Homework Assignment 5

Problem 1. For any based space (X, x_0) define a natural non-zero map $\alpha_{(X,x_0)}\colon \pi_1(X, x_0) \to H_1(X)$. The challenge will be to show that your definition is well-defined.
What means "non-zero"? We don't have the tools yet to prove that H_1 is ever non-zero! So I will be happy enough with a map that is non-zero as per our intuitive notion of $H_1(S^1)$.
What means "natural"? That if $f\colon (X, x_0) \to (Y, y_0)$, then the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{\alpha_{(X,x_0)}} & H_1(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(Y,y_0) & \xrightarrow{\alpha_{(Y,y_0)}} & H_1(Y) \end{array}$$

(In other words, α should be a "natural transformation").

Problem 2. Show that the set $\Delta'_n \coloneqq \{(s_1, s_2, \ldots, s_n)\colon 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\} \subset I^n \subset \mathbb{R}^n$ is homeomorphic to the standard n simplex Δ_n via a map of the form $[v_0, \ldots, v_n]\colon \Delta_n \to \Delta'_n$ where $v_0, \ldots, v_n \in I^n$.

Problem 3. Using the alternative model of the n-simplex presented in the previous problem, show that

1. The n-cube I^n can be presented as a union of size $n!$ of n-simplices.
2. The product $\Delta_p \times \Delta_q$ can be presented as a union of size $\binom{p+q}{q}$ of $(p+q)$-simplices.

Problem 4. "Homotopies between maps" define an "ideal" within the category of topological spaces and continuous maps between them: the homotopy relation is an equivalence relation, and if $f_1 \sim f_2$, then $f_1 \circ g \sim f_2 \circ g$ and $g \circ f_1 \sim g \circ f_2$ whenever these compositions make sense. Show that the same is true for the notion "homotopy of morphisms between chain complexes", within the category Kom of chain complexes.