the analogous definition for $(w_1, w_2) \in \mathbb{Z}^{X_1 \cup X_2} = \mathbb{Z}^X$.

We now verify that ϕ is an isomorphism.

[1] X is not necessarily finite even though $\mathbb{Z}^G$ is finitely generated: Who said a subgroup of a finitely generated group must be finitely generated?

(i) ϕ is well-defined since if $(u_1\text{im } A_1, v_1\text{im } A_2) = (u_2\text{im } A_1, v_2\text{im } A_2)$, then $u_1 - u_2 \in \text{im } A_1, v_1 - v_2 \in \text{im } A_2$, for some $w_1 \in \mathbb{Z}^{X_1}, w_2 \in \mathbb{Z}^{X_2}$, we have $u_1 - u_2 = A_1w_1, v_1 - v_2 = A_2w_2$ and hence

$$\begin{aligned}(u_1, v_2) - (u_2, v_2) &= (u_1 - u_2, v_1 - v_2)\\ &= (A_1w_1, A_2w_2)\\ &= A(w_1, w_2)\\ &\in \text{im A}\end{aligned}$$

hence $(u_1, v_1)\text{im } A = (u_2, v_2)\text{im } A$.

(ii) Repeating the same argument as above but in the reverse direction shows that ϕ is injective.

(iii) ϕ is clearly surjective.

(iv) ϕ is clearly a homomorphism.

Hence ϕ is an isomorphism, completing the proof. □

Proposition 2. *Let $A \in M_{G\times X}(\mathbb{Z})$. If $A' = PAQ$, where $P \in M_{G\times G}(\mathbb{Z}), Q \in M_{X\times X}(\mathbb{Z})$ are both invertible (in $M_{G\times G}(\mathbb{Z}), M_{X\times X}(\mathbb{Z})$ respectively), then*

$$M_A \cong M_{A'}$$

Proof.

$$\begin{tikzcd} \mathbb{Z}^X \arrow[r, "A"] & \mathbb{Z}^G \arrow[r, "\pi"] \arrow[d, "P"'] & \mathbb{Z}^G/\text{im } A \\ \mathbb{Z}^X \arrow[u, "Q"] \arrow[r, "A'"] & \mathbb{Z}^G \arrow[r, "\pi"] & \mathbb{Z}^G/\text{im } A' \end{tikzcd}$$

Define $\phi : \mathbb{Z}^G/\text{im } A \to \mathbb{Z}^G/\text{im } A'$ by

$$\phi(v\text{im } A) = Pv\text{im } A'$$

(i) ϕ is well-defined. If $v_1\text{im } A = v_2\text{im } A$, then for some $w \in \mathbb{Z}^X$, we have

$$\begin{aligned}v_1 - v_2 &= Aw\\ Pv_1 - Pv_2 &= PAw\\ Pv_1 - Pv_2 &= PAQw' \text{ for some } w \in \mathbb{Z}^X \qquad (1)\\ Pv_1\text{im } A' &= Pv_2\text{im } A'\end{aligned}$$

we can do (1) because Q is invertible.

(ii) Similarly, ϕ is injective. Indeed, if $Pv_1\text{im } A' = Pv_2\text{im } A'$, then

$$\begin{aligned}Pv_1 - Pv_2 &\in \text{im } A'\\ P(v_1 - v_2) &= PAQw, \quad \text{for some } w \in \mathbb{Z}^X\\ v_1 - v_2 &= AQw \qquad (2)\\ v_1\text{im } A &= v_2\text{im } A\end{aligned}$$

we can do (2) because P is invertible.

(iii) Since P is surjective, ϕ is clearly surjective.

(iv) Clearly ϕ is a homomorphism.

We conclude that $\mathbb{Z}^G/\text{im } A \cong \mathbb{Z}^G/\text{im A}'$, that is, $M_A \cong M_{A'}$. □

Just like ordinary matrices, we can perform row/column operations on a matrix $A \in M_{G\times X}(\mathbb{Z})$. For example, if A' is obtained from A by adding 3 times column x_1 to column x_2, then $A'_{\cdot,x_2} = A_{\cdot,x_2} + 3A_{\cdot,x_1}$. What's different is that now we might add a column to infinitely many others at once.

Proposition 3. *Let $A \in M_{G\times X}(\mathbb{Z})$, then we can add an integer multiple of column/row to another by multiplying an invertible $Q \in M_{X\times X}(\mathbb{Z})/P \in M_{G\times G}(\mathbb{Z})$ on the right/left. We can also add multiples of a column to possibly infinitely many other columns all at once by doing the same.*

Proof. Suppose we want to add $c_\alpha \in \mathbb{Z}$ times column $A_{\cdot,x_0}$ to columns $A_{\cdot,x_\alpha}$, for $\alpha \in \Lambda$ a possibly infinite set, all at once, consider the matrix $Q \in M_{X\times X}(\mathbb{Z})$ given by

$$Q_{x,x'} = \begin{cases} 1, & x = x' \\ \\ c_\alpha, & x = x_0, x' = x_\alpha \\ \\ 0, & \text{otherwise} \end{cases}$$

It is easy to verify that Q does the desired column operation, it remains to verify that Q is invertible. In fact, it is also easy to verify that the inverse of Q is given by

$$Q^{-1}_{x,x'} = \begin{cases} 1, & x = x' \\ \\ -c_\alpha, & x = x_0, x' = x_\alpha \\ \\ 0, & \text{otherwise} \end{cases}$$

both Q and Q^{-1} are in $M_{X\times X}(\mathbb{Z})$.

The row case are left to the readers as an exercise. □

We can now prove the original theorem, which classifies all finitely generated Abelian group.

Proof. Let M be a finitely generated Abelian group, then $M \cong M_A$ for some $A \in M_{G\times X}(\mathbb{Z})$ where G is finite, say it has order $|n|$, by Lemma 1.

Consider the collection $\mathcal{C} = \{PAQ : P \in M_{g\times g}(\mathbb{Z}), Q \in M_{X\times X}(\mathbb{Z}) \text{ both invertible}\}$. Let $A_1 \in \mathcal{C}$ be such that $(A_1)_{g_1,x_1}$ is positive but smallest among all other entries in all matrices in the collection $\mathcal{C}$, for some $g_1 \in G, x_1 \in X$. Such A_1 must exist, otherwise A is clearly the zero matrix, $M_A = \mathbb{Z}^G/\{0\} \cong \mathbb{Z}^n$ and we are done.

Now all entries in the same row or column as $(A_1)_{g_1,x_1}$ is a multiple of $(A_1)_{g_1,x_1}$, otherwise, performing a row/column operation, which corresponds to multiplications of invertible matrices on the left/right by Proposition 3, we can make some entry even smaller but remain positive as the remainder of the division algorithm, which contradicts the minimality of $(A_1)_{g_1,x_1}$. Without loss of generality, we will assume that $(A_1)_{g_1,x_1}$ is the only nonzero entry in its column, since we can eliminate all others by performing either a finite row operation, or a possibly infinite column operation, by multiplication of invertible matrices on the left/right.

But then $A_1 = ((A_1)_{g_1,x_1}) \oplus A_2$ for some $A_2 \in M_{G\setminus\{g_1\}\times X\setminus\{x_1\}}(\mathbb{Z})$. It is easy to verify that

$$M_{((A_1)_{g_1,x_1})} \cong \mathbb{Z}/(A_1)_{g_1,x_1}\mathbb{Z}$$

hence

$$M_{A_1} \cong \mathbb{Z}/(A_1)_{g_1,x_1}\mathbb{Z} \times M_{A_2}$$

that is,

$$M_A \cong \mathbb{Z}/(A_1)_{g_1,x_1}\mathbb{Z} \times M_{A_2}$$

by Proposition 2. Repeating this process, if the process stops somewhere, that is, some A_k is the zero matrix, then

$$M_A \cong \mathbb{Z}/(A_1)_{g_1,x_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/(A_{n-k})_{g_{n-k},x_{n-k}}\mathbb{Z} \times \mathbb{Z}^k$$

otherwise we have

$$M_A \cong \mathbb{Z}/(A_1)_{g_1,x_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/(A_n)_{g_n,x_n}\mathbb{Z}$$

which is essentially the same form.

Anyways, we have

$$M_A \cong \mathbb{Z}^r \times \prod_{i=1}^{l} \mathbb{Z}/(A_i)_{g_i,x_i}\mathbb{Z}$$

for some nonnegative l, r and positive $(A_i)_{g_i,x_i}$'s. But previously we proved a theorem that says if $(a, b) = 1$, then $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/ab\mathbb{Z}$. Hence we have the desired representation

$$M \cong M_A \cong \mathbb{Z}^r \times \prod_{i=1}^{k} \mathbb{Z}/p_i^{s_i}\mathbb{Z}$$

The uniqueness of this representation is a homework question. $\square$