Homework Assignment 1

Solve and submit your solutions of the following problems. Note that the last problem is worth the same as the previous 5! Also note that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, September 19, 2025 11:59 pm (Eastern Daylight Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Determine which of the following operations are associative:

1. On $\mathbb{Z}$, $a * b := a - b$.
2. On $\mathbb{R}$, $a * b := a + b + ab$.
3. On $\mathbb{Q}$, $a * b := (a + b)/5$.
4. On $\mathbb{Z} \times \mathbb{Z}$, $(a, b) * (c, d) := (ad + bc, bd)$.
5. On $\mathbb{Q} \setminus \{0\}$, $a * b := a/b$.

Q2 (10 points)

Let $S := [0, 1)$ and on it define

$$a * b := \begin{cases} a + b & \text{if } a + b < 1 \\ a + b - 1 & \text{otherwise.} \end{cases}$$

Show that with this operation S is an Abelian (commutative) group.

Q3 (10 points)

Let $\sigma = [13, 2, 15, 14, 10, 6, 12, 3, 4, 1, 7, 9, 5, 11, 8]$ and $\tau = [14, 9, 10, 2, 12, 6, 5, 11, 15, 3, 8, 7, 4, 1, 13]$ be permutations in S_{15}. The cycle decomposition of σ is $(1, 13, 5, 10)(3, 15, 8)(4, 14, 11, 7, 12, 9)$. Find the cycle decompositions of τ, σ^2, $\sigma\tau$, $\tau\sigma$, and $\tau^2\sigma$.

Q4 (10 points)

Let $\mathbb{Q}[\sqrt{2}]$ denote the set $\{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

1. Show that $\mathbb{Q}[\sqrt{2}]$ is a group under addition.
2. Show that $\mathbb{Q}[\sqrt{2}] \setminus \{0\}$ is a group under multiplication.

Q5 (10 points)

Let G and H be group, and let $G \times H$ be taken with the operation $(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2)$.

1. Show that $G \times H$ is a group. (It is called "the direct product of G and H").
2. If A and B are groups, show that $A \times B$ is Abelian if and only if both A and B are Abelian.
3. Prove that in $G \times H$, the elements $(g, 1)$ and $(1, h)$ commute.

Q6 (50 points)

Let $G = \langle g_1 = (1, 2, 3),\ g_2 = (2, 3, 4) \rangle \subset S_4$.

1. Use the NCGE algorithm discussed in class to write a "table of tricks" T for G.
2. Use T to determine $|G|$.
3. Use T to decide if $\sigma_1 = (12) \in G$.
4. Use T to decide if $\sigma_2 = (13)(24) \in G$.

Ready to submit?

→ Please ensure all pages are in order and rotated correctly before you submit

→ You will not be able to resubmit your work after the due date has passed.

Please wait...