1. If $C \triangleleft A$ and $D \triangleleft B$, then show that $(C \times D) \triangleleft (A \times B)$ and $(A/C) \times (B/D) \cong (A \times B)/(C \times D)$.

Let's show $(C \times D) \triangleleft (A \times B)$. Let (g,h) be an element of $C \times D$, and (a,b) be an element of $(A \times B)$. Then, our aim is to show that

$$(a,b)^{-1}(g,h)(a,b) \in C \times D$$

Observe that $(a,b)^{-1} = (a^{-1}, b^{-1})$, so $(a,b)^{-1}(g,h)(a,b) = (a^{-1}ga, b^{-1}hb)$. But because $C \triangleleft A$ and $D \triangleleft B$, so $a^{-1}ga \in C$, and $b^{-1}hb \in D$. So $(a,b)^{-1}(g,h)(a,b) \in C \times D$.

Since this is true for any $(g,h) \in C \times D$ and any $(a,b) \in A \times B$, we conclude $(C \times D) \triangleleft (A \times B)$.

Now, we show that $(A/C) \times (B/D) \cong (A \times B)/(C \times D)$. Consider the map

$L: ([a],[b]) \mapsto [(a,b)]$ where $[a], [b], [(a,b)]$ are equivalence classes

We see that this map is a group homomorphism, because since C, D are normal subgroups of A, B respectively, it means that $[a_1] \cdot [a_2] = [a_1 a_2]$, and also that this is well-defined; in addition

Let's show that it is injective and surjective. Note that the cosets in $(A \times B)/(C \times D)$ can be written as

$(a,b)(C \times D)$ or (aC, bD) to make things easier

Because a, b can be any two elements of G, we see that L is surjective, because all elements (a, b) and their equivalence classes can be achieved in the image of L

Let's show L is injective. Let $L([a_1],[b_1]) = L([a_2],[b_2])$.
Our goal is to show that $[a_1] = [a_2]$ and $[b_1] = [b_2]$.

This means that $(a_1, b_1)(C \times D) = (a_2, b_2)(C \times D)$, and also that $(a_1 a_2^{-1}, b_1 b_2^{-1})(C \times D) = C \times D$. So that means $a_1 a_2^{-1} \in C$, $b_1 b_2^{-1} \in D$.

Now this in turn means that $a_1 C = a_2 C$, $b_1 D = b_2 D$, so $[a_1] = [a_2]$, $[b_1] = [b_2]$. So we have shown that L is injective.

Since L is injective and surjective, it is a bijective homomorphism
let's quickly check L is a homomorphism:

$L([a],[b]) \cdot L([a'],[b']) = [(aa', bb')]$
$L(([a],[b]) \cdot ([a'],[b'])) = [(aa', bb')]$

so L is a bijective homomorphism, so L is a isomorphism and thus $(A/C) \times (B/D) \cong (A \times B)/(C \times D)$.

2. If $A \trianglelefteq G$, $B \trianglelefteq G$, and $G = AB$, prove that $G/(A \cap B) \cong (G/A) \times (G/B)$.

Note that by the 2nd Isomorphism Theorem,

$AB/A \cong A/A\cap B$
$AB/B \cong B/A\cap B$

so it is also true that $AB/A \times AB/B \cong A/A\cap B \times B/A\cap B$.
If $L_1: AB/A \to A/A\cap B$, $L_2: AB/B \to B/A\cap B$ are isomorphisms, then the map

$L: (a(AB), b(AB)) \mapsto (L(a(AB)), L(b(AB)))$

is again a isomorphism from $(AB/A) \times (AB/B)$ to $A/A\cap B \times B/A\cap B$.

If we can show that $AB/A\cap B \cong A/A\cap B \times B/A\cap B$, then based on our earlier discussion earlier $AB/A\cap B \cong A/A\cap B \times B/A\cap B$.

Consider the map

$R: (a(A\cap B), b(A\cap B)) \to ab(A\cap B)$

Because $AB \trianglerighteq A$, $AB \trianglerighteq B$, G/A, G/B, $G/A\cap B$ are all groups. Because a, b are arbitrary elements of A, B, we see that R is a surjective map.

We also show that R is injective, which would also entail that R is well-defined. Suppose that $a_1 b_1 (A\cap B) = a_2 b_2 (A\cap B)$.
Our goal is to show that $a_1(A\cap B) = a_2(A\cap B)$, $b_1(A\cap B) = b_2(A\cap B)$.
This requires that $a_1 a_2^{-1} \in A\cap B$, $b_1 b_2^{-1} \in A\cap B$.

First, we observe that $a_1 b_1 (A\cap B) = a_2 b_2 (A\cap B)$ entails

$a_1 b_1 (a_2 b_2)^{-1} \in A \cap B$, which in turn entails that $a_1 b_1 b_2^{-1} a_2^{-1} \in A \cap B$.

From this, we want to show $a_1 a_2^{-1} \in A \cap B$. Notice that

$a_1 b_1 b_2^{-1} a_2^{-1} = a_1 a_2^{-1} a_2 b_1 a_2^{-1} a_2 b_2^{-1} a_2^{-1} = (a_1 a_2^{-1})(a_2 b_1 a_2^{-1})(a_2 b_2^{-1} a_2^{-1})$

Because $A \trianglelefteq AB$ and $B \trianglelefteq AB$, we see that $a_2 b_1 a_2^{-1} \in B$, $a_2 b_2^{-1} a_2^{-1} \in B$. So that means that because $(a_1 a_2^{-1})(a_2 b_1 a_2^{-1})(a_2 b_2^{-1} a_2^{-1}) \in B$, $a_1 a_2^{-1} \in B$ also. Because $a_1 a_2^{-1} \in A$ because $a_1, a_2 \in A$, $a_1 a_2^{-1} \in A \cap B$. So $a_1(A \cap B) = a_2(A \cap B)$.

Then, we show that $b_1 b_2^{-1} \in A \cap B$ also. We see that because $b_1, b_2 \in B$, $b_1 b_2^{-1} \in B$ by definition. Then, we also see that because

$a_1 b_1 b_2^{-1} a_2^{-1} = a \in A$, because $a_1 b_1 b_2^{-1} a_2^{-1} \in A \cap B$. Then, $b_1 b_2^{-1} a_2^{-1} = a_1^{-1} a \in A$, and $b_1 b_2^{-1} = a_1^{-1} a a_2 \in A$, so $b_1 b_2^{-1} \in A$. Then, that means $b_1 b_2^{-1} \in A$, and $b_1 b_2^{-1} \in B$ so $b_1 b_2^{-1} \in A \cap B$. Now, $b_1(A \cap B) = b_2(A \cap B)$, as desired.

We have shown that $a_1 b_1 (A \cap B) = a_2 b_2 (A \cap B)$ entails $a_1(A \cap B) = a_2(A \cap B)$, $b_1(A \cap B) = b_2(A \cap B)$, which means that R is injective. By the way R was defined, R is also well-defined. Because $A \trianglelefteq AB$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq AB$ as well. So that means because $A \cap B$ is a normal subgroup, $[a]_{A \cap B} \cdot [b]_{A \cap B} = [ab]_{A \cap B}$ is a well-defined operation. So by the way R is defined, R is also a homomorphism. So it is surjective, injective, and well-defined so R is a isomorphism.

Now, $A/A \cap B \times B/A \cap B \cong AB/A \cap B$. Because $AB/A \times AB/B \cong A/A \cap B \times B/A \cap B$, we conclude that $AB/A \cap B \cong AB/A \times AB/B$.

3. Show that if A is abelian and simple, then $A \cong \mathbb{Z}/p$ for some prime p.

Our goal is to show that A is cyclic with prime order, because $\mathbb{Z}/p$ is cyclic with prime order.

With this in mind, take some $x \in A$, with $x \neq e$. Consider the cyclic group $\langle x \rangle$. Because A is abelian, every subgroup is a normal subgroup, so $\langle x \rangle$ is a normal subgroup.

But A is simple, so it has no normal subgroups other than A and $\{e\}$. Because $x \neq e$, we conclude that $A = \langle x \rangle$.

We show that $|A| < \infty$ first. Suppose for the sake of contradiction that $|A| = \infty$. That means that $x^m \neq x^n$ for all $m, n \in \mathbb{Z}$ with $m \neq n$. This means that we can take the ~~so~~ normal subgroup $\langle x^2 \rangle$. Now, observe that $\langle x^2 \rangle \lhd \langle x \rangle$, because $\langle x \rangle$ contains elements of the form x^{2k+1}, where $k \in \mathbb{Z}$, which are not in $\langle x^2 \rangle$. But that means A has a proper normal subgroup, which is a contradiction.

So that means G has finite order. Let $|A| = n$, with $x^n = e$. We show that n is prime. Suppose n was not prime; then, let $m \mid n$, with $m \neq 1$, $m \neq n$, and $m < n$. Now I can take the normal subgroup $\langle x^m \rangle$. Notice that because $m > 1$, $x \notin \langle x^m \rangle$, so $\langle x^m \rangle \lhd \langle x \rangle$. But this contradicts the simplicity of A, as A has a proper normal subgroup. So that means n must be prime.

So $A = \{1, x, \ldots, x^{n-1}\}$. Construct the map $\phi : \mathbb{Z}/n \to A$ defined as

$\phi : [a] \to x^a$ (where $0 < a \leq n-1$ for convenience)

Here, we see that $\phi([a]+[b]) = x^{[a+b]} = x^{[a]+[b]} = \phi([a])\phi([b])$.

We also see that ϕ is injective, as $x^a \neq x^b \Leftrightarrow [a] \neq [b]$, and ϕ is surjective, as $\operatorname{im}\phi$ contains all elements of A. So ϕ is a isomorphism so A is isomorphic to $\mathbb{Z}/n$ for n prime.

4. A group G is called solvable if there exists a sequence of groups $G_0, G_1, \ldots, G_n$ such that $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$ and G_k / G_{k+1} is abelian for all $1 \le i < n$.

a) Show that subgroups and quotient groups of solvable groups are solvable.

Let S be a subgroup of G; note that $S = G \cap S$. Consider the series of groups

$$S = G_0 \cap S \triangleright G_1 \cap S \triangleright \cdots \triangleright G_n \cap S = \{e\}$$

$G_k \cap S \triangleright G_{k+1} \cap S$ is true because for any $h \in G_{k+1} \cap S$, $h \in G_{k+1}$. So that means if $g \in G_k \cap S$, then $g \in G_k$, so because $G_k \triangleright G_{k+1}$, $g^{-1}hg \in G_{k+1}$. But $g, h \in S$ also, so $g^{-1}hg \in S$. So $G_k \cap S \triangleright G_{k+1} \cap S$, so $G_k \cap S \triangleright G_{k+1} \cap S$.

We also have that $G_k \cap S / G_{k+1} \cap S$ is abelian. From the definition recall that

$$G_k \cap S / G_{k+1} \cap S = \{a(G_{k+1} \cap S) : a \in G_k \cap S\}$$

Let $h, g \in G_k \cap S$ be group elements. I claim that $ab(G_{k+1} \cap S) = ba(G_{k+1} \cap S)$. Notice that this would be true if

$$aba^{-1}b^{-1}(G_{k+1} \cap S) = G_{k+1} \cap S$$

Because G_k / G_{k+1} is abelian, $aba^{-1}b^{-1} \in G_{k+1}$. But $aba^{-1}b^{-1} \in S$, because $a, b \in S$ as well. So $G_k \cap S / G_{k+1} \cap S$ is also abelian.

Hence, the composition we found satisfies the requirements to be solvable, so the subgroup S is solvable.

Let's now show that a quotient group of a solvable group is solvable. Suppose $G \triangleright N$; we show G/N is solvable.

Consider a decomposition of G, $G_0, G_1, \dots, G_n$ defined so that

$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$. with G_k/G_{k+1} abelian.
Consider the subgroups $H_0, H_1, \dots, H_n$ defined by $H_k = G_k N$. Now, we see that in general, if $A, B \triangleleft G$ then $AB \triangleleft G$ also. We see this because if $ab \in AB$, and $h \in G$, then

$h^{-1}abh = h^{-1}ahh^{-1}bh = (h^{-1}ah)(hbh^{-1})$

Now, because $A \triangleleft G$, $h^{-1}ah \in A$; in addition, $hbh^{-1} \in B$. So that means $h^{-1}(ab)h \in AB$, so because h was arbitrary $AB \triangleleft G$.
We then conclude that all H_i, $0 \leq i \leq n$ are normal.

We also see that $H_i \triangleright H_{i+1}$. Because $G_i \triangleright G_{i+1}$, and $G \triangleright N$, we see that $G_iN \triangleright G_{i+1}N$, as $G_{i+1}N$ is a subgroup of G_iN, as all elements of $G_{i+1}N$ are contained in G_iN, and $G_{i+1}N$ is a group (as $G_i \triangleleft N_G(N) = G$). It is also a normal subgroup of G_iN, because for any $g \in G$, $g(G_{i+1}N) = (G_{i+1}N)g$ so $H_i \triangleright H_{i+1}$.

Now we show that H_i/H_{i+1} is abelian. Let $a_1, a_2 \in G_i$ and $b_1, b_2 \in N$. We want to show that

$(a_1b_1)(a_2b_2)H_{i+1} = (a_2b_2)(a_1b_1)H_{i+1}$ so that is equivalent to

$(a_2b_2)(a_1b_1)(b_2^{-1}a_2^{-1})(b_1^{-1}a_1^{-1}) \in H_{i+1}$. Now this is equivalent to

$(a_2b_2)(a_1b_1)(a_1^{-1}a_2^{-1})(a_2a_1b_2^{-1}a_1^{-1}a_2^{-1})(a_2a_1a_2^{-1}a_1^{-1})(a_1b_1^{-1}a_1^{-1}) =$
$(a_2b_2a_2^{-1})(a_2a_1b_1a_1^{-1}a_2^{-1})(a_2a_1b_2^{-1}a_1^{-1}a_2^{-1})(a_2a_1a_2^{-1}a_1^{-1})(a_1b_1^{-1}a_1^{-1})$

Now, notice that this is again equivalent to proving that

$(a_2b_2a_2^{-1})(a_2a_1b_1a_1^{-1}a_2^{-1})(a_2a_1b_2^{-1}a_1^{-1}a_2^{-1})(a_2a_1a_2^{-1}a_1^{-1})(a_1b_1^{-1}a_1^{-1})H_{i+1}$
$= H_{i+1}$

Because $a_2a_1a_2^{-1}a_1^{-1} \in G_{k+1}$ (because G_k/G_{k+1} is abelian) and $a_1b_1a_1^{-1} \in N$, that means $(a_2a_1a_2^{-1}a_1^{-1})(a_1b_1^{-1}a_1^{-1}) \in G_{i+1}N = H_{i+1}$. Then, this is again equivalent to

$(a_2b_2a_2^{-1})(a_2a_1b_1a_1^{-1}a_2^{-1})(a_2a_1b_2^{-1}a_1^{-1}a_2^{-1})H_{i+1} = H_{i+1}$

Now, $a_2b_2a_2^{-1} \in N$, $a_2a_1b_1a_1^{-1}a_2^{-1} = (a_2a_1)b_1(a_2a_1)^{-1} \in N$, and $a_2a_1b_2^{-1}a_1^{-1}a_2^{-1} = (a_2a_1)b_2^{-1}(a_2a_1)^{-1} \in N$ also. So that means $(a_2b_2)(a_1b_1)(b_2^{-1}a_2^{-1})(b_1^{-1}a_1^{-1}) \in H_{i+1}$, so H_i/H_{i+1} is abelian.

Now let's return to the original decomposition:

$G = H_0 \rhd H_1 \rhd \cdots \rhd H_n = \{e\}$

By construction, H_i/H_{i+1} is abelian. Now, I claim that

$G/N = H_0/N \rhd H_1/N \rhd \cdots \rhd H_n/N = \{e\}$

is also abelian. Because $H_i \rhd H_{i+1} \rhd N$ for all i, and $H_i \rhd N$,

$(H_i/N)/(H_{i+1}/N) \cong H_i/H_{i+1}$

by the 3rd Isomorphism Theorem. So $(H_i/N)/(H_{i+1}/N)$ is abelian, and

$G/N = H_0/N \rhd H_1/N \rhd \cdots \rhd H_n/N = \{e\}$ is the desired decomposition.

b) Show that if $N \triangleleft G$ and N and G/N are solvable, then so is G.

If G/N is solvable, then there exists a series of normal subgroups

$$G/N = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_n = \{e\}$$

with each H_i/H_{i+1} is abelian. The Lattice Isomorphism Theorem tells us that H_1 is of the form G_1/N, where $G \triangleright G_1$. Likewise, we see that H_2 is of the form G_2/N, where G_2 is a normal subgroup of G_1. We repeat this for all of the H_is and get the decomposition

$$G/N = G_0/N \triangleright G_1/N \triangleright \cdots \triangleright G_n/N = \{e\} \text{ where } G_i \triangleright G_{i+1} \text{ for all } i.$$

Here, we may take $G_n = N$, as $G_n/N = \{e\}$. Here, because normality is preserved, we see that $G_i \geq N$, and also that $G_i \triangleright G_{i+1}$, and $G_i \triangleright N$ (or else the quotient would no longer be a group). The Third Isomorphism Theorem tells us that

$$\frac{G_i/N}{G_{i+1}/N} \cong G_i/G_{i+1}$$

Because $H_i/H_{i+1} = \frac{G_i/N}{G_{i+1}/N}$ is abelian, and it is isomorphic to G_i/G_{i+1}, we conclude that G_i/G_{i+1} is also abelian. Then, we have a chain of subgroups $G_0, G_1, \ldots, G_n$ so that

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = N \text{ where } G_i/G_{i+1} \text{ is abelian}$$

But N is also solvable. So there exists another chain of subgroups $G_{n+1}, \ldots, G_m$ so that $N = G_{n+1} \triangleright \cdots \triangleright G_m = \{e\}$ where G_j/G_{j+1} is abelian. So

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = N \triangleright G_{n+1} \triangleright \cdots \triangleright G_m = \{e\} \text{ and } G_i/G_{i+1}$$
is abelian, so we're done.

5. Show that if $n \geq 3$, then A_n contains a subgroup isomorphic to S_{n-2}.

Let $\tau: \{1, 2, \ldots, n-2\} \rightarrow \{1, 2, \ldots, n-2\}$ be a bijection in S_{n-2}. I will construct one map $\phi: S_{n-2} \mapsto A_n$ so that

$$\phi(\tau) = \begin{cases} \tau \cdot (n \ n-1) & \text{if } \tau \notin A_n \\ \tau & \text{if } \tau \in A_n \end{cases}$$

I claim that ϕ is a homomorphism, which would mean that is a subgroup of A_n. Then, we'll show ϕ is one-to-one.

Let's show that ϕ is a homomorphism, or for $\varphi, \tau \in S_{n-2}$, $\phi(\varphi)\phi(\tau) = \phi(\varphi \cdot \tau)$ we have 4 main cases:

- $\varphi \in A_{n-2}$, $\tau \in A_{n-2}$

In this case, recall that φ is even iff $\text{sign}\,\varphi = 1$, where sign: $S_n \rightarrow S_n$ as defined as

$$\text{sign}\,\varphi = \prod_{\substack{\{i,j\} \subset \{1, \ldots, n\} \\ i \neq j}} \text{sign}\left(\frac{\varphi(i) - \varphi(j)}{i - j}\right)$$

In class, we showed that sign is a homomorphism and that $A_n = \ker \varphi$. So that means that because sign is a homomorphism, $\text{sign}(\varphi \cdot \tau) = \text{sign}(\varphi)\text{sign}(\tau)$. So $\text{sign}(\varphi \cdot \tau)$ is positive, and hence $\varphi \cdot \tau \in A_{n-2}$. Hence, $\phi(\varphi \cdot \tau) = \varphi \cdot \tau$. But this is the same as $\phi(\varphi)\phi(\tau) = \varphi\,\tau$, so ϕ sa

- $\varphi \in A_{n-2}$, $\tau \notin A_{n-2}$

Using similar reasoning, we see that since $\text{sign}(\varphi\cdot\tau) = \text{sign}\varphi\,\text{sign}\tau$, $\text{sign}(\varphi\cdot\tau) = -1$, so $\varphi\cdot\tau \notin A_{n-2}$. That means

$$\Phi(\varphi\cdot\tau) = ((\varphi\cdot\tau)\cdot(n\ n-1))$$

Now, $\Phi(\varphi) = \varphi$, and $\Phi(\tau) = \tau\cdot(n\ n-1)$ so now, we get that $\Phi(\varphi)\Phi(\tau) = \varphi\cdot\tau\cdot(n\ n-1)$

However, because the cycle decompositions of $(\varphi\cdot\tau)$ and $(n\ n-1)$ are disjoint, this becomes

$$(\varphi\cdot\tau)\cdot(n\ n-1)$$

so in this case $\Phi(\varphi\cdot\tau) = \Phi(\varphi)\Phi(\tau)$.

- $\varphi \notin A_{n-2}$, $\tau \in A_{n-2}$

This is the same as the previous case, just with τ, φ reversed, so we omit the proof for brevity.

- $\varphi \notin A_{n-2}$, $\tau \notin A_{n-2}$

Here, $\text{sign}(\varphi\cdot\tau) = \text{sign}(\varphi)\text{sign}(\tau) = (-1)\cdot(-1) = 1$, so $\varphi\cdot\tau \in A_{n-2}$. Hence, $\Phi(\varphi\cdot\tau) = \varphi\cdot\tau$.

We take $\Phi(\varphi) = \varphi\cdot(n\ n-1)$, $\Phi(\tau) = \tau\cdot(n\ n-1)$ Then,

$\Phi(\varphi)\Phi(\tau) = (\varphi\cdot(n\ n-1))\cdot(\tau\cdot(n\ n-1))$. But because the cycle decompositions of τ and $(n\ n-1)$ are disjoint, $\Phi(\varphi)\Phi(\tau) = (\varphi\cdot\tau\cdot(n\ n-1)\cdot(n\ n-1))) = \varphi\cdot\tau$

so Φ is a homomorphism.

Note that by how we constructed ϕ, $\phi(\varphi)$ is always a even permutation. So that means $\phi : S_{n-2} \to A_n$ is well defined.

This means $\text{im}\,\phi$ is a subgroup of A_n, as:

- $\phi(xy) = \phi(x)\phi(y)$, so $\text{im}\,\phi$ is closed under the group operation
- $\phi(x)^{-1} = \phi(x^{-1})$, from the previous line by taking $y = x^{-1}$: ~~identity transformation~~.
- $\text{im}\,\phi$ is associative

Note that $\phi : S_{n-2} \to \text{im}\,\phi \subset A_n$ is also one-to-one. If $\phi(x) = \phi(y)$, either $x \in A_{n-2}, y \in A_{n-2}$ or $x \notin A_{n-2}, y \notin A_{n-2}$. In the first case, $\phi(x) = x$ and $\phi(y) = y$, so $x = y$. If not, that means

$$x \cdot (n\ n-1) = y \cdot (n\ n-1)$$

But $(n\ n-1)$ is invertible, so we cancel it to get $x = y$, as

$$x \cdot (n\ n-1) \cdot (n\ n-1) = y \cdot (n\ n-1) \cdot (n\ n-1) \Rightarrow x = y$$

Hence, $\phi : S_{n-2} \to \text{im}\,\phi \subset A_n$ is one-to-one and onto, so since $\text{im}\,\phi \subset A_n$, S_{n-2} is isomorphic to a subgroup of A_n

6. ~~Let G be a group such that~~

a) Let G be a group so that there is a infinite chain of simple groups $G_1 \leq G_2 \leq \cdots$ such that $G = \bigcup G_i$. Show that G is simple.

Suppose that there exists a normal subgroup N of G.
Our goal is to show that N must be the entire group G if N is nontrivial (i.e. $N \neq \{e\}$).
~~Consider the quotient homomorphism $\phi: G \to G/N$ which~~
Consider the quotient homomorphism $\phi: G \to G/N$. Because N is nontrivial, there exist g_1, g_2 so that $g_1 \neq g_2$ and

$\phi(g_1) = \phi(g_2)$.

Because the chain of groups is infinite, there exists $k_0 \in \mathbb{N}$ so that $g_1 \in G_k$ for $k > k_0$; in addition, there exists $n_0 \in \mathbb{N}$ so that $g_2 \in G_n$ for $n > n_0$. Choose m so that $m > \max(k_0, n_0)$

Now, consider the restriction of ϕ on G_m, $\phi|_{G_m}: G_m \to G/N$.
Now, we see that $\phi|_{G_m}(g_1) = \phi|_{G_m}(g_2)$, so $\phi|_{G_m}$ has a nontrivial kernel, as $g_1 \neq g_2$, hence, $\ker \phi|_{G_m}$ is a nontrivial normal subgroup.

But because G_m is simple, it must mean that $\ker \phi|_{G_m}$ is all of G_m. So that means that $G_m \cap N = G_m$, as we are only considering elements of N that are also in G_m.

Because our choice of m was an arbitrary integer $m > \max(k_0, n_0)$ so that G_m contains both g_1 and g_2, $G_j \cap N = G_j$ for every $j > m$. But because $G_{m+1} \geq G_m \geq \cdots \geq G_i$, $G_j \cap N = G_j$ when $j \leq m+1$ as well. So $G_j \cap N = N$ for all $j \in \mathbb{N}$, so N contains every element in G.

But because N was a normal subgroup of G, $G \supseteq N$ so we conclude $N = G$, so the only possible non-trivial normal subgroup of G is G itself, so G is also simple.

b) Let A_∞ be the set of bijections $\tau: \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that are equal to the identity beyond some point and that are even up to that point. Namely, for each τ there exists some n (that may depend on τ) so that if $k > n$, $\tau(n) = n$ and $\tau|_{\{1,\dots,n\}} \in A_n$. Prove that A_∞ is a infinite simple group.

Note that if $\tau \in A_\infty$, that means that if τ is fixed after n, then we can treat τ as an element of A_n as τ is fixed after n. This is true for any $\tau \in A_\infty$, so A_∞ can be expressed as $A_\infty = \bigcup A_k$, where A_k is the symmetric group for k elements.

Since $A_1 \leq A_2 \leq A_3 \leq \cdots$ and $A_\infty = \bigcup A_k$, that means that since each A_k is simple, so is A_∞ by part a).

(Note that for any $n \in \mathbb{N}$, $A_n \leq A_\infty$, so $A_\infty \supseteq \bigcup A_n$. But as discussed earlier, any element of A_∞ can be expressed as a element of A_n for some $n \in \mathbb{N}$, so