Homework Assignment 17

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, March 27, 2026 11:59 pm (Eastern Daylight Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Let $f \in \mathbb{Q}[x]$ be cubic. If $\mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ is cyclic of order 3, show that all the roots of f are real.

Q2 (10 points)

Give an example of fields $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset F_3$ (with all the inclusions strict) such that $[F_3 : F_0] = 8$ and such that if $i < j$ then F_j/F_i is splitting, with one exception, that F_2/F_0 is not splitting.

Q3 (10 points)

Determine $\mathrm{Gal}\big(\mathbb{Q}(x^4 - 14x^2 + 9)\big/\,\mathbb{Q}\big)$.

Q4 (15 points)

Let p be a prime. Prove that $\mathrm{Gal}\big(\mathbb{Q}(x^p - 2)\big/\,\mathbb{Q}\big)$ is isomorphic to the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{F}_p$ and $a \neq 0$.

Q5 (15 points)

Let E/F be a splitting extension with $\mathrm{char}(F) = 0$, let $a \in E$, let $G := \mathrm{Gal}(E/F)$, and let $H := \mathrm{Gal}(E/F(a))$. Show that $\prod_{\beta \in G/H}(x - \beta(a))$ is the minimal polynomial of a over F. (First make sure that "$\beta(a)$" makes sense for $\beta \in G/H$, noting that G/H may not even be a group).

Q6 (20 points)

Let E/F be a splitting extension with $\mathrm{char}(F) = 0$ and let $G := \mathrm{Gal}(E/F)$. Define the "norm function" $N : E \to E$ by $N(a) := \prod_{\sigma \in G} \sigma(a)$ and the "trace function" $T : E \to E$ by $T(a) := \sum_{\sigma \in G} \sigma(a)$.

1. Prove that for any $a \in E$, $N(a) \in F$ and $T(a) \in F$, so really, $N, T : E \to F$.
2. Prove that N is multiplicative, $N(ab) = N(a)N(b)$ and T is additive, $T(a + b) = T(a) + T(b)$.
3. What are N and T if E/F is $\mathbb{C}/\mathbb{R}$?
4. If $E = F(\sqrt{D})$ is a degree 2 extension, show that $N(a + b\sqrt{D}) = a^2 - Db^2$ and $T(a + b\sqrt{D}) = 2a$.
5. If $m(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_0 \in F[x]$ is the minimal polynomial of $a \in E$ over F and $n := [E : F]$, show that $N(a) = (-1)^n c_0^{n/d}$ and $T(a) = -\frac{n}{d} c_{d-1}$.

Ready to submit?

- Please ensure all pages are in order and rotated correctly before you submit
- You will not be able to resubmit your work after the due date has passed.

Please wait...