The Cayley-Hamilton Theorem

Theorem. Let R be a commutative ring, let $A \in M_{n\times n}(R)$ be a matrix and let $\chi_A(t) := \det(tI-A)$ be the characteristic polynomial of A. Then $\chi_A(A)=0$.

Froof. Substitute A into t in $\det(tI-A)$, get $\chi_A(A) = \det(AI-A) = \det(0) = 0$. □

Alas, the same froof also froves that $\rho_A(A)=0$, where $\rho_A(t) := \operatorname{tr}(tI-A)$, and indeed, *evaluation does not commute with taking the determinant.*

Proof of Theorem. For any matrix M over any commutative ring there is "the adjugate matrix $\operatorname{adj}(M)$ of M", defined using the minors of M, which satisfies $\det(M)I = (\operatorname{adj} M)M$. Use this with $M = tI-A$, over the ring $R[t]$, and find that in the ring $M_{n\times n}(R[t])$ we have

$$\chi_A(t)I = \det(tI-A)I = (\operatorname{adj}(tI-A))(tI-A).$$

Now note that the ring $M_{n\times n}(R[t])$ is isomorphic to the ring $M_{n\times n}(R)[t]$, and on the latter there is a linear "evaluation at $t=A$" map $\operatorname{ev}_A\colon M_{n\times n}(R)[t] \to M_{n\times n}(R)$, defined by "putting A to the right of all the coefficients"; namely, by $\sum B_k t^k \mapsto \sum B_k A^k$. This evaluation map ev_A is *not* multiplicative, but nevertheless it annihilates anything that has a right factor of $(tI-A)$ (exercise!). Hence under ev_A the above equality becomes $\chi_A(A)I=0$. □

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