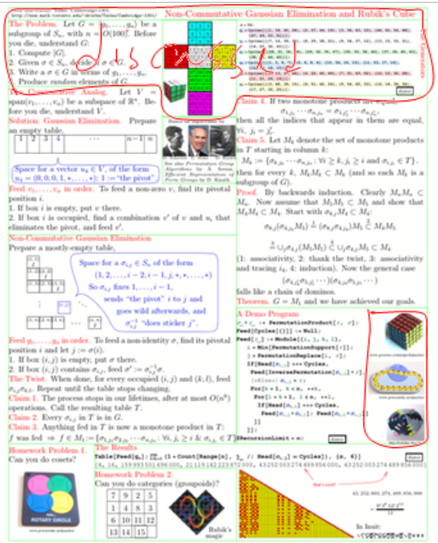
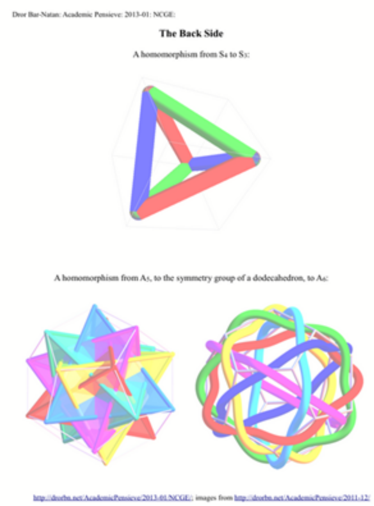
Hours 1-2 on Wednesday September 3: The NCGE handout and the very basics of groups.





+ Def of a group

uniqueness of e

Uniqueness of g^{-1}

Cancellation

$(ab)^{-1} = b^{-1}a^{-1}$

Examples $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, 2\mathbb{Z}, V$

$\mathbb{Q}^\times, \mathbb{R}^\times, \mathbb{C}^\times, F^\times$

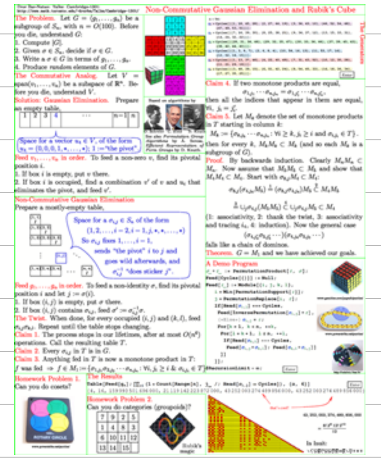
The Rubik's cube

S_n, [ ] notation

$S(\text{cube})$

$O(3)$

Hour 3 on Friday September 5: More NCGE.
http://drorbn.net/25-347 → It's a tough class!
The syllabus is still not public!
Say hello to Jacob, wish luck to Matt!
Permutation product is "usual" composition! → $\sigma \cdot \tau = \sigma \circ \tau$



Goal: within your lifetime, understand

$G = \langle g_1, \dots g_\alpha \rangle \subset S_n$:

1. $|G| = ?$ 2. $\sigma \in G?$
3. write $\sigma \in G$ in terms of $g_1 \dots g_\alpha$
4. Random σ.

on board!

Classical row reduction as in handout.

Describe the NCGE algorithm as in handout.

Claim 0. The process ends after at most n^6 steps. Call the resulting table T.

Claim 1 Every σ_{ij} in T is in G.

Claim 2 Anything fed to T is now a monotone product $\sigma_{1j_1} \sigma_{2j_2} \sigma_{3j_3} \dots$ $j_i \geq i$ —

Claim 3 If two monotone products are equal,

$$\sigma_{1j_1} \dots \sigma_{nj_n} = \sigma_{1j_1'} \dots \sigma_{nj_n'}$$

then all the indices are equal, $\forall i\ j_i = j_i'$.

Claim 4 Let $M_k = \left\{ \begin{matrix} \text{monotone products} \\ \text{beginning with } k \end{matrix} \right\} = \{ \sigma_{kj_k} \cdots \sigma_{nj_n} \}$, then for every k, $M_k \cdot M_k \subset M_k$ (and so each M_k is a subgroup of S_n.

Proof. By backwards induction. Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4} M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4}) M_5 \overset{2}{\subset} M_4 M_5$$

$$\overset{3}{=} \cup_j \sigma_{4,j}(M_5 M_5) \overset{4}{\subset} \cup_j \sigma_{4,j} M_5 \subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case

$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

Theorem. $G = M_1$ and we have achieved our goals.

<u>Example</u> $\sigma_1 = (123) = 2314$ $\quad \sigma_2 = (12)(34) = 2143$, in S_4

11 I			
12 $\sigma_1 = 2314$ 1	22 I		
13 $\sigma_{12}^2 = 3124$ 2	23 $\sigma_{12}^{-1}\sigma_2 = 1342$ 3	33 I	
14 $\sigma_{23}\sigma_{13} = 4132$ 5	24 $\sigma_{13}^{-1}\sigma_{23}\sigma_{12} = 1423$ 4	34	44 I

on board (minus fills)

Feed $\sigma_1 = 2314$ ... Fed @ σ_{12}

Feed $\sigma_{12}^2 = 3124$ ... Fed @ σ_{13}

Feed $\sigma_2 = 2143$ ... Feed $\sigma_{12}^{-1}\sigma_2 = 1342$ .. Fed @ σ_{23}

Feed $\sigma_{12}\sigma_{23} = 2143$ ... Feed $\sigma_{12}^{-1}\sigma_{12}\sigma_{23} = \sigma_{23}$ ....

No point feeding $\sigma_{ij}\sigma_{kl}$ if $i<k$!

Feed $\sigma_{23}\sigma_{12} = 3412$ ... Feed $\sigma_{13}^{-1}\sigma_{23}\sigma_{12} = 1423$ ... to σ_{24}

Feed $\sigma_{23}\sigma_{13} = 4132$ .. to σ_{14}

Feed $\sigma_{24}\sigma_{12} = 4213$ ... Feed $\sigma_{14}^{-1}\sigma_{24}\sigma_{12} = 1423$ ... drop.

$\Rightarrow |G| = 4 \cdot 3 \cdot 1 \cdot 1 = 12$. Is $4123 \in G$?

write 2431 in terms of $\sigma_{1,2}$.