MAT 347 on 260121-23, H43-45: Classes by Matt and Jacob on Tensor Products

Tensor products

Given M, N modules

- Can form $M \oplus N$, a new module
- Want mutiplication.

Def: A tensor product of M, N is a module $M \otimes_R N$ and an R-bilinear map $i: M\times N \longrightarrow M\otimes_R N$ s.t. $\forall$ R-module P, and $p: M\times N \to P$ $\exists! \, \hat{p}: M\otimes_R N \longrightarrow P$ s.t. $p = \hat{p}\circ i$

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M\otimes_R N \arrow[r, dashed, "\hat{p}"'] & P \end{tikzcd}$$

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M\otimes_R N \arrow[r, dashed, "\hat{p}"'] & P \end{tikzcd}$$

Q1) When is i injective?

Q2) When is i surjective?

$V\times W \xrightarrow{i} U$

1) $i(v,0) = 0 \Longrightarrow$ Never
$= i(0,0)$

2) Trickier

Lemma: $\mathrm{Span}(\mathrm{Im}(i)) = M\otimes_R N$

proof:

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \arrow[dr, "p(v,w)=0"] & \\ M\otimes N \arrow[r, dashed, "\hat{p}=0"] \arrow[r, bend right, "\pi"'] & M\otimes N/\mathrm{span}(\mathrm{Im}(i)) \end{tikzcd}$$

$\pi: M\otimes N \longrightarrow M\otimes N/\mathrm{span}(\mathrm{Im}(i))$

$\pi\circ i(v,w) = 0 = p$

By uniqueness, $\pi = \hat{p} = 0$

$\square$

Example:

$M = \mathbb{R}^m$, $N = \mathbb{R}^n$

$i(v,w) = v^t w = (m\times 1)(1\times n)$
$= m\times n$

v $1\times m$

w $1\times n$

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \\ \mathrm{Mat}_{\mathbb{R}}(m,n) \end{tikzcd}$$

5

4

$5\times 4 = 20$

Thm: Existence and uniqueness

Tensor products exist and are unique up to unique isomorphism.

proof: Existence. Let G be free abelian group generated by $\{mn, m\in M, n\in N\}$.

Module axioms:
- $(r+s)v = rv + sv$
- $(rs)v = r\cdot sv$
- $r(v+w) = r\cdot v + r\cdot w$
- $1v = v$

Let $H < G$ be generated by:
- $(m_1+m_2)\cdot n - m_1\cdot n - m_2\cdot n$
- $m\cdot(n_1+n_2) - m\cdot n_1 - m\cdot n_2$
- $mr\cdot n - m\cdot rn$

Define
- $M\otimes N = G/H$
- $i(v,w) = [v\otimes w]$
- $r\cdot[v,w] := [rv\cdot w] \overset{\text{why?}}{=} [v\cdot rw]$

Let $v\otimes w := [v\cdot w] \in M\otimes N \; (= G/H)$

$M\otimes N$ is a module.

- $M\otimes N$ is an abelian group ✓
- $(r+s)\cdot(v\otimes w) \overset{\text{def}}{=} (r+s)v\otimes w$
$= (rv+sv)\otimes w$
$= rv\otimes w + sv\otimes w$
$= r\cdot(v\otimes w) + s\cdot(v\otimes w)$ $\square$

$\cdot\ (rs)\cdot v\otimes w = rs.v\otimes w$

$= (r\cdot(s.v))\otimes w$

$(v,w) = \quad 0 = r\cdot(sv\otimes w)$

$\square$

$\cdot\ r\cdot(v_1\otimes w_1 + v_2\otimes w_2) = r\cdot v_1\otimes w_1 + r\cdot v_2\otimes w_2$

$= r\cdot(v_1\otimes w_1) + r\cdot(v_2\otimes w_2)$ $\square$

$\cdot\ 1\cdot(v\otimes w) = 1\cdot v\otimes w = v\otimes w$ $\square$

$\cdot\ i$ is bilinear:

$i(v_1+v_2, w) = (v_1+v_2)\otimes w$

$= v_1\otimes w + v_2\otimes w$ prop 1

$i(v, w_1+w_2) = v\otimes(w_1+w_2)$

$= v\otimes w_1 + v\otimes w_2$ prop 2

$i(rv, w) = rv\otimes w = r\cdot v\otimes w$

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M\otimes N \arrow[r, dashed, "\hat{p}"] & P \end{tikzcd}$$

$\cdot\ \hat{p}(i(v,w)) = p(v,w)$

$\cdot$ "By the lemma", $\hat{p}$ uniquely defined on $M\otimes N$

Uniqueness:

$$\begin{tikzcd} & M\times N \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ A_1 \arrow[rr, dashed, shift left, "\hat{i}_2"] & & A_2 \arrow[ll, dashed, shift left, "\hat{i}_1"] \end{tikzcd}$$

$\cdot\ \overbrace{i_2(v,w)}^{x} = \hat{i}_2(i_1(v,w))$

$\cdot\ i_1(v,w) = \hat{i}_1(i_2(v,w))$

$= \hat{i}_2(\hat{i}_1(i_2(v,w)))$

$= (\hat{i}_2\circ\hat{i}_1)(\underbrace{i_2(v,w)}_{x})$

$\hat{i}_2\circ\hat{i}_1(x) = x$

Examples:

$\cdot\ \mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}\ (= \mathrm{Mat}_{\mathbb{R}}(n,m))$

$\cdot\ R$ a ring, $I, J \trianglelefteq R$

$$\begin{tikzcd} R/I \times R/J \arrow[d] \arrow[dr, "p"] & \\ R/I\otimes R/J \arrow[r, dashed, "\hat{p}"] & M = R/(I+J) \end{tikzcd}$$

$$\begin{tikzcd} R/I \times R/J \arrow[d, "i"'] \arrow[dr, "p"] & \\ R/I\otimes R/J \arrow[r, dashed, "\hat{p}"'] & R/(I+J) \end{tikzcd}$$

$\cdot\ p([r_1],[r_2])$

$= [r_1 r_2]$ (well defined?)

$\cdot\ \hat{p}$ surjective.

Ker $\hat{p}$?

$\hat{p}([r]\otimes[1]) = p([r],[1])$

$= [r] \overset{\text{suppose}}{=} [0]$

$\Longleftrightarrow r \in I + J$

$\Longleftrightarrow r = x+y,\ x\in I,\ y\in J$

$$[r]_I \otimes [1]_J = [x+y]_I \otimes [1]_J$$

$$= \underbrace{[x]_I \otimes [1]_J}_{0} + [y]_I \otimes [1]_J$$

$$= 0 + [y]_I \otimes [1]_J$$

$$= [y \cdot 1]_I \otimes [1]_J$$

$$= y[1]_I \otimes [1]_J$$

$$= [1]_I \otimes [y]_J = 0$$

Ker $\hat{p}$? Lemma: Every simple tensor has this form

$$\hat{p}([r] \otimes [1]) = p([r],[1])$$

$$= [r] \overset{\text{suppose}}{=} [0]$$

$$\iff r \in I + J$$

$$\iff r = x + y,\ x \in I,\ y \in J$$

$$\begin{tikzcd} R/I \times R/J \arrow[d, "i"] \arrow[dr, "p"] & \\ R/I \otimes R/J \arrow[r, dashed, "\hat{p}"'] & R/(I+J) \end{tikzcd}$$

$p([r_1],[r_2])$

$= [r_1 r_2]$ (well defined?)

$\hat{p}$ surjective.

$I = (a)$, $J = (b)$ $\quad R/(a) \otimes R/(b) \simeq R/((a)+(b)) = R/\gcd(a,b)$

Thm $\quad (R\text{-mod}, \oplus, \otimes, 0, R)$ "forms a ring" up to isomorphism:

(i) R is an identity for $\otimes$ $\rightarrow R \otimes_R M \cong M \cong M \otimes_R R$

(ii) 0 is an identity for $\oplus$ $\quad 0 \oplus M \cong M$

(iii) $\oplus$ and $\otimes$ are associative

(iv) $\otimes$ distributes over $\oplus$

(v) $\oplus$ is commutative

I will focus on the $\otimes$ parts

(i) $R \otimes_R M \cong M$

we will do this by showing M satisfies the universal property for $R \otimes_R M$, namely,

$$\begin{tikzcd} R \times M \arrow[r] \arrow[d, "f"] & R \otimes_R M \arrow[dl, dashed, "\exists!\tilde{f}"] \\ N & \end{tikzcd}$$

If we have, $f : R \times M \to N$ R-bilinear, note $f(r,m)$

$= r f(1,m)$

$= f(1, rm)$

If we consider the map $\quad R \times M \to M$

$(r,m) \mapsto rm$

$$\begin{tikzcd} R \times M \arrow[r, "\ell", "(r{,}m) \mapsto rm"'] \arrow[d, "f"] & M \arrow[dl, "\tilde{f}(m) = f(1{,}m)"] \\ N & \end{tikzcd}$$

$f(1, rm) = f(r,m)$

$\tilde{f}$ is unique, R-linear, and makes the diagram commute, so (M, ℓ) satisfies the universal property. By uniqueness of tensor product, $M \cong R \otimes_R M$

$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

Pf

$$\begin{tikzcd} A \times (B \times C) \arrow[r] \arrow[d, "\cong\ (a{,}(b{,}c)) \mapsto (a \otimes b) \otimes c"] & A \otimes (B \otimes C) \arrow[dddl, bend left] \\ (A \times B) \times C \arrow[d] & \\ (A \otimes B) \times C \arrow[d] & \\ (A \otimes B) \otimes C \arrow[uuur, bend right] & \end{tikzcd}$$

for each a, we get a bilinear map

$$\begin{tikzcd} B \times C \arrow[r] \arrow[d, "\ell_a"] & B \otimes C \arrow[dl, "\exists!\tilde{\ell}_a"] \\ (A \otimes B) \otimes C & \end{tikzcd}$$

$\tilde{\ell}_{a+a'}(\sum b_i \otimes c_i) = \sum \tilde{\ell}_{a+a'}(b_i \otimes c_i)$

$\sum \ell_{a+a'}(b_i, c_i)$

so $\tilde{\ell}_a$ depends linearly on A

So, we get a bilinear map

$$\begin{tikzcd} A \times (B \otimes C) \arrow[r] \arrow[d, "\tilde{\ell}_a"] & A \otimes (B \otimes C) \arrow[dl] \\ (A \otimes B) \otimes C & \end{tikzcd}$$

$(a, b \otimes c) \mapsto \tilde{\ell}_a(b \otimes c)$

we can also go the other way to get a map

$(A \otimes B) \otimes C \to A \otimes (B \otimes C)$.

$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$

aside if $f : M_1 \to M_2$ is a R-module hom. N is an R-module, there is a map

$N \otimes_R M_1 \to N \otimes_R M_2$

$n \otimes m_1 \mapsto n \otimes f(m_1)$

(generally, if $N_1 \xrightarrow{g} N_2$ is a map,

$N_1 \otimes M_1 \to N_2 \otimes M_2$

$n_1 \otimes m_1 \mapsto g(n_1) \otimes f(m_1)$)

$$\begin{tikzcd} N \times M_1 \arrow[r] \arrow[d, "(n{,}m) \mapsto (n{,}f(m))"'] & N \otimes M_1 \arrow[d] \\ N \times M_2 \arrow[r] & N \otimes M_2 \end{tikzcd}$$

$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$

First, $B \to B \oplus C$, $C \to B \oplus C$

we get $A \otimes B \to A \otimes (B \oplus C)$, $A \otimes C \to A \otimes (B \oplus C)$

By universal property of $\oplus$, we get

$$\begin{tikzcd}
A \times B \arrow[rr] \arrow[dr] & & A \otimes B \oplus A \otimes C \arrow[d] & & A \times C \arrow[ll] \arrow[dl] \\
& A \otimes B \arrow[ur] \arrow[r] & A \otimes (B \oplus C) & A \otimes C \arrow[l] \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \times (B \oplus C) \arrow[r] \arrow[d] & A \otimes (B \oplus C) \arrow[dl] \\
(A \otimes B) \oplus (A \otimes C) &
\end{tikzcd}$$

$$\begin{tikzcd}
(a \otimes b, a \otimes c) \arrow[d, shift right] \\
(a \otimes (b, c)) \arrow[u, shift right]
\end{tikzcd}$$

$\ell_a : B \to (A \otimes B) \oplus A \otimes C \xleftarrow{r_a} C$

Summarize use the two universal properties in two different orders.

$\ell_a \oplus r_a : B \oplus C \to (A \otimes B) \oplus (A \otimes C)$

$a \mapsto (a, 0)$ $\qquad$ $b \mapsto (0, b)$

$$\begin{tikzcd}
& A \oplus B \arrow[d, "\exists ! \text{ hom.}"] & \\
A \arrow[ur, bend left] \arrow[r, "\text{hom}"'] & P & B \arrow[ul, bend right] \arrow[l, "\text{hom}"]
\end{tikzcd}$$