This is Dror Bar-Natan's 2025 MAT 1301 personal notebook. It is publically available but it comes with no guarantees whatsoever. Its content may or may not be correlated with the actual class content.

Bring a mug and a donut!

MAT 1301 Algebraic Topology

DROR BAR-NATAN, http://drorbn.net/25-1301

TA: Hadi Azizi.

This is a graduate course! [I will assume that you can complete details & that you do that!]

Topology (hard) $\xRightarrow[\text{MAT 1301}]{}$ Algebra (easy)

$\exists ?\, f : ◎ \to ○$ $\qquad\exists ?\, \varphi : G \to H ?$

$\pi_1(X, x_0) = \{f : [0,1] \to X : f(0) = f(1) = x_0\} / f_0 \sim f_1$ if $\exists f_t$ s.t.

$\qquad\qquad = \{f : (S^1, 1) \to (X, x_0)\}$

Prop $\sim$ is an equiv. relation.

Thm 1 w/ $[f]\cdot[g] := [f \cdot g]$, π_1 is a group.

done line

Prop if $\exists$ path h w/ $h(0) = x_0$ & $h(1) = x_1$,

then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Prop $\pi_1(\text{convex in } \mathbb{R}^n) = 0$ (the group w/ 1 element)

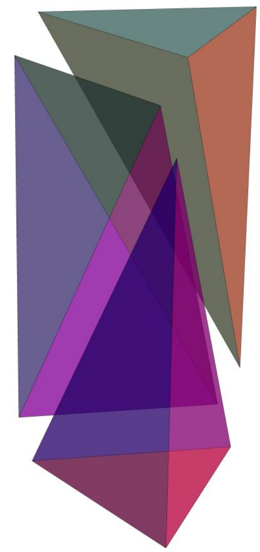
Def simply-connected.

Thm 2 $\pi_1(S^1, 1) \cong \mathbb{Z}$

Covering spaces, flat sets.

Tiny lifting, path lifting, homotopy lifting.

Proof of Thm 2.



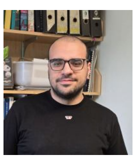
MAT1301S - Algebraic Topology

Toronto, Spring 2025

Agenda. Learn about the surprising relation between the easily deformed (topology) and the most rigid (algebra).

Ambition. Get to the Wirtinger presentation of the fundamental group of knot complements and to the definition of the Alexander polynomial as the order ideal of the first homology of the universal Abelian cover of a knot complement. Both of these goals are just a bit too far, yet they can serve as perfect motivators for all that isn't too far.

Instructor. Dror Bar-Natan, drorbn@math.toronto.edu (for course administration matters only; math on email is slow and prone to misunderstandings, so I generally avoid it). Office: Bahen 6178.



Hadi Azizi

Teaching Assistant. Hadi Azizi, hadi.azizi@mail.utoronto.ca.

Classes. Mondays 1-2 and Tuesdays 2:30-4:30, at Bahen 6183.

Office Hours. Tuesdays at 9:30-10:30 at Bahen 6178 and online at https://drorbn.net/vchat.

Text. Mostly Alan Hacther's Algebraic Topology, but also several specialized sources for specialized topics.

URL. https://drorbn.net/25-1301.

Course Calendar

#	Week of ...	Things
1	January 6-10	Blackboards for Monday January 6. Handout: About This Class. Tuesdsay or Wednesday: HW will likley be assigned.
2	January 13-17	Tuesdsay or Wednesday: HW will likley be assigned.
3	January 20-24	Tuesday: Class photo during break. Tuesdsay or Wednesday: HW will likley be assigned.
4	January 27-31	
5	February 3-7	Thursday (tentative): our term test will take place in the evening.
6	February 10-14	Tuesdsay or Wednesday: HW will likley be assigned. Friday is the last date for undergraduate students to drop this class without penalty.
R	February 17-21	Monday is Family Day, and the week is Reading Week - no classes and no office hours.
7	February 24-28	Tuesdsay or Wednesday: HW will likley be assigned. Friday is the last date for graduate students to drop this class without penalty.
8	March 3-7	Tuesdsay or Wednesday: HW will likley be assigned.
9	Mrach 10-14	
10	March 17-21	Tuesdsay or Wednesday: HW will likley be assigned.
11	March 24-28	Tuesdsay or Wednesday: HW will likley be assigned.
12	March 31 - April 4	

Further resources:

- The University of Toronto School of Graduate Studies Calendar.
- A similar class I gave in 2002 in Jeruasalem.
- A similar class I gave in 2004-05 in Toronto, which included also point-set topology.
- A similar classs I gave in 2007-08 in Toronto, which included also differential geometry.
- A summer class on homology given in Ghana in 2010.
- An informal summer class on homology given in Toronto in 2017.
- My 25-1301 Pensieve Folder.
- The source files used to create this page: sources.zip.

About This Class

Agenda. Learn about the surprising relation between the easily deformed (topology) and the most rigid (algebra).

Ambition. Get to the Wirtinger presentation of the fundamental group of knot complements and to the definition of the Alexander polynomial as the order ideal of the first homology of the universal Abelian cover of a knot complement. Both of these goals are just a bit too far, yet they can serve as perfect motivators for all that isn't too far.

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Optimistic Plan. 5 weeks of fundamental groups: paths and homotopies, the fundamental group, coverings and the fundamental group of the circle, Van-Kampen's theorem, the Wirtinger presentation, the general theory of covering spaces. Then 7 weeks of homology: simplices and boundaries, prisms and homotopies, abstract nonsense and diagram chasing, axiomatics, degrees, CW and cellular homology, subdivision and excision, the generalized Jordan curve theorem, salad bowls and Borsuk-Ulam, cohomology, products, the Alexander polynomial.

Warning and Recommendation. This will be a tough and very abstract class, designed for graduate students in mathematics. I will make every effort to make it understandable, but certain parts of the material require a very high level of mathematical sophistication. Many bits of material will only be sketched, on the understanding that students should be able to complete the details on their own. Don't take this class unless you are ready to put in the intellectual effort that will be involved! Every bit of this class absolutely makes sense. But you'll have to think hard at all times, and be ready to repeatedly adjust your perspective, to see that this is so. Don't let go! If you'll fall behind you'll find it nearly impossible to catch up. This actually does not mean "do your homework in time" (highly recommended anyway). It means "do your deep thinking in time".

Marking Scheme. There will be one term test (up to 25% of the total grade) and about 7 homework assignments (also up to 25%). There will then be a final exam counting for whatever points you did not pick up from the term test and the homework assignments, yet counting for no more than 85%. So for example, if you got 20/25 on the term test and 22/25 on the homework, your total so far is 42% and thus the final is worth 58%. But if you got 0/25 on both the term test and the homework assignments, the maximal grade you can get in this class will be 80%. A renormalization function of the form $m \mapsto 100(m/100)^\gamma$ will then be applied to your overall mark, for a constant $0 < \gamma \leq 1$ which will be chosen later with a bias in favour of $\gamma = 1$. Similar renormalizations may or may not also be applied to individual homework assignments or the tests.

The Term Test will likely take place in the evening on Thursday Februatry 6. There will be no make-up term test, though note that if you miss the term test for any reason, the weight of the final automatically expands.

Homework. Assignments will be posted on the course web page and on Crowdmark (usually on Tuesdays or Wednesdays) approximately on the weeks shown in the class timeline. They will be due a week later and they will be (at least partially) marked by the TA. All students (including those who join the course late) will receive a mark of 0 on each assignment not handed in; though in computing the homework grade, your worst 2 assignments will not count. I encourage you to discuss the assignments with other students or even browse the web, so long as you do at least some of the thinking on your own and you write up your own solutions.

Solution Sets. No "official" solution sets for homework assignments and for the term test will be provided. However, I encourage students who got 90% or more on any given assignment (or test) to scan and send me their marked assignments, and I will post their solutions on the class web site as a service to everybody else. Notes:

- Please hide student ID numbers in all such scans! You may or may not wish to also suppress your name.
- Scans must of of good quality: they must be of high resolution and contrast, and the paper must look "flat". Use a flatbed scanner or one of those phone apps that simulate a flatbed scanner. Do not use a cellphone camera directly.
- I prefer to receive PDF files, but I'll also take .jpg images if a solution is only 1-3 pages long.
- You may fix and improve your solution set before sending it to me, yet please keep a clear distinction between what was written before submission and what was written after; for example, use a pen of a different colour for the later edits.
- I will remove all solution sets from the class web site sometime in July 2025.

Class Photo. Just for fun, on the Tuesday of the third week of classes I will take a class photo and post it here. If you are shy or worried about your privacy, don't be in it.

Accessibility Needs. The University of Toronto is committed to accessibility. If you require accommodations for a disability, or have any accessibility concerns about the course, the classroom or course materials, please contact Accessibility Services as soon as possible.

Quercus Other than for email announcements and perhaps for grade distribution, Quercus will not be used in this class.

How to Succeed in this Class

- Keep up! Don't fall behind on reading, listening, and doing assignments! MAT1301 will move at a very high pace. New material is covered once and just once. There will be no going over the same thing again and again - if you fall behind, you stay behind.
- Unless you are an Einstein, there is no way to do well in this class merely by attending lectures - you must think about the material much more than just 3 hours a week if you want it to sink in. And if you are planning on not attending lectures, well, think again. Most people find it very hard to pace their own studies without a human contact; if you'll try, you are likely to discover the hard way that you belong to the majority.
- Take your own class notes, in your own handwriting, and strive to make them as complete as possible. Writing "burns" things into your brain and forces you to keep from daydreaming. And nothing beats reading your own notes when you review the material later on.
- Math is about understanding, not about memorizing. To understand is to internalize; it is to come to the point where whatever the professor does on the blackboard or whatever is printed in the books becomes yours; it is to come to the point where you appreciate why everything is done the way it is done, what does it mean, what are the reasons and motivations and what is it all good for. Don't settle for less!
- This said, you are expected know all definitions and all proofs, and memorizing helps. Memorizing is sometimes the first step towards understanding. If you remember something, you can think about it on the subway ride back home instead of reading advertisements.
- Keep asking yourself questions; many of them will be answered in class, but not all. Remember the old Chinese proverb:

Teachers open the door, but you must enter by yourself

师傅领进门,修行靠个人!

Print 25 copies of index & About.

$\pi_1(X, x_0) = \{f : [0,1] \to X : f(0)=f(1)=x_0\} / \sim$ homotopy

Thm w/ $[f]\cdot[g] := [f\cdot g]$, π_1 is a group.

Prop if $\exists$ path h w/ $h(0)=x_0$ & $h(1)=x_1$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Prop $\pi_1(\text{convex in } \mathbb{R}^n) = 0$ (the group w/ 1 element)

Def simply-connected.

Thm $\pi_1(S^1, 1) \cong \mathbb{Z}$

Covering spaces, flat sets.

Tiny lifting, path lifting, homotopy lifting.

done line.

Proof of Thm

Thm (The Fundamental theorem of algebra) Every non-constant polynomial in $\mathbb{C}[z]$ has a root in $\mathbb{C}$.

Let $P(z) = z^n + \sum_{k<n} a_k z^k$ and assume it has no roots

Consider

$$f(s) = \frac{P(Re^{2\pi i s})}{|P(Re^{2\pi i s})|} \cdot \frac{|P(R)|}{P(R)}$$

I claim that $0 = [f] = n$ so $n = 0$.

$R \to r$ $\qquad$ $P \to P_t = z^n + t\sum a_k z^k$

MAT 1301 on 250113: The Fundamental Group of S^1, the fundamental theorem of algebra, functoriallity.
HW01 is online and is due on Wednesday at 11:59pm!

Reminder

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "p"] \\ (X,x_0) \arrow[r, "f"] \arrow[ur, dotted, "\tilde f"] & (S^1,1) \end{tikzcd}$$

$p(s)=e^{2\pi i s}$

$\exists!\ \tilde f$ if 1. f is tiny $f(X)\subset$ f.k.t.

2. $(X,x_0)=(I,0)$

3. $(X,x_0)=(I^2, 0=(0,0))$

Thm $\pi_1(S^1,1) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}$

Pf $\phi: n \mapsto \begin{pmatrix} z\mapsto z^n \\ \| \\ p\circ(s\mapsto ns)\end{pmatrix}$ $\quad \psi: [f] \longmapsto \tilde f(1)$

1. $\psi([f])\in\mathbb{Z}$? 2. ψ is well-def? 3. $\phi/\!/\psi = Id_{\mathbb{Z}}$

4. $\psi/\!/\phi = Id_{\pi_1}$ 5. ϕ is a homo?

Thm (The fundamental theorem of algebra) Every non-constant polynomial in $\mathbb{C}[z]$ has a root in $\mathbb{C}$.

Let $P(z)=z^n+\sum_{k<n} a_k z^k$ and assume it has no roots.

Consider

$$F(s)=\frac{P(Re^{2\pi i s})}{|P(Re^{2\pi i s})|}\cdot\frac{|P(R)|}{P(R)}$$

I claim that $0=[F]=n$ so $n=0$.

done line

$R\to r$ $\qquad P\to P_t=z^n+t\sum a_k z^k$

If $|z|>1, \sum|a_k|$

$|\sum a_k z^k| \leq \sum |a_k||z|^k \leq \sum|a_k||z|^{n-1} < |z|^n$

MAT 1301 on 250114: Functoriallity.
HW01 is online and is due tomorrow at 11:59pm!

Aside: Piano $\downarrow$ octave

Assume $q(z)=z^n+\sum_{k<n} a_k z^k$ has no roots. Then

replace R with tR $\quad 0 = \left[f(s) = \frac{q(Re^{2\pi i s})}{|\ \ |} \cdot \frac{|\ \ |}{q(R)} \right] = n \quad$ Replace $a_k \to t a_k$ w/ $R>1, \sum|a_k|$

$|\sum a_k z^k| \le \sum |a_k||z|^k \le \sum|a_k||z|^{n-1} < |z|^n$

[V.S., L.T, compositions] [Top spaces, cont. fnctns, comp]

[groups, homo, comp] [sets, fnctns, comp]

Def A Category $\mathcal{C} = (\mathrm{Obj}_{\mathcal{C}}, \mathrm{Mor}, \circ, \mathrm{Id})$

s.t. Associativity, identity

Examples The above + based spaces + homotopy classes of paths, + the game of 15 + tangles.

Def $F:\mathcal{C}\to\mathcal{D}$ where $\mathcal{C}$ & $\mathcal{D}$ are categories

1. preserves compositions
2. preserves identities.

Examples: Forget $\times 3$, $*: \mathrm{Vect}\to\mathrm{Vect}^{op}$, $**: \mathrm{Vect} \circlearrowleft$

$\boxed{\pi_1 : \mathrm{Top}_\circ \longrightarrow \mathrm{GRP}}$

Def $A\subset X$ in Top. A retract $r:X\to A$ is

Examples  $\to A$; yet $\nexists$ retract $r: D^1\to S^0$

Thm $\nexists$ retract $r: D^2\to S^1$.

Thm The Brouwer F.P. Theorem.

done line

Thm $\pi_1(S^n) = 0$ for $n \geq 2$

Aside: $[X, Y] = \ldots$

Prop If $\gamma \in [S^1, S^1]$ is odd, then $[\gamma]$ is odd.

Pf WLOG, $\gamma(1) = 1$. Also regard $\gamma : [0,1] \to S^1$ w/ $\gamma(0) = 1$,

$\gamma(\frac{1}{2} + s) = -\gamma(s)$.

Let $\bar{\gamma} : [0, 1/2] \to \mathbb{R}$ be the lift of γ on $[0, 1/2]$. Then

$p(\bar{\gamma}(1/2)) = \gamma(1/2) = -1$, so $\bar{\gamma}(1/2) = \frac{k}{2}$, k odd.

Let $\bar{\bar{\gamma}} : [0, 1] \to \mathbb{R}$ be
$$\bar{\bar{\gamma}}(s) = \begin{cases} \bar{\gamma}(s) & 0 \leq s \leq 1/2 \\ \frac{k}{2} + \bar{\gamma}(s - \frac{1}{2}) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

then $\bar{\bar{\gamma}} = \tilde{\gamma}$ so $\tilde{\gamma}(1) = k$.

Thm (Borsuk–Ulam for $n=2$) Given $f : S^2 \to \mathbb{R}^2$,

$\exists x \in S^2$ s.t. $f(x) = f(-x)$.

Pf Let $\gamma = \dfrac{f(x) - f(-x)}{|\quad|}\Big|_{S^1 \subset S^2}$ The γ is odd & $\gamma = 0$.

Thm If S^2 is presented as a union of 3 closed sets $A_1 \cup A_2 \cup A_3$, then at least one of them contains an antipodal pair.

Pf $f(x) = (d(x, A_1), d(x, A_2))$

Thm $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ $\rightsquigarrow$ $\pi_1(T^2) = \mathbb{Z}^2$

Conjecture $\pi_1(X)$ is always Abelian.

MAT 1301 on 250120 H7: Applications of the fundamental group of the circle.
HW02 is online and is due on Wednesday at 11:59pm!

Last class: $\pi_1 : \mathrm{Top}_0 \to \mathrm{GRP}$ is a Functor! $\nexists$ retract $r : D^2 \to S^1$ Brouwer F.P.

Prop If $\gamma \in [S^1, S^1]$ is even, then $[\gamma]$ is even.

Aside: $[X, Y] = \ldots$

Prop If $\gamma \in [S^1, S^1]$ is odd, then $[\gamma]$ is odd.

Pf 1 WLOG, $\gamma(1) = 1$. Also regard $\gamma : [0,1] \to S^1$ w/ $\gamma(0) = 1$,

$\gamma(\frac{1}{2} + s) = -\gamma(s)$.

Let $\bar{\gamma} : [0, 1/2] \to \mathbb{R}$ be the lift of γ on $[0, 1/2]$. Then

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then $\bar{\bar{\gamma}} = \tilde{\gamma}$ so $\tilde{\gamma}(1) = k$.

Pf 2 Def top group; **Thm**: If G is a topological group, $\gamma_{1,2} \in \pi_1(G, 1)$

then $[\gamma_1 \cdot \gamma_2] = [\gamma_1 * \gamma_2]$. **Pf** $\gamma_1 \cdot \gamma_2 = (\gamma_1 \cdot e) * (e \cdot \gamma_2) \sim \gamma_1 * \gamma_2$
$\qquad = (e \cdot \gamma_2) * (\gamma_1 \cdot e) \sim \gamma_2 * \gamma_1$

Not completed!

Thm (Borsuk–Ulam for $n=2$) Given $f : S^2 \to \mathbb{R}^2$,

$\exists x \in S^2$ s.t. $f(x) = f(-x)$.

Pf Let $\gamma = \dfrac{f(x) - f(-x)}{| \quad |}\Big|_{S^1 \subset S^2}$ the γ is odd & $\gamma = 0$.

Thm If S^2 is presented as a union of 3 closed sets $A_1 \cup A_2 \cup A_3$, then at least one of them contains an antipodal pair.

Pf $f(x) = (d(x, A_1), d(x, A_2))$

done line.

Thm $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) \rightsquigarrow \pi_1(T^2) = \mathbb{Z}^2$

MAT 1301 on 250121 H8-9: van Kampen examples.
HW02 is online and is due on Wednesday at 11:59pm!
HW03 will be online by tomorrow.

Class photo during break!

Borsuk–Ulam: Given $f: S^2 \to \mathbb{R}^2$, $\exists x \in S^2$ $f(x) = f(-x)$

Thm 1 There's a straight line cutting across Toronto, that cuts in half both its population and its area.

Thm 2 Let $\gamma: S^1 \to S^1$

0. γ even $\Rightarrow [\gamma] \in \mathbb{Z}$ is even

1. γ odd $\Rightarrow [\gamma] \in \mathbb{Z}$ is odd.

Thm 3 If G is a top. group, $\gamma_1 \cdot \gamma_2 \sim \gamma_1 * \gamma_2 \sim \gamma_2 * \gamma_1$

Thm Let $g_1, g_2: \mathbb{R}^2 \to \mathbb{R}$ be integrable w/ compact support. Then $\exists H \subset \mathbb{R}^2$, a half plane, s.t. $\int_H g_i = \int_{H^c} g_i$ $i=1,2$.

Pf $F(r,\theta) = \left(\int_{H_{r,\theta}} g_1, \int_{H_{r,\theta}} g_2 \right) \ldots$

Pf of thm 2 part 1 from thm 3

Thm $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ $\rightsquigarrow$ $\pi_1(T^2) = \mathbb{Z}^2$

Homotopy of maps; homotopy equivalence,

$\pi_1: \mathrm{Top}_0/\sim \longrightarrow \mathrm{GRP}$.

Example:

 $\sim S^1 \sim$ Mobius band.

Q. Is $\pi_1(X)$ is always Abelian?

Van-Kampen Plan: statement for 2 sets, examples, } Sources: 060110, next pages.
categorical comments, proof, statement for many sets.

van Kampen's thm: If $X = U_1 \cup U_2$ is a union of open sets & $U_1 \cap U_2$ is connected & $x_0 \in U_1 \cap U_2$, then

lousy version: $\pi_1(X) \cong \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$

better version: The obvious map

$$\pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$$

is an isomorphism.

Definition: Given G_1, G_2, define

$$G_1 * G_2 = \left(\begin{array}{l} \text{words in } G_1 \sqcup G_2 = \{\bar g : g \in G_1\} \cup \{\hat g : g \in G_2\} \\ \text{under concat, mod } \bar e = \hat e = (),\ \bar g_1 \bar g_2 = \overline{g_1 g_2} \\ \quad \hat g_1 \hat g_2 = \widehat{g_1 g_2} \end{array} \right)$$

Example: $\mathbb{Z} * \mathbb{Z} = \{a^n\} * \{b^n\} = F(a,b)$

$\mathbb{Z}/2 * \mathbb{Z}/2 = \ldots$

Definition: Given $H \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \begin{matrix} G_1 \\ G_2 \end{matrix}$, complete it to

$$\begin{tikzcd} & G_1 \arrow[dr, "\psi_1 : g \mapsto \bar g"] & \\ H \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & G_1 *_H G_2 \\ & G_2 \arrow[ur, "\psi_2 : g \mapsto \hat g"'] & \end{tikzcd}$$

$$G_1 *_H G_2 := G_1 * G_2 \Big/ \begin{array}{c} \forall h \in H, \\ \overline{\varphi_1(h)} = \widehat{\varphi_2(h)} \end{array}$$

$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & & \\ U_1 \cap U_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X = U_1 \cup U_2 \arrow[r, "\pi_1"] & {} \\ & U_2 \arrow[ur, "j_2"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & \pi_1(U_1) \arrow[dr] \arrow[drr, bend left, "j_{1*}"] & & \\ \pi_1(U_1 \cap U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_2(U_2) \arrow[r] & \pi_1(U_1 \cup U_2) \\ & \pi_1(U_2) \arrow[ur, "\psi_2"] \arrow[urr, bend right, "j_{2*}"'] & & \end{tikzcd}$$

Math 1300 Topology, Tuesday Jan 10 2006 (2 hours)

http://katlas.math.toronto.edu/0506-Topology

in ward

<u>Van-Kampen' Theorem</u> If $X = U_1 \cup U_2$, U_1 & U_2 are open $b \in U_1 \cap U_2$ and $U_1 \cap U_2$ is connected, then

$$\pi_1(X) \simeq \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

$$\begin{tikzcd} & U_1 \arrow[dr, hook, "j_1"] & \\ U_1 \cap U_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & U_1 \cup U_2 \\ & U_2 \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$

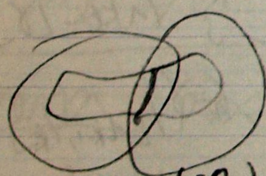
$$\Downarrow$$

$$\begin{tikzcd} & \pi_1(U_1) \arrow[r] & {} \\ \pi_1(U_1 \cup U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \\ & \pi_1(U_2) \arrow[r] & {} \end{tikzcd}$$

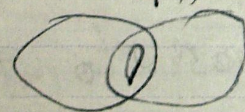
If $H \xrightarrow{\varphi_1} G_1$, $H \xrightarrow{\varphi_2} G_1$ then $G_1 * G_2 = \left(\begin{array}{l} \text{Words in } G_1 \sqcup G_2 = \{\bar g : g \in G_1\} \cup \{\hat g : g \in G_2\} \\ \text{under concat, mod } \bar e = \hat e = 1,\ \bar g_1 \bar g_2 = \overline{g_1 g_2},\ \hat g_1 \hat g_2 = \end{array} \right.$

$$G_1 *_H G_2 = G_1 * G_2 \Big/ \overline{\varphi_1(h)} = \widehat{\varphi_2(h)} \quad \forall h \in H$$

<u>Idea</u>



yet



<u>Examples</u> 0. $\pi_1(S^1)$

1. $\pi_1(T^2)$
2. $\pi_1(\Sigma_g)$ / Abelinization
4. Puncturing a 3-d nbd in X
3. $\pi_1(S^3)$ via $S^3 =$ Union of two solid tori
5. $\pi_1(\mathbb{R}P^3)$
6. $\pi_1(T_{8,3}^c)$