Do not open this notebook until instructed.

MAT 347 Groups, Rings, Fields

Term Test 1

University of Toronto, November 4, 2025

Solve all 5 problems in this booklet.
The problems are of equal weight.
You have an hour and fifty minutes to write this test.

Notes

- No outside material allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- Write your solution to each problem on its page and on the back of that page. If you run out of space you may continue into the scratch pages, but you **must** indicate this on the problem page or else the scratch pages will not be read.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook – neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- **In Red.** Added after the test took place.
- Please return this booklet intact, without tearing out any pages. If you use some of the pages as scratch, mark them SCRATCH and return them too.

Good Luck!

Problem 1. In this question, all permutations are written using cycles notation.

Last Thursday I wanted to analyze the group $G = \langle (12345),\ (2354) \rangle < S_5$, so I ran the NCGE algorithm on it. The output was the following table:

1	I				
2	(12345)	I			
3	(13524)	(2354)	I		
4	(14253)	(2453)	–	I	
5	(15432)	(25)(34)	–	–	I
	1	2	3	4	5

1. What is $|G|$? Show your calculation.
2. Is the permutation (354) a member of G? Justify your answer.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.
Tip. Neatness, cleanliness and organization count, here and everywhere else!

Problem 2. Let G be a group.

1. Prove that if $a, b \in G$ are conjugate, then their orders are the same.
2. Prove that if $x, y \in G$, then the order of xy is equal to the order of yx.

Problem 3. Let G be a group and $N \lhd G$ be a normal subgroup thereof. We know from class that G/N is a group.

Explain why it was necessary to assume that N was normal for G/N to be a group. What exactly breaks if N is not normal? How does the normality condition fix that?

Tip. Often in mathematics, writing the right things is more valued than writing a lot of things.

Problem 4. Prove that if a group G is Abelian and simple then it is isomorphic to $\mathbb{Z}/p$ for some prime p. Note that we are not assuming that G is finite.

Problem 5. Assuming that a group G has a Sylow-p subgroup, prove that the number of Sylow-p subgroups of G is congruent to 1 modulo p.

Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.
Tip. Once you have finished writing an exam, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.