Homework Assignment 15

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, March 6, 2026 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Let F be a field and let $f \in F[x]$ be a polynomial of degree 2 or 3. Show that f is irreducible iff it has no roots in F.

Q2 (10 points)

Show that the polynomial $x^2 - 2$ is irreducible in $\mathbb{F}_3[x]$. Use that to give a complete descrption of a field E that has 9 elements. Namely, for some basis $B = \{u_1, u_2\}$ of $E/\mathbb{F}_3$, compute all the products $u_i u_j$ as linear combinations of elements of B. Likewise, compute the inverses of all the non-zero linear combinations of elements of B.

Q3 (10 points)

Let G be a finite Abelian group with the property that $\forall k\, |\{g \in G : g^k = e\}| \leq k$.

1. Show that if $G \simeq \prod \mathbb{Z}/p_i^{s_i}$, then the primes p_i are distinct.
2. Show that G is cyclic.
3. Show that the multiplicative group of a finite field is always cyclic. You may wish to consider the roots of the polynomials $x^k - 1$.

Q4 (10 points)

1. Let R be a UFD and let $p \in R$ be a prime. Let $f \in R[x]$ and let $\bar{f}$ be the image of f in $(R/\langle p\rangle)[x]$. Assume that $\deg \bar{f} = \deg f$ and that $\bar{f}$ is irreducible in $(R/\langle p\rangle)[x]$. Show that f is irreducible in $Q[x]$, where Q is the field of fractions of R.
2. Use $p = 5$ to show that $21x^3 - 3x^2 + 2x + 8$ is irreducible in $\mathbb{Q}[x]$.
3. Yet note that the same polynomial is not irreducible mod 2.

Q5 (10 points)

1. Show that a polynomial $f(x)$ is irreducible if and only if the polynomial $f(x+1)$ is irreducible.
2. For a prime p, show that the pth cyclotomic polynomial $\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

Ready to submit?

→ Please ensure all pages are in order and rotated correctly before you submit

→ You will not be able to resubmit your work after the due date has passed.

Please wait...