Homework Assignment 3

Solve and submit your solutions of the following problems. Note that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, October 3, 2025 11:59 pm (Eastern Daylight Time)

Late penalty

5% deducted per hour

Q1 (10 points)

1. A permutation $\sigma \in S_n$ has a cycle decomposition consisting of k cycles, of lengths $l_1, \dots, l_k$. What is the order of σ?
2. What is the smallest n for which S_n contains an element of order 18?
3. What is the maximal order of an element in S_{13}? In S_{26}?

Not for credit, and not so easy: If you're in the mood, figure out and plot the maximal order of an element in S_n as a function of n.

Q2 (10 points)

1. Let $\sigma, \tau \in S_n$ be permutations, and assume that σ has a cycle decomposition $\sigma = (a_{11}, a_{12}, \dots, a_{1l_1})(a_{21}, a_{22}, \dots, a_{2l_2}) \cdots (a_{k1}, a_{k2}, \dots, a_{kl_k})$. Prove that the cycle decomposition of $\sigma^\tau = \tau^{-1}\sigma\tau$ is $(\tau^{-1}a_{11}, \tau^{-1}a_{12}, \dots, \tau^{-1}a_{1l_1})(\tau^{-1}a_{21}, \tau^{-1}a_{22}, \dots, \tau^{-1}a_{2l_2}) \cdots (\tau^{-1}a_{k1}, \tau^{-1}a_{k2}, \dots, \tau^{-1}a_{kl_k})$.
2. Prove that two permutations σ_1 and σ_2 in S_n are conjugate iff the sequences of lengths of cycles in their cycle decompositions are the same (up to a permutation of these lengths).

Q3 (10 points)

1. Define a relation on permutations in S_n by $\sigma_1 \sim \sigma_2$ if σ_1 is a conjugate of σ_2. Show that $\sim$ is an equivalence relation.
2. We call the equivalence classes for the relation $\sim$ "the conjugacy classes of S_n". How many conjugacy classes are there in S_2? In S_3? In S_4? In S_5?

Q4 (10 points)

Let $\sigma \in G = S_{20}$ be a permutation whose cycle decomposition consists of one 5-cycle, two 3-cycles, and one 2-cycle. What is the order of the centralizer of σ, $C_G(\sigma) := \{\tau \in G : \sigma\tau = \tau\sigma\}$.

Q5 (10 points)

Let H be a subgroup of index 2 in a group G. Show that H is normal in G.

Hint. There's $G/H = \{gH : g \in G\}$, and there's $H \backslash G = \{Hg : g \in G\}$.

Q6 (10 points)

1. A group G is called *cyclic* if it is generated by a single element inside G. In other words, if there is a single element $a \in G$ whose set of powers, positive and negative, is G. Show that if G is cyclic then it is isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}/n$ for some n
2. Show that if $G/Z(G)$ is cyclic then G is Abelian. (Recall that $Z(G)$ is the centre of G, the set of all elements in G that commute with all elements of G).

Ready to submit?

→ Please ensure all pages are in order and rotated correctly before you submit

→ You will not be able to resubmit your work after the due date has passed.

Please wait...