Homework Assignment 8

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Warning. It's a tough one! Give it the time that it deserves.

Due date
Friday, November 21, 2025 11:59 pm (Eastern Standard Time)

Late penalty
5% deducted per hour

Q1 (10 points)

Suppose a group G is a product $A_1 \times \cdots \times A_m$, a group H is a product $B_1 \times \cdots \times B_n$, and a group K is a product $C_1 \times \cdots \times C_k$, with all the groups assumed to be Abelian.

1. Show that the set of homomorphisms $\Phi : G \to H$ can be put in a natural bijection with the set of $n \times m$ matrices (ϕ_{ij}), where ϕ_{ij} is a homomorphism $A_j \to B_i$. Let the matrix corresponding to Φ be denoted by M_Φ.
2. Suppose we also have a homomorphism $\Psi : H \to K$. Show that $M_{\Psi \circ \phi} = M_\Psi \cdot M_\Phi$, where the right hand side is a simple modification of the notion of matrix multiplication, which you have to make explicit.

Q2 (10 points)

Show that if r and r' are natural numbers and if A and A' are finite Abelian groups and $\Phi : \mathbb{Z}^r \times A \to \mathbb{Z}^{r'} \times A'$ is an isomorphism of groups, then $\phi_{11} : \mathbb{Z}^r \to \mathbb{Z}^{r'}$, in the sense of the previous question, is also an isomorphism.

Q3 (10 points)

Show that if the groups $\mathbb{Z}^r$ and $\mathbb{Z}^{r'}$ are isomorphic, then $r = r'$.

Q4 (10 points)

Show that if the groups $G = \prod_{i=1}^{m} \mathbb{Z} \big/ p_i^{s_i} \mathbb{Z}$ and $H = \prod_{j=1}^{n} \mathbb{Z} \big/ q_j^{t_j} \mathbb{Z}$ are isomorphic, where m, n, s_i, and t_j are natural numbers and where p_i and q_j are primes, then the sequences $((p_i, s_i))$ and $((q_j, t_j))$ are the same up to a permutation.

Hint. Pick another prime l and another natural number k, and compare the counts of elements of order l^k in G and in H.

Q5 (10 points)

Prove the uniqueness statement within the structure theorem of finite generated Abelian groups.

Q6 (14 points)

A tough bonus question.

1. (1 point) Show that if G is finitely generated, then so is its Abelianization A (see HW2 for "Abelianization").
2. (6 points) Let $F_2 = F(x, y)$ be the free group on two generators x and y, and let G be its subgroup consisting of the elements in which the total power of x is 0 (so $x^5 y^2 x^{-3} y^{-1} x^{-2}$ is in, but $x^5 y^2 x^{-3} y^{-1} x^{-1}$ is not). Show that G is generated by the (infinitely many) elements $\left\{ y^{x^n} = x^{-n} y x^n : n \in \mathbb{Z} \right\}$.
3. (6 points) Show that the Abelianization of G is $\mathbb{Z}_{FS}^{\mathbb{Z}}$, where "FS" means "Finite Support".
4. (1 point) Deduce that there exists a finitely generated group that has a non-finitely generated subgroup.