The Simplicity of A_n

This handout is to be read twice: First read red only, to ascertain that everything in red is easy and boring. Then read black and red, to actually understand the proof.

Theorem. The alternating group A_n is simple for $n \neq 4$.

Remark. Easy for $n \leq 3$ and false for $n = 4$ as there is a $\phi\colon A_4 \to A_3$ (see below). So we assume that $n \geq 5$.

Reminder (from HW3). Two permutations in S_n are conjugate iff the sequences of lengths of cycles in their cycle decompositions are the same (up to a permutation of these lengths).

Lemma 1. Every element of A_n is a product of 3-cycles.

Proof. Every element of A_n is a product of an even number of 2-cycles, and $(12)(23) = (123)$ and $(12)(34) = (123)(234)$. □

Lemma 2. If $N \lhd A_n$ contains a 3-cycle, then $N = A_n$.

Proof. WLOG, $(123) \in N$. Then for all $\sigma \in S_n$, $(123)^\sigma \in A_n$. Indeed, if $\sigma \in A_n$, this is clear. Otherwise $\sigma = (12)\sigma'$ with $\sigma' \in A_n$, and then as $(123)^{(12)} = (123)^2$, we have that $(123)^\sigma = (123)^{(12)\sigma'} = ((123)^2)^{\sigma'} \in N$. And so N contains all the 3-cycles, and so by Lemma 1, $N = A_n$. □

Proof of the Theorem. We now assume that $N \lhd A_n$ is not trivial, and check a few cases. In each case we find that $N = A_n$:

Case 1. N contains an element whose cycle decomposition has a cycle of length ≥ 4.

Resolution. $\sigma = (123456)\sigma'$ (with σ' fixing 1,2,3,4,5,6) implies $\sigma^{-1}\sigma^{(123)} = (236) \in N$. □

Case 2. N contains an element with two cycles of length 3.

Resolution. $\sigma = (123)(456)\sigma'$ implies $\sigma^{-1}\sigma^{(124)} = (12436) \in N$. □

Case 3. N contains $\sigma = (123) \cdot (\text{disjoint 2-cycles})$.

Resolution. $\sigma^2 = (132) \in N$. □

Case 4. N contains a disjoint product of 2-cycles.

Resolution. $\sigma = (12)(34)\sigma' \in N$ implies $\sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$ implies $\tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$. □ □

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{p0, p1, p2, p3} = {{0, 0, 0}, {0, 1, 1}, {1, 0, 1}, {1, 1, 0}};
tube[a_, b_] := Tube[{a, b}, .1];
ImageCrop@Graphics3D[{
   {Red, tube[p0, p1], tube[p2, p3]},
   {Green, tube[p0, p2], tube[p1, p3]},
   {Blue, tube[p0, p3], tube[p2, p1]}
  }, ViewPoint → {7, 6, 4}, Boxed → False]
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