This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 9

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, November 28, 2025 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

1. Prove that in any ring, $(-1)^2 = 1$.
2. Prove that even in a rng (a ring without a unit), for any a we have $(-a)^2 = a^2$.

(In this question, "prove" means "from the most basic axioms").

Q2 (10 points)

An "integral domain" is a commutative ring that has no zero divisors. Namely, a commutative ring in which $ab = 0$ implies that $a = 0$ or $b = 0$.

Prove that a finite integral domain is field.

Q3 (10 points)

A ring R is called *Boolean* if for every $a \in R$, we have that $a^2 = a$.

1. Prove that every Boolean ring is commutative.
2. Prove that if a Boolean ring is an integral domain, then it is isomorphic to $\mathbb{Z}/2$.

Q4 (10 points)

In a ring R, an element x is called "nilpotent" if for some positive n, $x^n = 0$. Let $\eta(R)$ be the set of all nilpotent elements of R.

1. Prove that if R is commutative then $\eta(R)$ is an ideal.
2. Find an example of a non-commutative ring R for which $\eta(R)$ is not an ideal.

Q5 (10 points)

Let A be a commutative ring. The constant term (the coefficient of x^0) of a polynomial $f \in A[x]$ is invertible in A and all its other coefficients are nilpotent. Show that f is invertible in $A[x]$.

Q6 (0 points)

No bonus question! I was going to ask: If J is a maximal ideal in ℓ^∞ containing the ideal $\{(a_i) : a_i \to 0\}$, prove that ℓ^∞ / J is isomorphic to $\mathbb{R}$. But I couldn't find a simple enough proof, and so it's out.

Yet hey, during office hours today I learned a simple proof that $2 = 3$! This really does open up math to a whole world of new possibilities. Here's how it goes: Clearly, the group $2\mathbb{Z}$ is isomorphic to the group $3\mathbb{Z}$, as both of them are isomorphic to the group $\mathbb{Z}$. And so the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are isomorphic. As the former has 2 elements and the latter has 3 elements, it follows that $2 = 3$.

Is this truly a revolution in mathematics or am I making a silly mistake? Is there a moral to learn?

Ponder that, but no bonus is involved no matter what is your conclusion.

Ready to submit?

- Please ensure all pages are in order and rotated correctly before you submit
- You will not be able to resubmit your work after the due date has passed.

Please wait...